So sortiert man heute!

Sebastian Wild

www.wild-inter.net

based on joint work with Markus Nebel, Conrado Martínez, Ian Munro, Tamio Nakajima

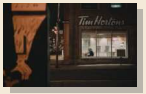
Marburg
University

Outline

1 Context

2 Java's Sorting Methods

3 Dual-Pivot Quicksort



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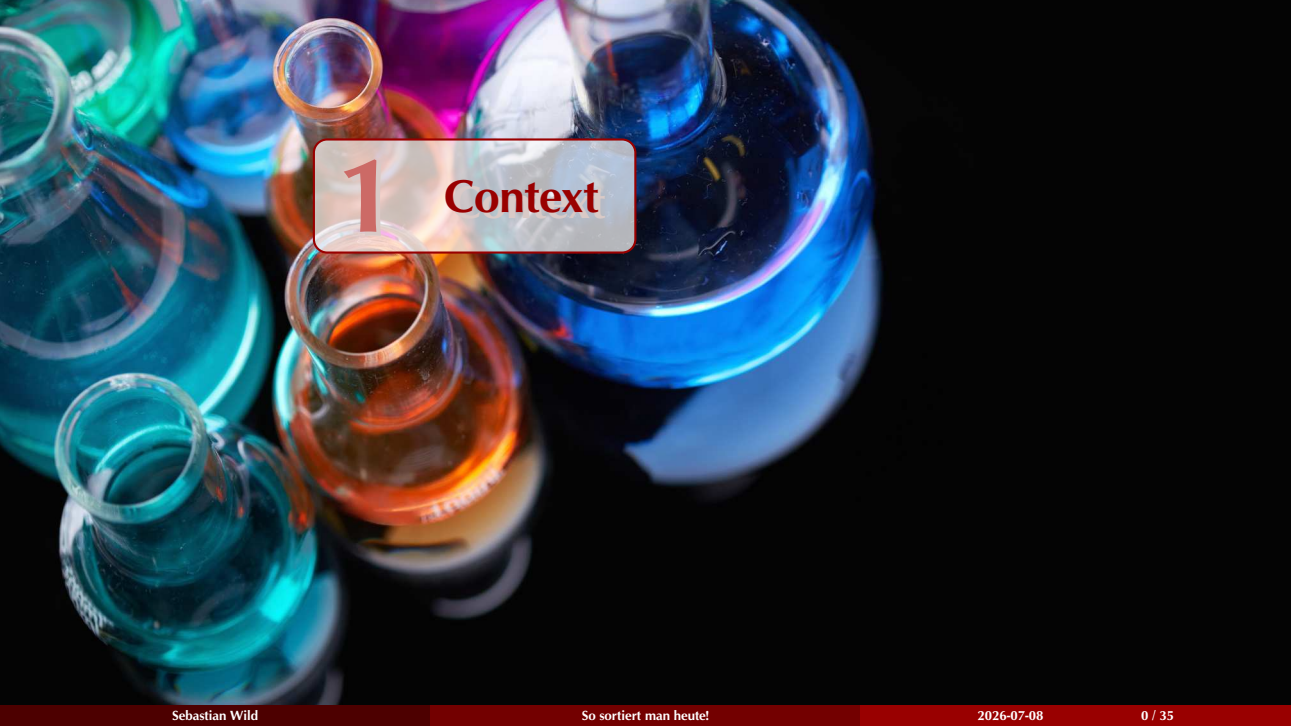
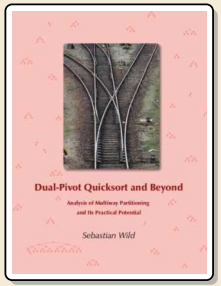


1 Context

Persönliche Geschichte

1. Dual-Pivot Quicksort

- Zufällig von Neuerung für Java 7 erfahren
- offensichtliche Fragen
  - *Warum ist Dual-Pivot Quicksort schneller?*
  - *Warum erst jetzt entdeckt?*
  - *Geht's noch besser?*
- Einige Antworten in meiner Doktorarbeit



2. Timsort

- 10+ Jahre im Einsatz
- **Aber:** Wiederholter Ärger

2015: subtiler algorithmischer Bug entdeckt!
Fix in Python und Java unterschiedlich

2018: subtiler Bug in Java-Fix entdeckt!

- 2018: Timsort bis zu 50% verschwenderisch!

⇝ **Fundamentales Problem in Merge-Policy!**

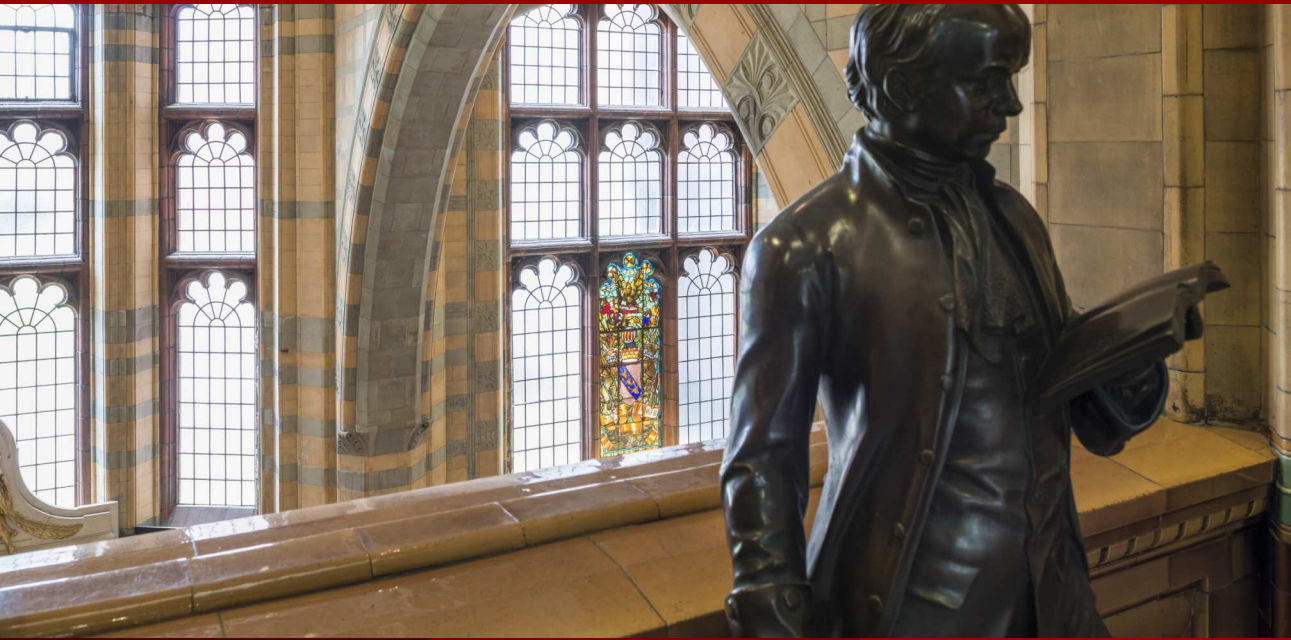
- *Unsere Merge-Policy* ***Powersort*** *behebt all das!*

⇝ ersetzt sukzessive alte Policy in Libraries

POWER sort=↑

Munro, Wild: *Nearly-Optimal Mergesorts: Fast, Practical Sorting Methods That Optimally Adapt to Existing Runs*, ESA **2018**

What Is a University?



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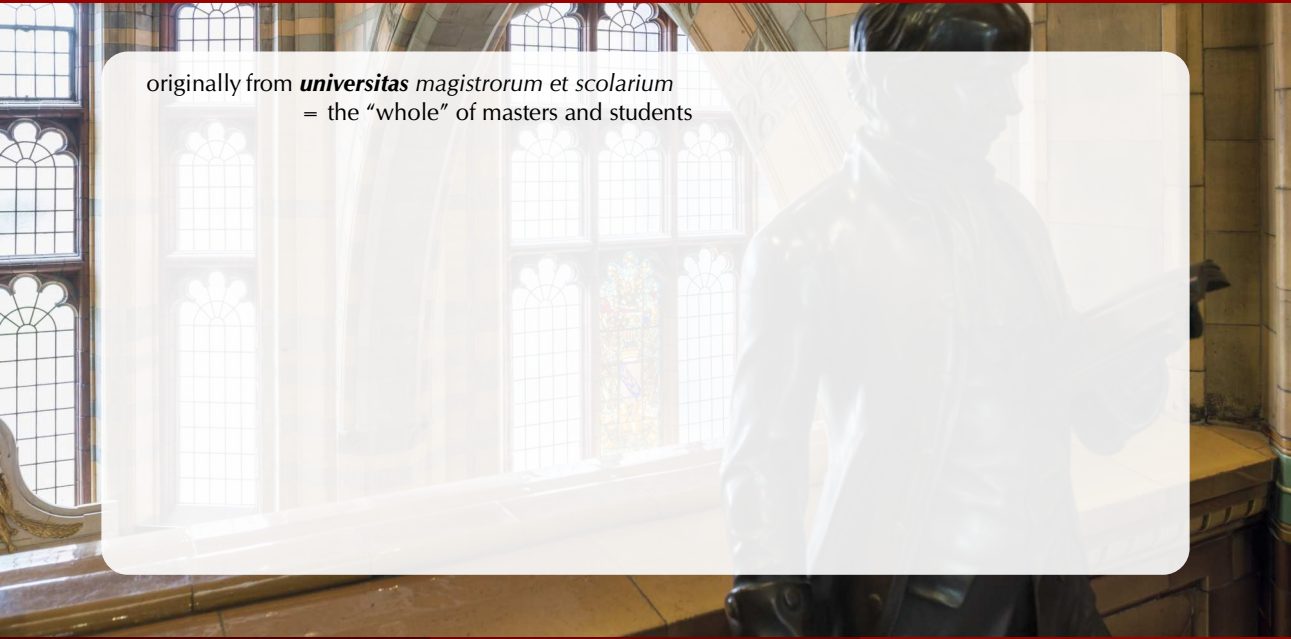

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= the “whole” of masters and students

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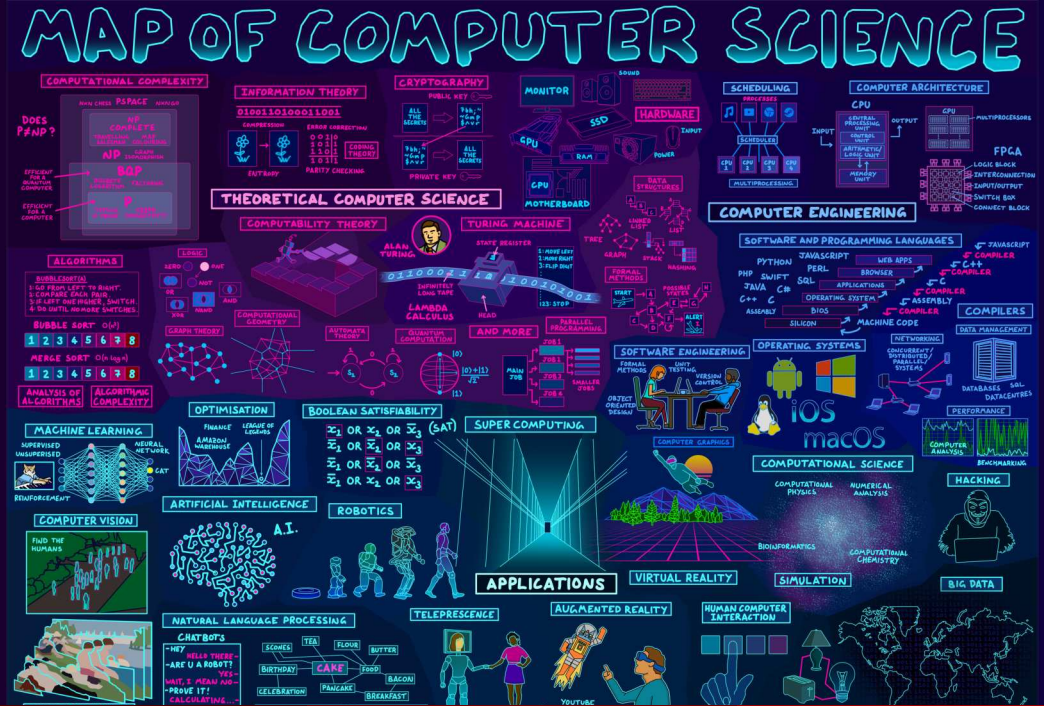
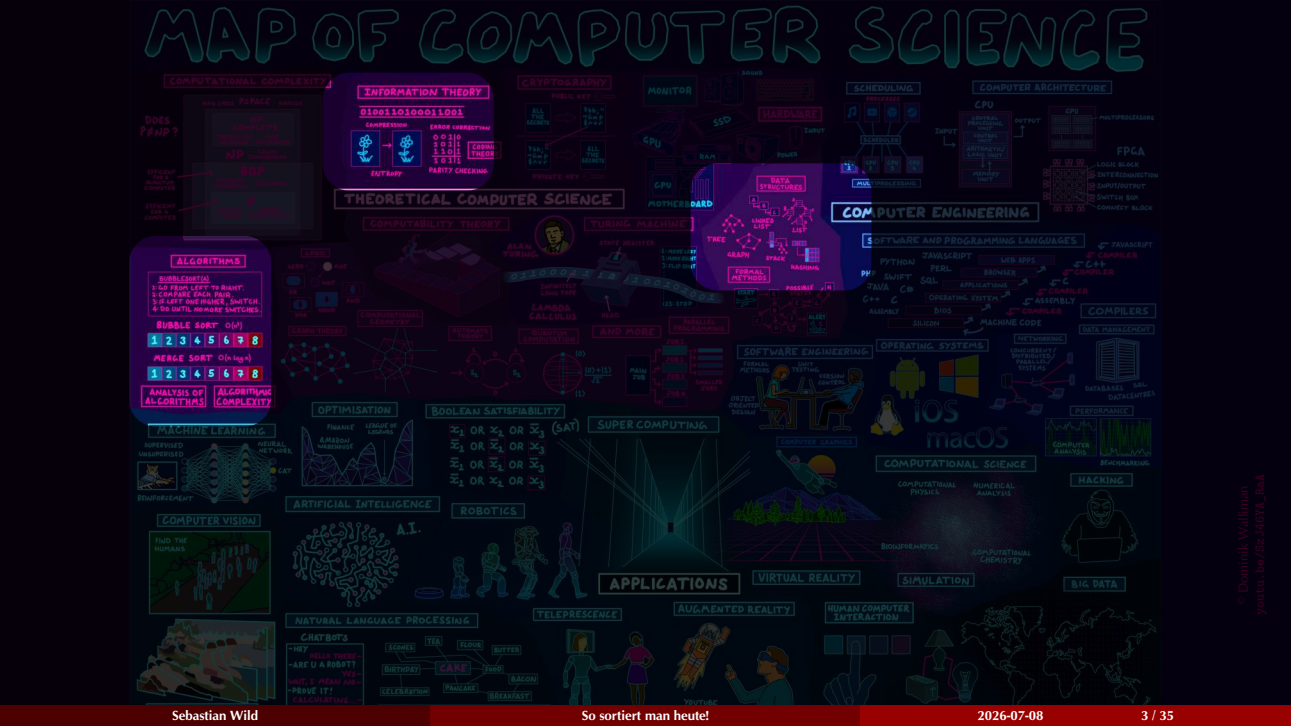
- Chance to get involved in forefront of human inquiry!

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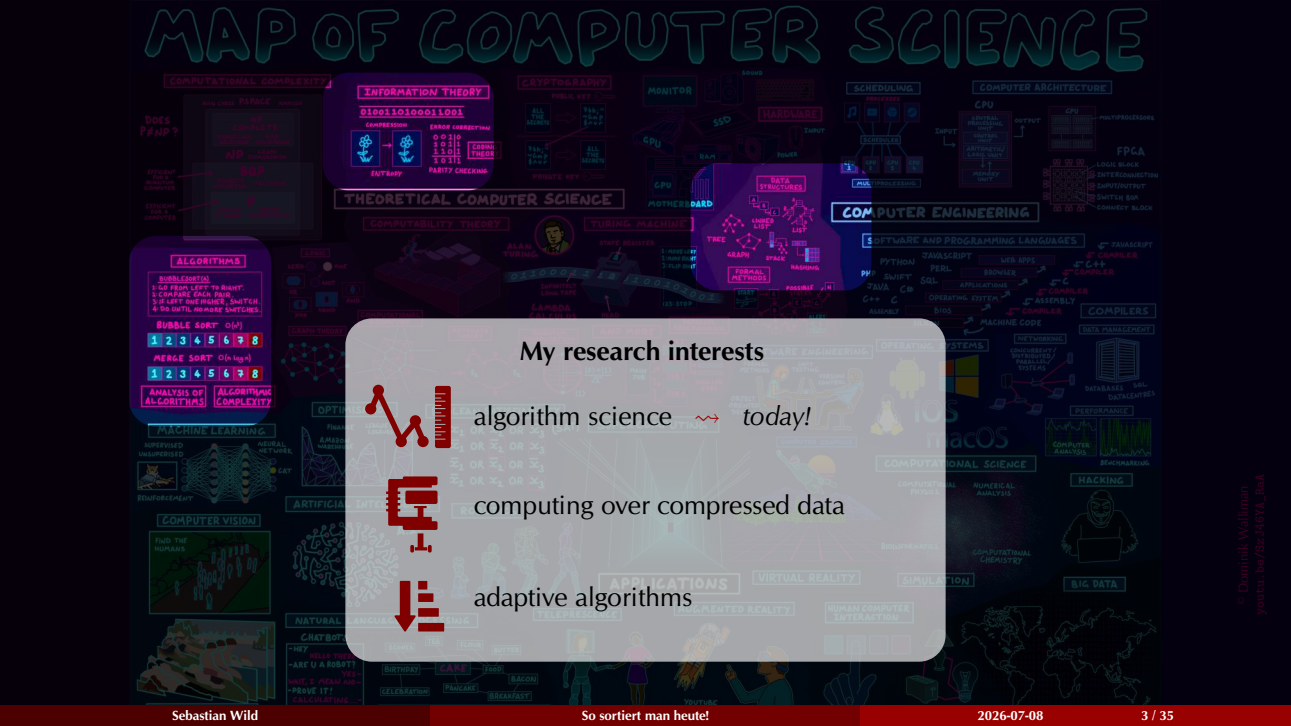
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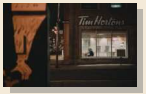

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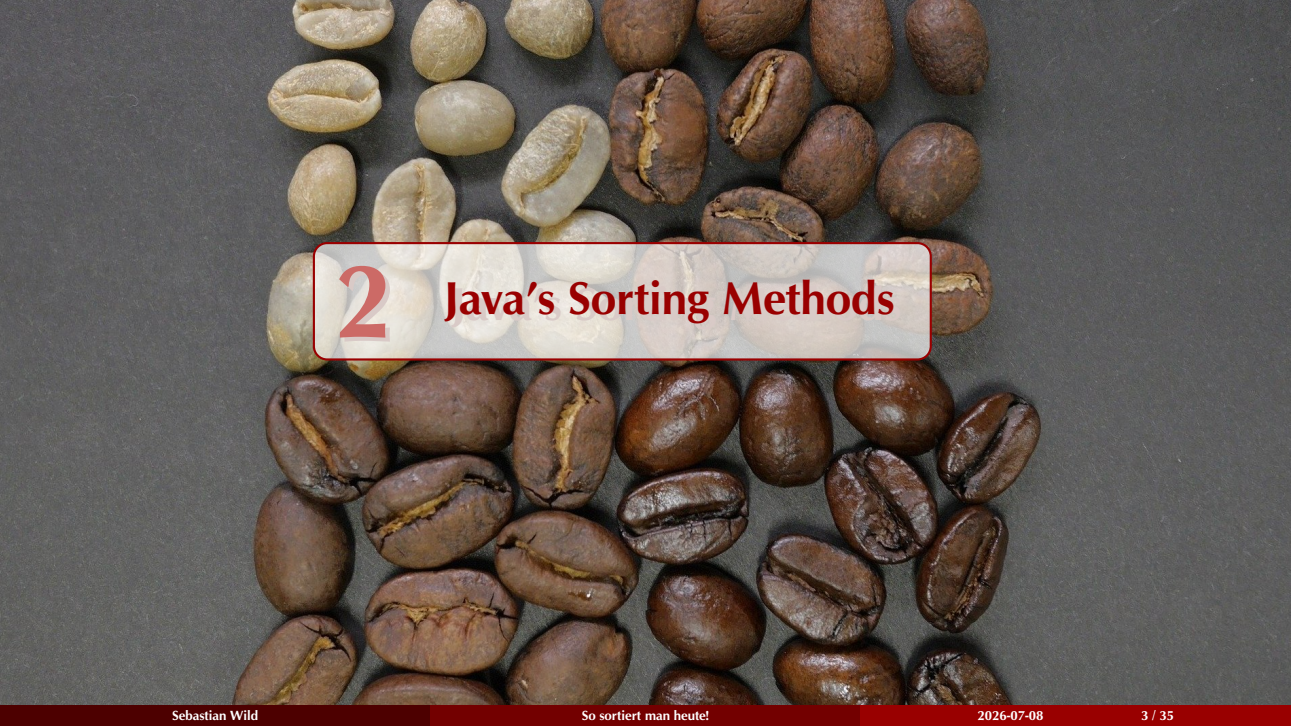

2 Java's Sorting Methods

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Adaptive algorithm: exploit “structure” of input

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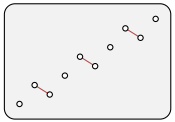

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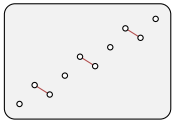

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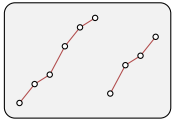

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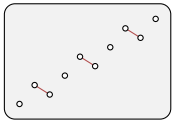

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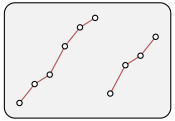

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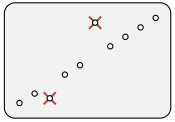

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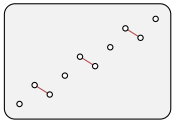

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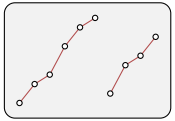

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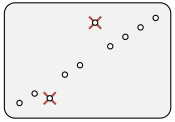

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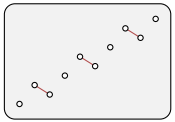

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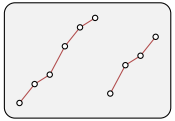

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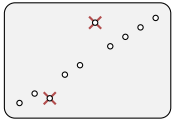

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optimal algorithms known for many measures of presortedness

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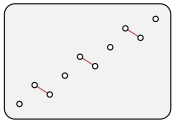

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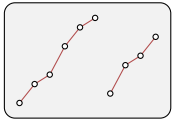

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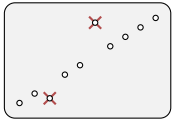

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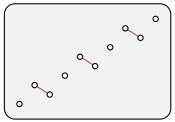

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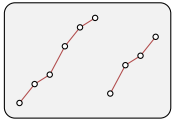

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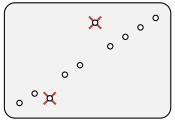

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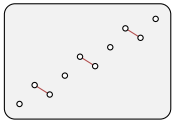

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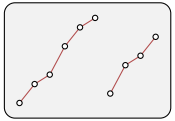

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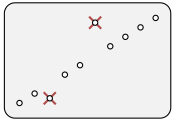

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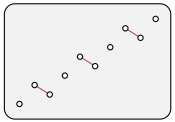

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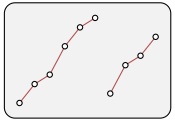

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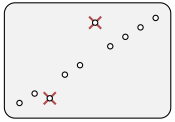

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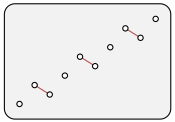

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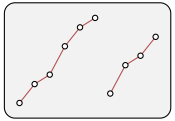

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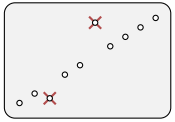

- . . . many more

up to constant factors!

optimal algorithms known for many measures of presortedness

Want:

- Optimal up to lower order terms

practical methods

- **low overhead** for detecting presortedness

Adaptive Sorting

Adaptive algorithm: exploit “structure” of input

⇝ **adaptive sorting**: exploit “presortedness”

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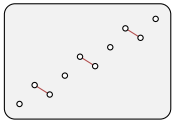

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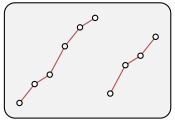

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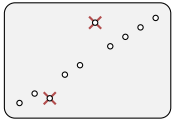

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Algorithm Science

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Algorithm & Systems Engineering

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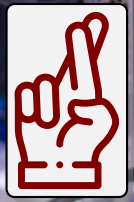

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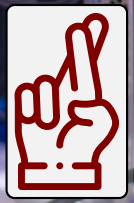

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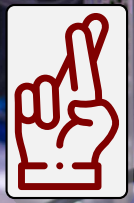

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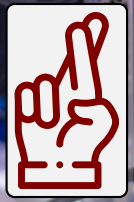

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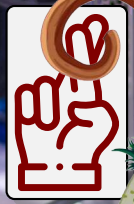

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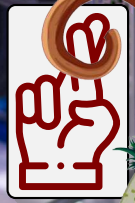

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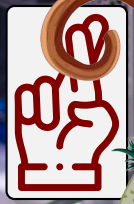

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Platform-independent performance guarantees

- concrete quantitative model
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Case Study: Sorting in Java Library

Why Java?

Java™

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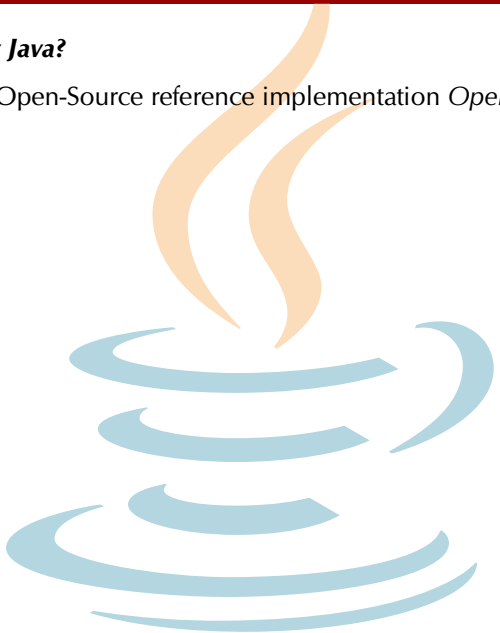
- Open-Source reference implementation *OpenJDK*



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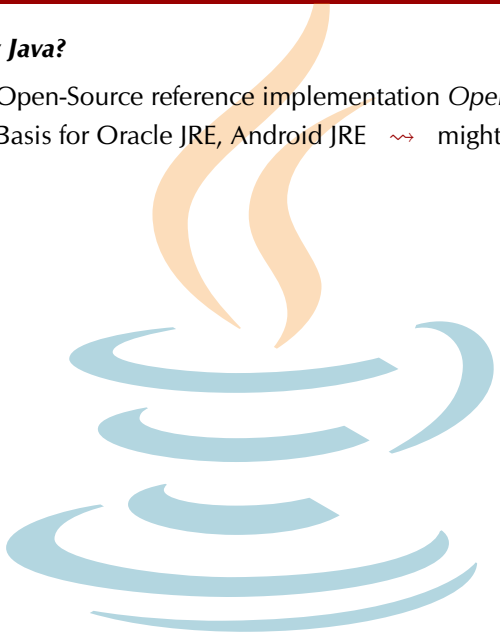

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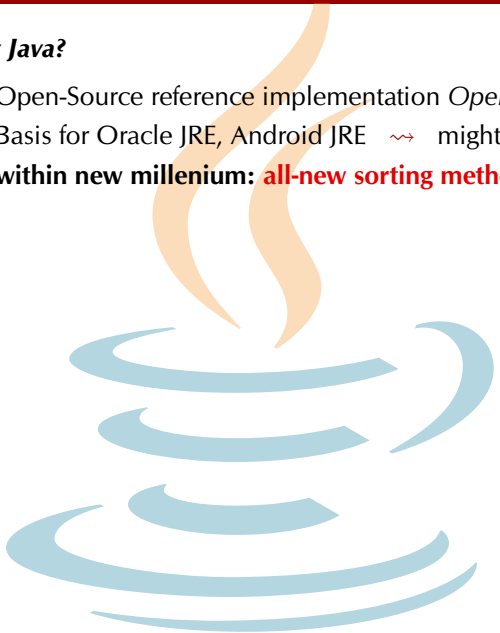

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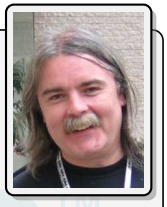

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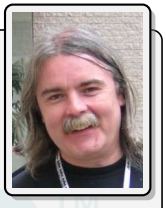

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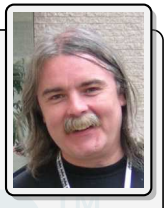

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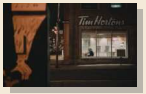
$\rightsquigarrow$ **Focus:** Benchmarks & Unit-Tests not on proofs or analysis

Outline

1 Context

2 Java's Sorting Methods

3 Dual-Pivot Quicksort



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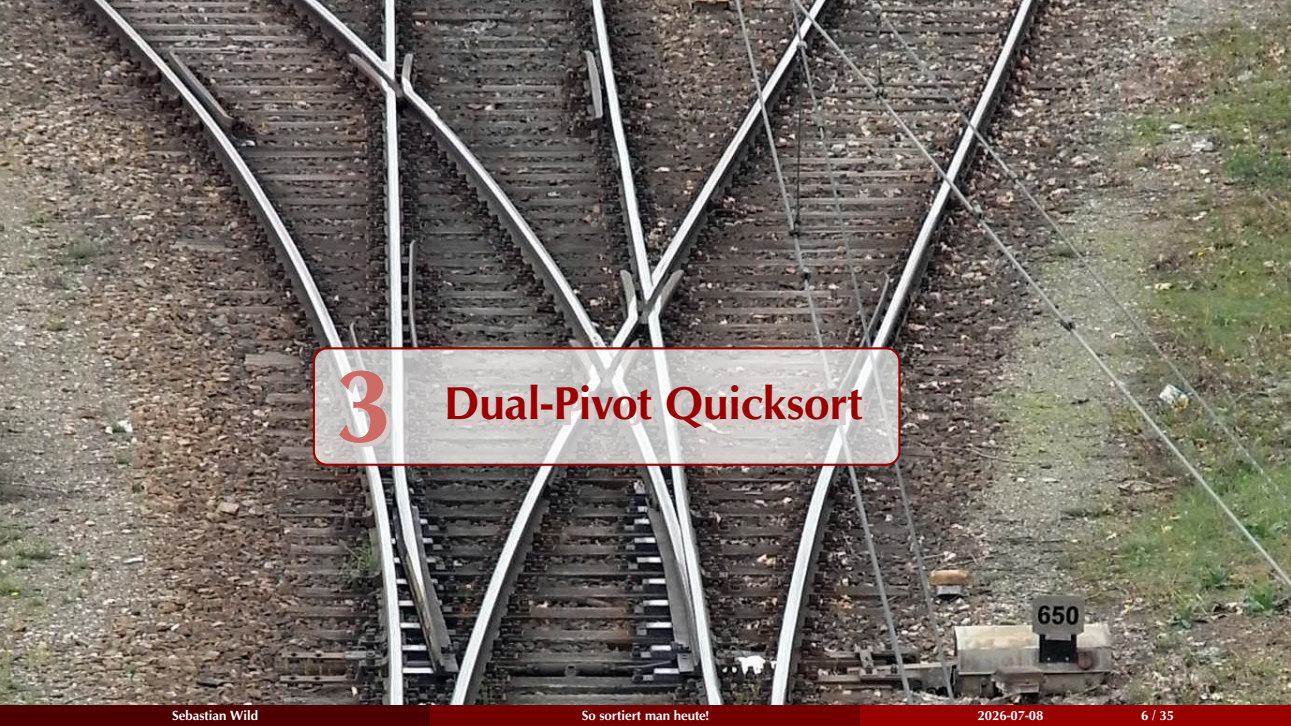

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Quicksort Implementations in Libraries

Context: Internal, unstable, general-purpose sorting methods

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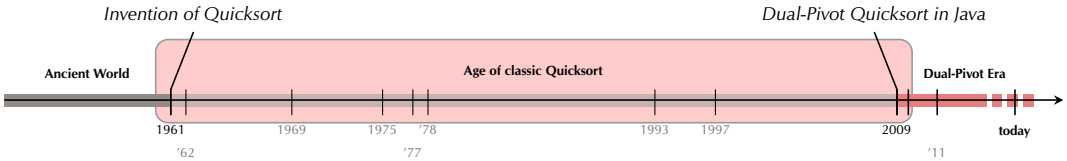
1997 **Musser:** $\mathcal{O}(n \log n)$ worst case by truncating recursion



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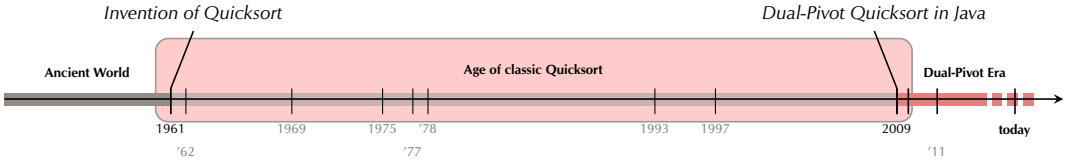

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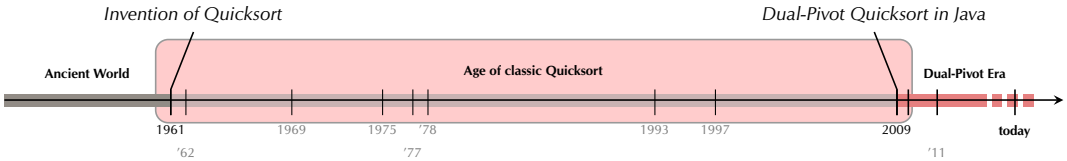
Since then: Almost identical in all programming libraries!



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experiments with Quicksort with **two pivots**

2009 – 2011 optimizations by Joshua Bloch, Jon Bentley and others

11 Sep 2009 announcement on Java core library mailing list

29 Oct 2009 **inclusion** in development version of OpenJDK

28 July 2011 **public release** of Java 7 with ***Yaroslavskiy-Bentley-Bloch (YBB) Quicksort***

13 Nov 2019 Major update of Dual-Pivot Quicksort for Java 14

OpenJDK's Dual-Pivot Quicksort

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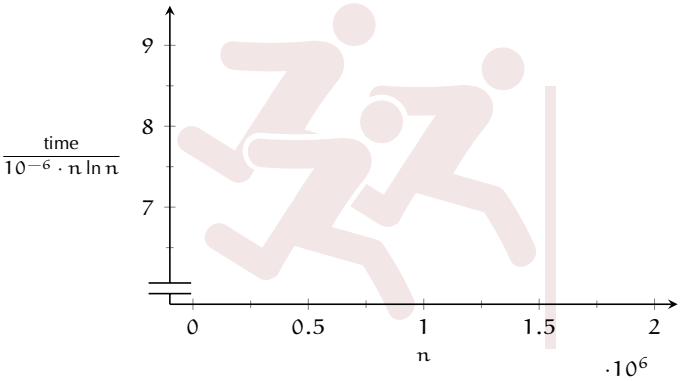

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Average of 1000 random permutations per size (`int[]`).

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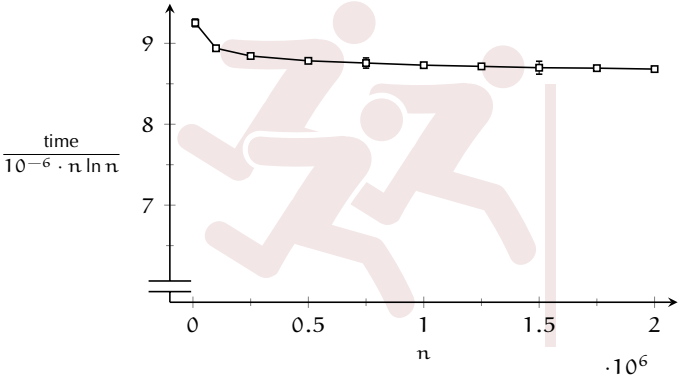

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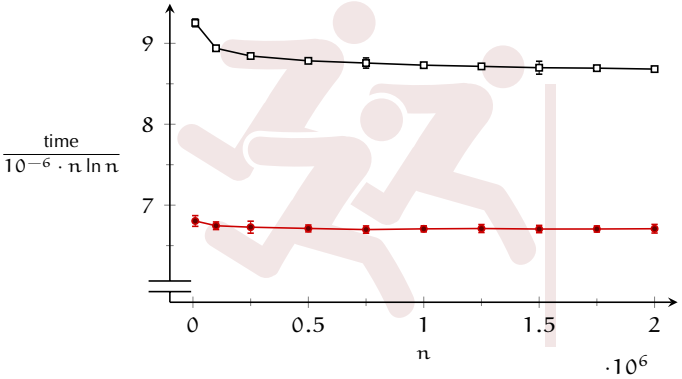

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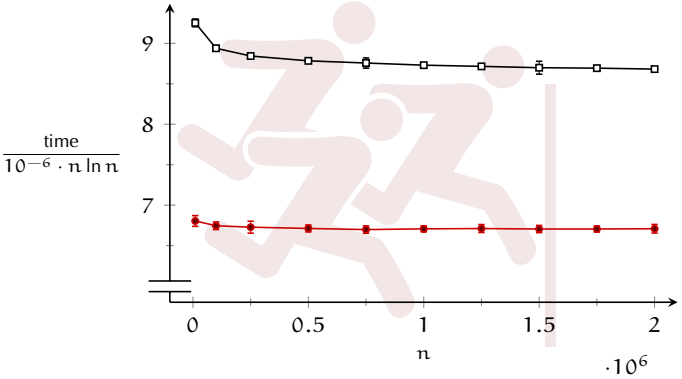

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⇝ Another Algorithm-Science case study!

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Single-Pivot Quicksort

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Dual-Pivot Quicksort: Conceptual View

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   (assuming distinct elements in this talk)

   by comparing x to P.
3. Arrange elements according to classes:

4. Sort subarrays recursively.

Dual-Pivot Quicksort: Conceptual View

Single-Pivot Quicksort

1. Choose pivots P.
2. For each element x, determine its **class**
   - **small** for $x < P$
   - **large** for $P < x$

   ← assuming distinct elements in this talk

   by comparing x to P.
3. Arrange elements according to classes:

4. Sort subarrays recursively.

Dual-Pivot Quicksort

Dual-Pivot Quicksort: Conceptual View

Single-Pivot Quicksort

1. Choose pivots P.
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   - **small** for $x < P$
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   assuming distinct elements in this talk

   by comparing x to P.
3. Arrange elements according to classes:

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Dual-Pivot Quicksort

1. Choose **two pivots** $P_1 \leqslant P_2$

Dual-Pivot Quicksort: Conceptual View

Single-Pivot Quicksort

1. Choose pivots P.
2. For each element x, determine its **class**
   - **small** for $x < P$
   - **large** for $P < x$

   assuming distinct elements in this talk

   by comparing x to P.
3. Arrange elements according to classes:

4. Sort subarrays recursively.

Dual-Pivot Quicksort

1. Choose **two pivots** $P_1 \leqslant P_2$
2. For each element x, determine its **class**
   - **small** for $x < P_1$
   - **medium** for $P_1 < x < P_2$
   - **large** for $P_2 < x$

   by comparing x to **pivots** P_1 and P_2

Dual-Pivot Quicksort: Conceptual View

Single-Pivot Quicksort

1. Choose pivots P.
2. For each element x, determine its **class**
   - **small** for $x < P$
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   assuming distinct elements in this talk

   by comparing x to P.
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   by comparing x to **pivots** P_1 and P_2
3. Arrange elements according to classes:

Dijkstra's Dutch National Flag Problem

Dual-Pivot Quicksort: Conceptual View

Single-Pivot Quicksort

1. Choose pivots P.
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   by comparing x to P.
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   - **small** for $x < P_1$
   - **medium** for $P_1 < x < P_2$
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   by comparing x to **pivots** P_1 and P_2
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Dijkstra's Dutch National Flag Problem

Dual-Pivot Quicksort: Conceptual View

Single-Pivot Quicksort

1. Choose pivots P
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   - **small** for $x < P$
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   by comparing x to P.

   assuming distinct elements in this talk
3. Arrange elements according to classes:

4. Sort subarrays recursively.

How to do ❸ efficiently and in place?

Dual-Pivot Quicksort

2. For each element x, determine its **class**
   - **small** for $x < P_1$
   - **medium** for $P_1 < x < P_2$
   - **large** for $P_2 < x$

   by comparing x to **pivots** P_1 and P_2

Dijkstra's Dutch National Flag Problem

3. Arrange elements according to classes:

4. Sort subarrays recursively.

Dual-Pivot Quicksort: Conceptual View

Single-Pivot Quicksort

1. Choose pivots P
2. For each element x, determine its **class**
   - **small** for $x < P$
   - **large** for $P < x$

   by comparing x to P.

   assuming distinct elements in this talk
3. Arrange elements according to classes:

4. Sort subarrays recursively.

How to do ❸ efficiently and in place? ⇝ ***YBB Partitioning***

Dual-Pivot Quicksort

2. For each element x, determine its **class**
   - **small** for $x < P_1$
   - **medium** for $P_1 < x < P_2$
   - **large** for $P_2 < x$

   by comparing x to **pivots** P_1 and P_2

Dijkstra's Dutch National Flag Problem

3. Arrange elements according to classes:

4. Sort subarrays recursively.

Basic Algorithms

```
ClassicQuicksort(A, left, right)
 1  if right ⩽ left then return
 2  P := A[right]
 3  k := left − 1;  g := right
 4  while true
 5        do k := k + 1 while A[k] < P end while
 6        do g := g − 1 while A[g] > P end while
 7        if k ⩾ g then break while end if
 8        Swap A[k] and A[g]
 9  end while
10  Swap A[k] and A[right]
11  ClassicQuicksort(A,  left  , k − 1)
12  ClassicQuicksort(A, k + 1,  right )
```

Basic Algorithms

```
ClassicQuicksort(A, left, right)
 1  if right ⩽ left then return
 2  P := A[right]
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Basic Algorithms

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 9  end while
10  Swap A[k] and A[right]
11  ClassicQuicksort(A,  left  , k − 1)
12  ClassicQuicksort(A, k + 1,  right )
```

- **Classic Quicksort**

Basic Algorithms

```
ClassicQuicksort(A, left, right)
 1  if right ⩽ left then return
 2  P := A[right]
 3  k := left − 1;  g := right
 4  while true
 5        do k := k + 1 while A[k] < P end while
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11  ClassicQuicksort(A,  left  , k − 1)
12  ClassicQuicksort(A, k + 1,  right )
```

- **Classic Quicksort**
  - *crossing-pointer* scheme 

Basic Algorithms

```
ClassicQuicksort(A, left, right)
 1  if right ⩽ left then return
 2  P := A[right]
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 4  while true
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 8      Swap A[k] and A[g]
 9  end while
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11  ClassicQuicksort(A,  left  , k − 1)
12  ClassicQuicksort(A, k + 1,  right )
```

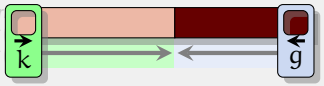
- **Classic Quicksort**
  - *crossing-pointer* scheme 
  - symmetric treatment 

Basic Algorithms

ClassicQuicksort(A, *left*, *right*)

```
1  if right ⩽ left then return
2  P := A[right]
3  k := left − 1;  g := right
4  while true
5      do k := k + 1 while A[k] < P end while
6      do g := g − 1 while A[g] > P end while
7      if k ⩾ g then break while end if
8      Swap A[k] and A[g]
9  end while
10 Swap A[k] and A[r
11 ClassicQuicksort
12 ClassicQuicksort
```

Invariant



- **Classic Quicksort**
  - *crossing-pointer* scheme
  - symmetric treatment

Basic Algorithms

ClassicQuicksort$(A, \mathit{left}, \mathit{right})$

```
1  if right ⩽ left then return
2  P := A[right]
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4  while true
5      do k := k + 1 while A[k] < P end while
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10 Swap A[k] and A[r
11 ClassicQuicksort
12 ClassicQuicksort
```

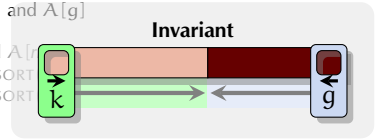

- **Classic Quicksort**
  - *crossing-pointer* scheme 
  - symmetric treatment 
- **YBB Quicksort**

YbbQuicksort$(A, \mathit{left}, \mathit{right})$

```
1  if right ⩽ left then return
2  P1 := A[left];   P2 := A[right]
3  if P1 > P2 then Swap P1 and P2 end if
4  ℓ := left + 1;   g := right − 1;   k := ℓ
5  while k ⩽ g
6      if A[k] < P1
7          Swap A[k] and A[ℓ] ;  ℓ := ℓ + 1
8      else if A[k] ⩾ P2
9          while A[g] > P2 and k < g
10             g := g − 1
11         end while
12         Swap A[k] and A[g] ;  g := g − 1
13         if A[k] < P1
14             Swap A[k] and A[ℓ] ;  ℓ := ℓ + 1
15         end if
16     end if
17     k := k + 1
18 end while
19 ℓ := ℓ − 1;   g := g + 1
20 Swap A[left] and A[ℓ] ; Swap A[right] and A[g]
21 YbbQuicksort(A,  left  , ℓ − 1)
22 YbbQuicksort(A, ℓ + 1, g − 1)
23 YbbQuicksort(A, g + 1,  right )
```

Basic Algorithms

ClassicQuicksort$(A, \mathit{left}, \mathit{right})$

```
1  if right ⩽ left then return
2  P := A[right]
3  k := left − 1;  g := right
4  while true
5      do k := k + 1 while A[k] < P end while
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7      if k ⩾ g then break while end if
8      Swap A[k] and A[g]
9  end while
10 Swap A[k] and A[r
11 ClassicQuicksort
12 ClassicQuicksort
```

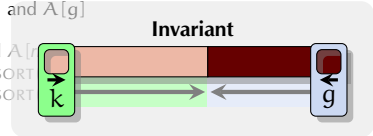

- **Classic Quicksort**
  - *crossing-pointer* scheme 
  - symmetric treatment 
- **YBB Quicksort**

YbbQuicksort$(A, \mathit{left}, \mathit{right})$

```
1  if right ⩽ left then return
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5  while k ⩽ g
6      if A[k] < P1
7          Swap A[k] and A[ℓ] ;  ℓ := ℓ + 1
8      else if A[k] ⩾ P2
9          while A[g] > P2 and k < g
10             g := g − 1
11         end while
12         Swap A[k] and A[g] ;  g := g − 1
13         if A[k] < P1
14             Swap A[k] and A[ℓ] ;  ℓ := ℓ + 1
15         end if
16     end if
17     k := k + 1
18 end while
19 ℓ := ℓ − 1;   g := g + 1
20 Swap A[left] and A[ℓ] ; Swap A[right] and A[g]
21 YbbQuicksort(A,  left  , ℓ − 1)
22 YbbQuicksort(A, ℓ + 1, g − 1)
23 YbbQuicksort(A, g + 1,  right )
```

Basic Algorithms

ClassicQuicksort$(A, left, right)$

```
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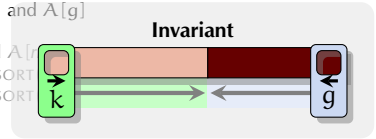

- **Classic Quicksort**
  - *crossing-pointer* scheme 
  - symmetric treatment 
- **YBB Quicksort**

YbbQuicksort$(A, left, right)$

```
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2  P1 := A[left];   P2 := A[right]
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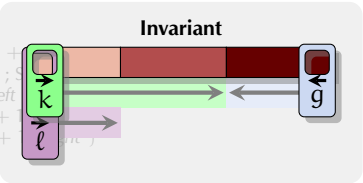
5  while k ⩽ g
6      if A[k] < P1
7          Swap A[k] and A[ℓ] ;  ℓ := ℓ + 1
8      else if A[k] ⩾ P2
9          while A[g] > P2 and k < g
10             g := g − 1
11         end while
12         Swap A[k] and A[g] ;  g := g − 1
13         if A[k] < P1
14             Swap A[k] and A[ℓ] ;  ℓ := ℓ + 1
15         end if
16     end if
17     k := k + 1
18 end while
19 ℓ := ℓ − 1;   g := g +
20 Swap A[left] and A[ℓ] ; S
21 YbbQuicksort(A,  left
22 YbbQuicksort(A, ℓ +
23 YbbQuicksort(A, g +
```



Basic Algorithms

ClassicQuicksort$(A, left, right)$

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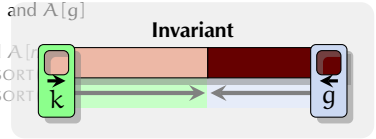

- **Classic Quicksort**
  - *crossing-pointer* scheme 
  - symmetric treatment 
- **YBB Quicksort**
  - . . . looks messy 

YbbQuicksort$(A, left, right)$

```
1  if right ⩽ left then return
2  P1 := A[left];   P2 := A[right]
3  if P1 > P2 then Swap P1 and P2 end if
4  ℓ := left + 1;   g := right − 1;   k := ℓ
5  while k ⩽ g
6      if A[k] < P1
7          Swap A[k] and A[ℓ] ;  ℓ := ℓ + 1
8      else if A[k] ⩾ P2
9          while A[g] > P2 and k < g
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14             Swap A[k] and A[ℓ] ;  ℓ := ℓ + 1
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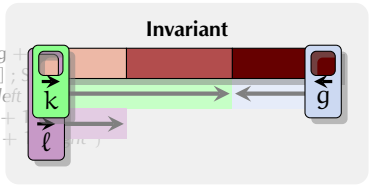
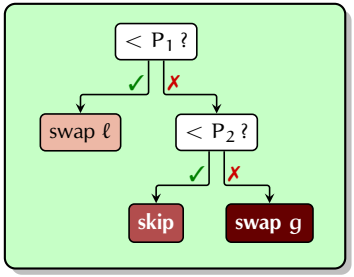
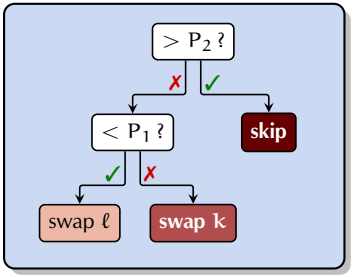
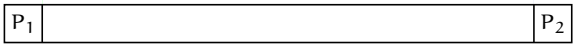


YBB Partitioning Example





Invariant: 


```
YbbQuicksort(A, left, right)
 1  if right ⩽ left then return
 2  P1 := A[left];   P2 := A[right]
 3  if P1 > P2 then Swap P1 and P2 end if
 4  ℓ := left + 1;   g := right − 1;   k := ℓ
 5  while k ⩽ g
 6      if A[k] < P1
 7          Swap A[k] and A[ℓ] ; ℓ := ℓ + 1
 8      else if A[k] ⩾ P2
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18  end while
19  ℓ := ℓ − 1;   g := g + 1
20  Swap A[left] and A[ℓ] ; Swap A[right] and A[g]
21  YbbQuicksort(A, left , ℓ − 1)
22  YbbQuicksort(A, ℓ + 1, g − 1)
23  YbbQuicksort(A, g + 1, right )
```
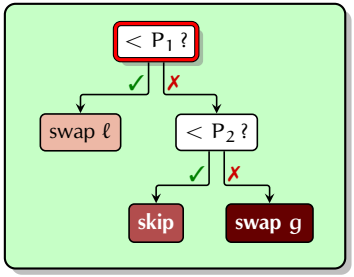
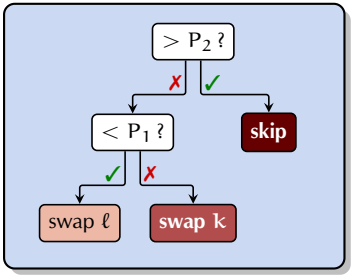
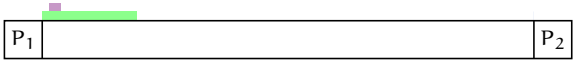

YBB Partitioning Example





Invariant:

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 3  if P1 > P2 then Swap P1 and P2 end if
 4  ℓ := left + 1;   g := right − 1;   k := ℓ
 5  while k ⩽ g
 6  → if A[k] < P1
 7        Swap A[k] and A[ℓ] ; ℓ := ℓ + 1
 8    else if A[k] ⩾ P2
 9        while A[g] > P2 and k < g
10            g := g − 1
11        end while
12        Swap A[k] and A[g] ; g := g − 1
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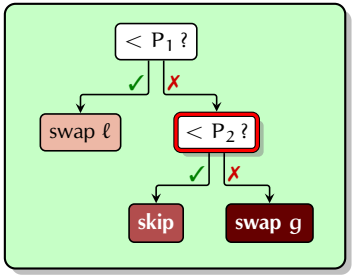
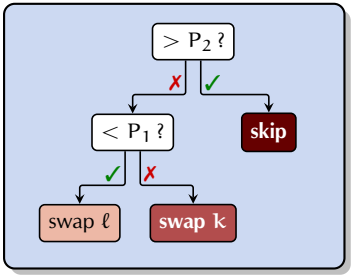
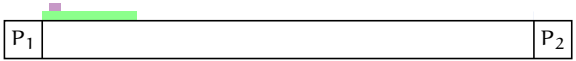
14            Swap A[k] and A[ℓ] ; ℓ := ℓ + 1
15        end if
16    end if
17    k := k + 1
18  end while
19  ℓ := ℓ − 1;   g := g + 1
20  Swap A[left] and A[ℓ] ; Swap A[right] and A[g]
21  YbbQuicksort(A, left , ℓ − 1)
22  YbbQuicksort(A, ℓ + 1, g − 1)
23  YbbQuicksort(A, g + 1, right )
```

YBB Partitioning Example





Invariant:

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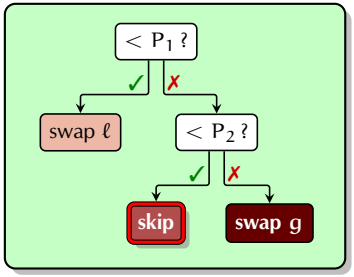

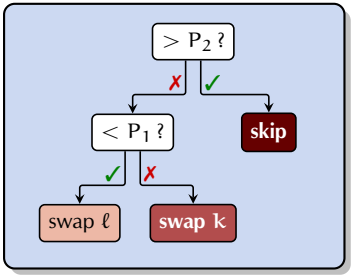

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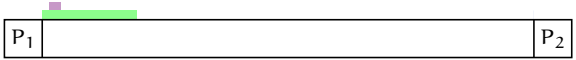


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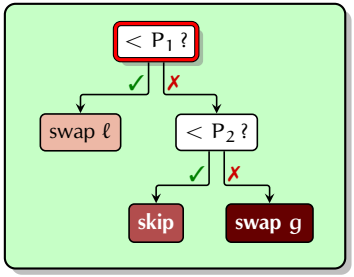
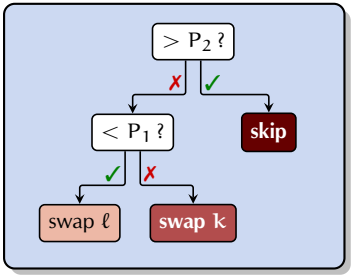
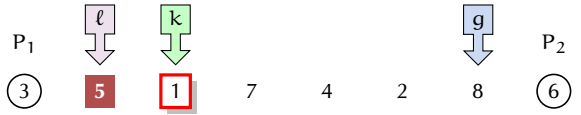
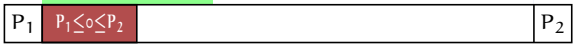






Invariant: 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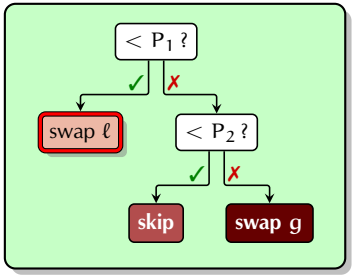

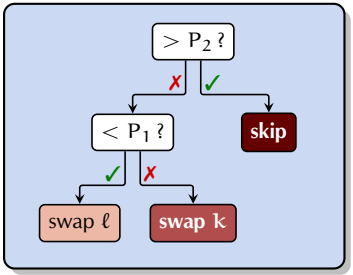

Invariant: $P_1 \leq o \leq P_2$

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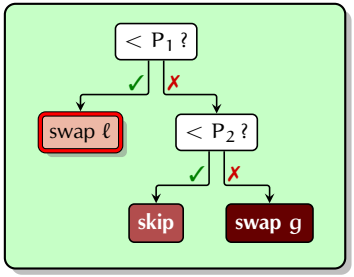
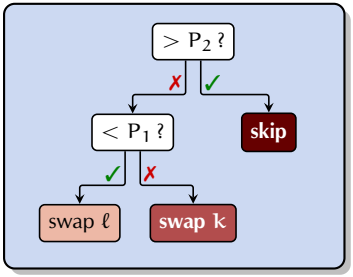
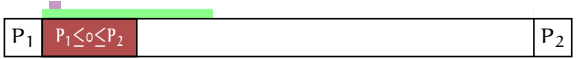




Invariant:

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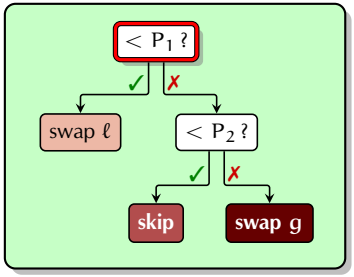
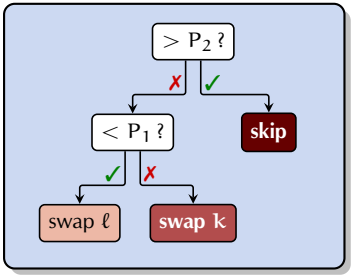
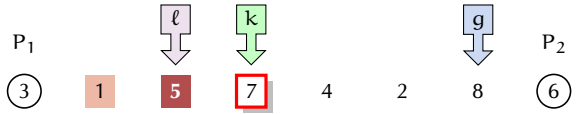
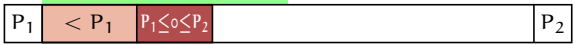






Invariant:

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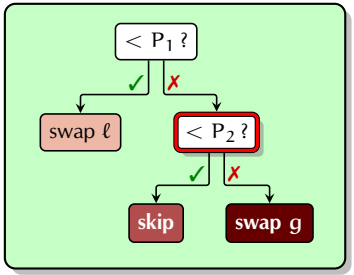

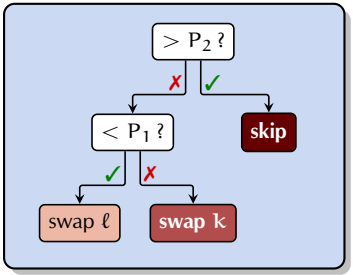

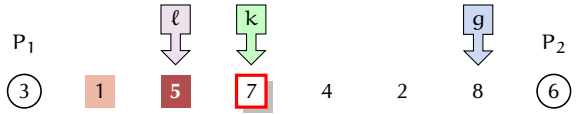

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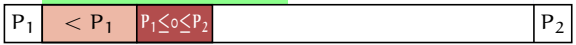


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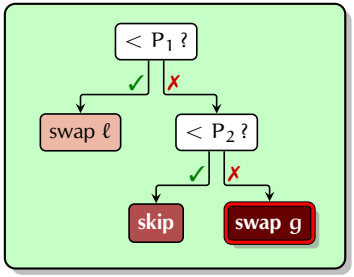

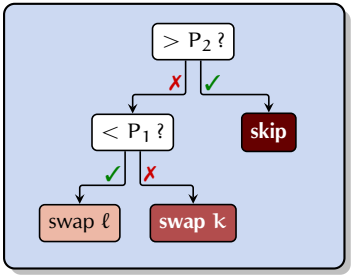

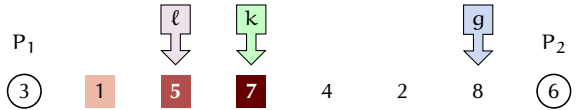

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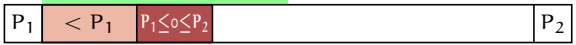


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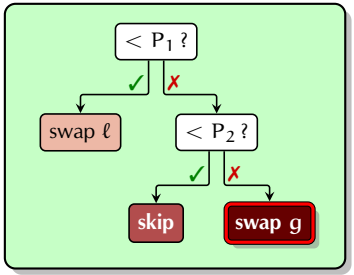
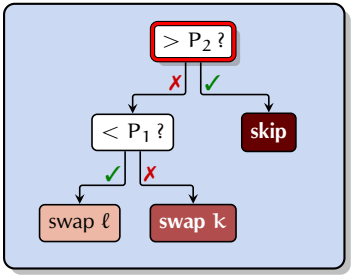
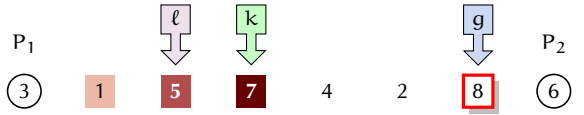
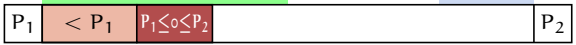






Invariant:

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 4  ℓ := left + 1;   g := right − 1;   k := ℓ
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 7          Swap A[k] and A[ℓ] ; ℓ := ℓ + 1
 8      else if A[k] ⩾ P2
 9          while A[g] > P2 and k < g
10              g := g − 1
11          end while
12          Swap A[k] and A[g] ; g := g − 1
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14              Swap A[k] and A[ℓ] ; ℓ := ℓ + 1
15          end if
16      end if
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18  end while
19  ℓ := ℓ − 1;   g := g + 1
20  Swap A[left] and A[ℓ] ; Swap A[right] and A[g]
21  YbbQuicksort(A, left , ℓ − 1)
22  YbbQuicksort(A, ℓ + 1, g − 1)
23  YbbQuicksort(A, g + 1, right )
```
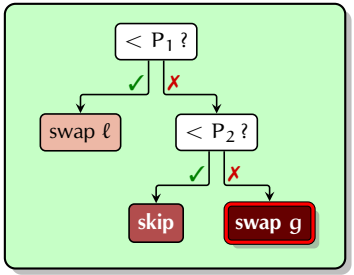
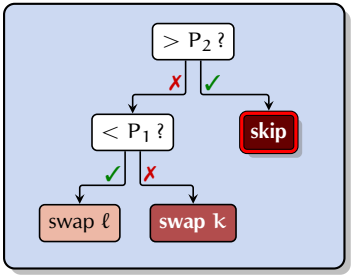
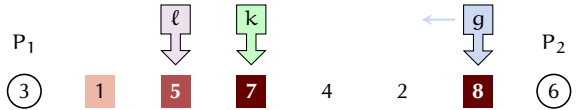
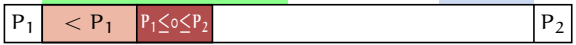

YBB Partitioning Example







Invariant:

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23  YbbQuicksort(A, g + 1,  right )
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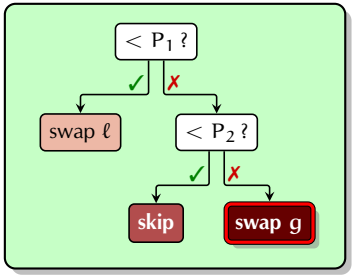
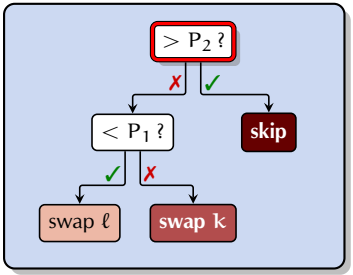
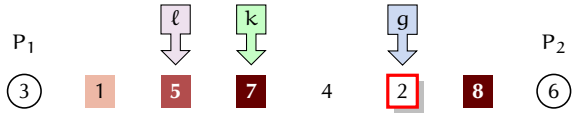
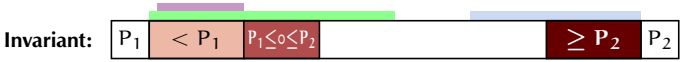
YBB Partitioning Example







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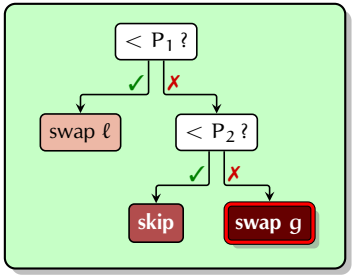
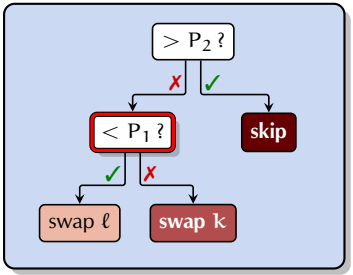
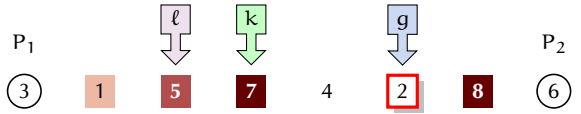
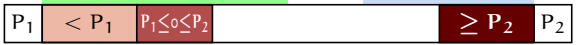
YBB Partitioning Example







Invariant:

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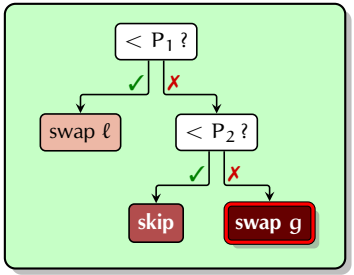
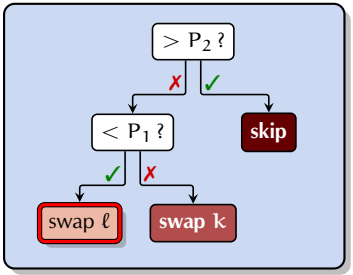
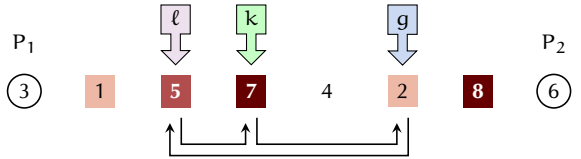
YBB Partitioning Example





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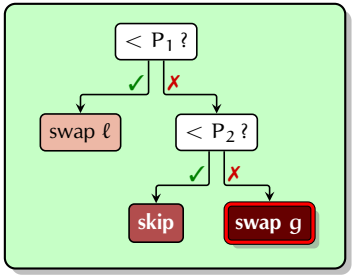
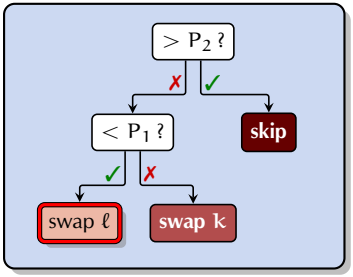
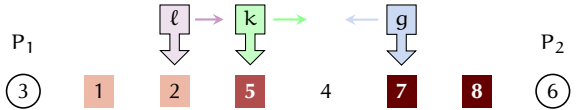
YBB Partitioning Example







Invariant:


```
YbbQuicksort(A, left, right)
 1  if right ⩽ left then return
 2  P1 := A[left];   P2 := A[right]
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 5  while k ⩽ g
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 7          Swap A[k] and A[ℓ] ; ℓ := ℓ + 1
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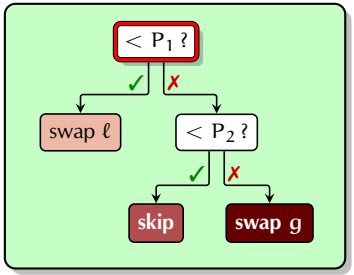
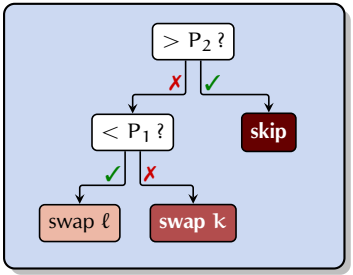
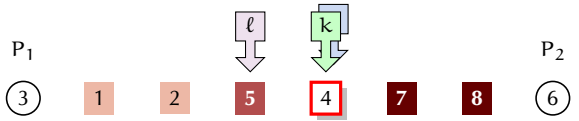
YBB Partitioning Example







YbbQuicksort$(A, left, right)$

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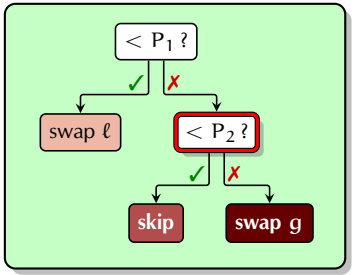
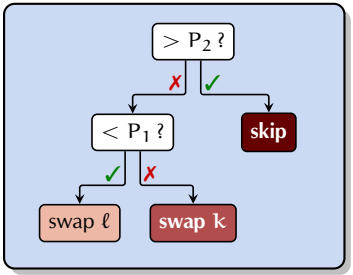
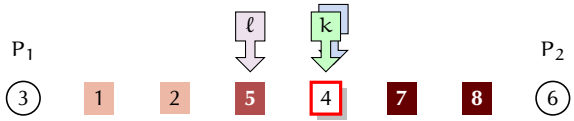
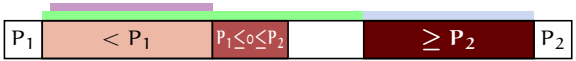
YBB Partitioning Example







Invariant:

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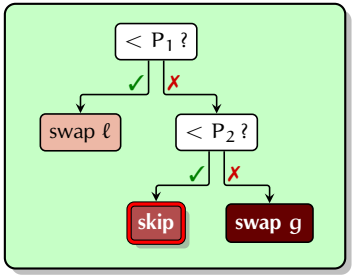
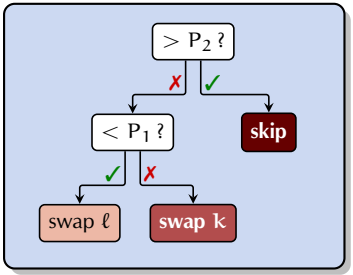
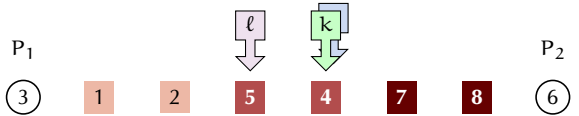
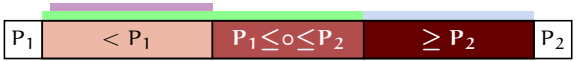
YBB Partitioning Example







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```
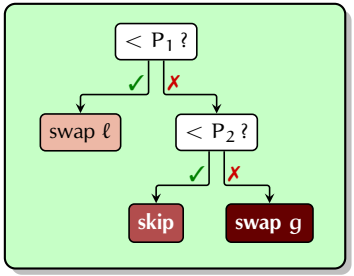
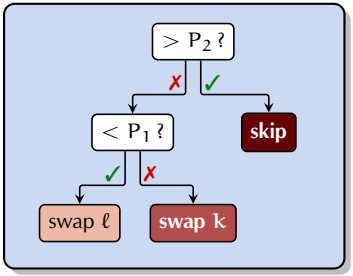

YBB Partitioning Example

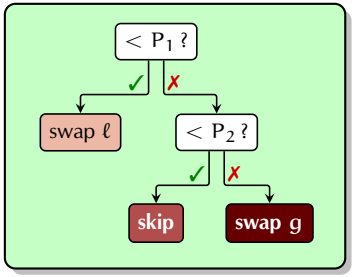


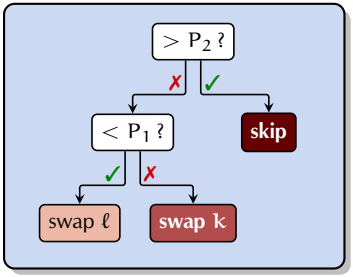


YBBQUICKSORT(A, *left*, *right*)

```
1  if right ⩽ left then return
2  P1 := A[left];   P2 := A[right]
3  if P1 > P2 then Swap P1 and P2 end if
4  ℓ := left + 1;   g := right − 1;   k := ℓ
5  while k ⩽ g
6      if A[k] < P1
7          Swap A[k] and A[ℓ] ;  ℓ := ℓ + 1
8      else if A[k] ⩾ P2
9          while A[g] > P2 and k < g
10             g := g − 1
11         end while
12         Swap A[k] and A[g] ;  g := g − 1
13         if A[k] < P1
14             Swap A[k] and A[ℓ] ;  ℓ := ℓ + 1
15         end if
16     end if
17     k := k + 1
18 end while
19 ℓ := ℓ − 1;   g := g + 1
20 Swap A[left] and A[ℓ] ; Swap A[right] and A[g]
21 YBBQUICKSORT(A,  left  , ℓ − 1)
22 YBBQUICKSORT(A, ℓ + 1, g − 1)
23 YBBQUICKSORT(A, g + 1,  right )
```

YBB Partitioning Example

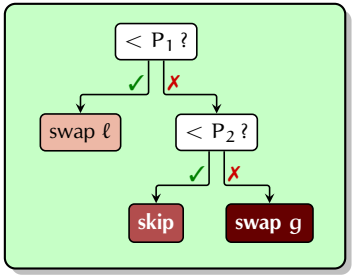


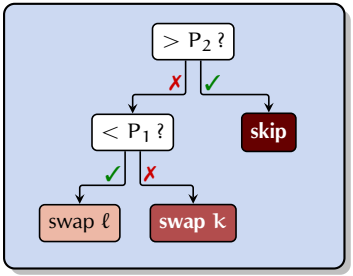


YBBQUICKSORT(A, *left*, *right*)

```
1  if right ⩽ left then return
2  P1 := A[left];   P2 := A[right]
3  if P1 > P2 then Swap P1 and P2 end if
4  ℓ := left + 1;   g := right − 1;   k := ℓ
5  while k ⩽ g
6      if A[k] < P1
7          Swap A[k] and A[ℓ] ;  ℓ := ℓ + 1
8      else if A[k] ⩾ P2
9          while A[g] > P2 and k < g
10             g := g − 1
11         end while
12         Swap A[k] and A[g] ;  g := g − 1
13         if A[k] < P1
14             Swap A[k] and A[ℓ] ;  ℓ := ℓ + 1
15         end if
16     end if
17     k := k + 1
18 end while
19 ℓ := ℓ − 1;   g := g + 1
20 Swap A[left] and A[ℓ] ; Swap A[right] and A[g]
21 YBBQUICKSORT(A, left , ℓ − 1)
22 YBBQUICKSORT(A, ℓ + 1, g − 1)
23 YBBQUICKSORT(A, g + 1, right )
```

YBB Partitioning Example





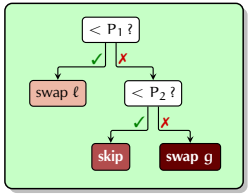
YbbQuicksort(A, *left*, *right*)

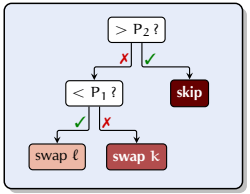
```
1  if right ⩽ left then return
2  P1 := A[left];   P2 := A[right]
3  if P1 > P2 then Swap P1 and P2 end if
4  ℓ := left + 1;   g := right − 1;   k := ℓ
5  while k ⩽ g
6      if A[k] < P1
7          Swap A[k] and A[ℓ] ;  ℓ := ℓ + 1
8      else if A[k] ⩾ P2
9          while A[g] > P2 and k < g
10             g := g − 1
11         end while
12         Swap A[k] and A[g] ;  g := g − 1
13         if A[k] < P1
14             Swap A[k] and A[ℓ] ;  ℓ := ℓ + 1
15         end if
16     end if
17     k := k + 1
18 end while
19 ℓ := ℓ − 1;   g := g + 1
20 Swap A[left] and A[ℓ] ; Swap A[right] and A[g]
21 YbbQuicksort(A,  left  , ℓ − 1)
22 YbbQuicksort(A, ℓ + 1, g − 1)
23 YbbQuicksort(A, g + 1,  right )
```

Comparisons

How many comparisons on average?

- Consider **first** partitioning step.





Comparisons

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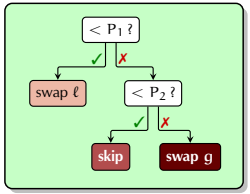
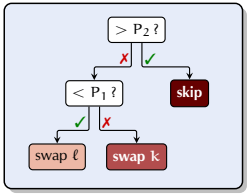
- Consider **first** partitioning step.
- How many #cmps per element?
  Well, either 2 or 1.





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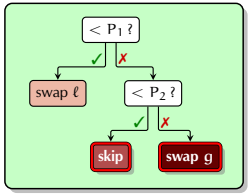

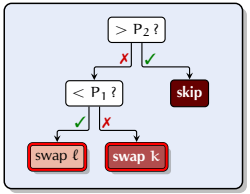

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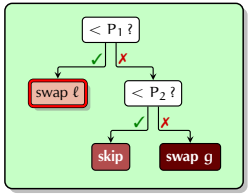

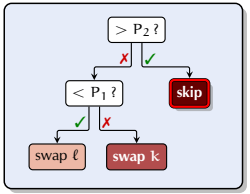

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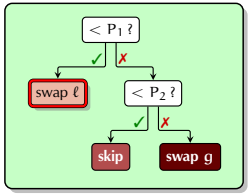

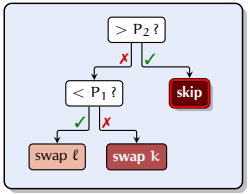

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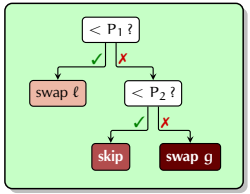

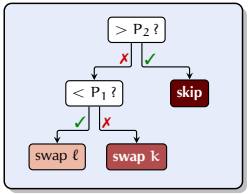

Analysis Sketch:

Comparisons

How many comparisons on average?

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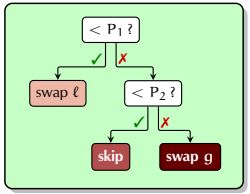

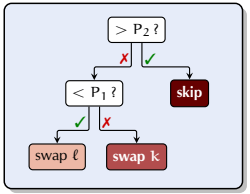

Analysis Sketch:

- Have to compare with one pivot first, say P_1.
  (Similar if P_2 used first.)

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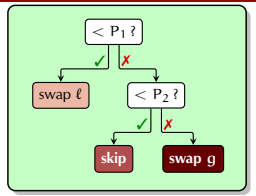

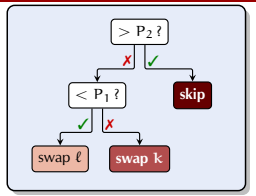

Analysis Sketch:

- Have to compare with one pivot first, say P_1.
  (Similar if P_2 used first.)

$\rightsquigarrow$ 1 cmp for small elements,
2 cmps for medium and large.

Comparisons

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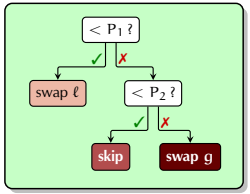

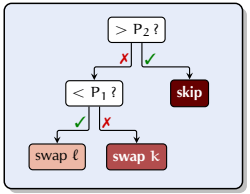

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- Have to compare with one pivot first, say P_1.
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$\rightsquigarrow$ 1 cmp for small elements,
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- On average, $\sim \frac{1}{3}n$ elements each kind.

Comparisons

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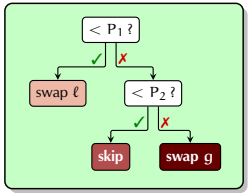

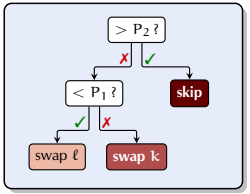

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$\rightsquigarrow$ on average $2 \cdot \frac{2}{3} + 1 \cdot \frac{1}{3} = \frac{5}{3}$ cmps per element

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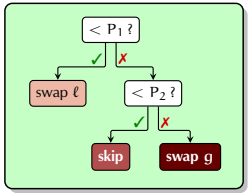

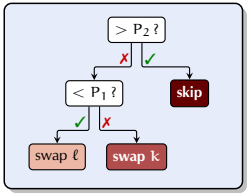

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- Sounds plausible?

Comparisons

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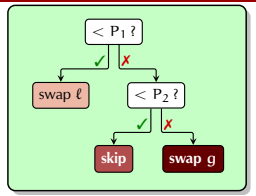

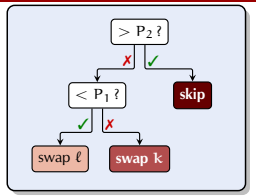

Analysis Sketch:

- Have to compare with one pivot first, say P_1.
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⇝ 1 cmp for small elements,
2 cmps for medium and large.

- On average, $\sim \frac{1}{3}n$ elements each kind.

⇝ on average $2 \cdot \frac{2}{3} + 1 \cdot \frac{1}{3} = \frac{5}{3}$ cmps per element

- Sounds plausible? . . . but it's wrong!

Comparisons

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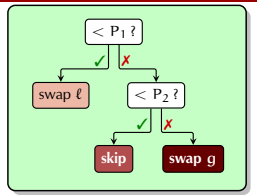

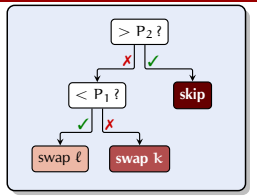

Problem: There are correlations.

Comparisons

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Analysis Sketch:

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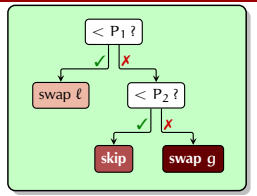

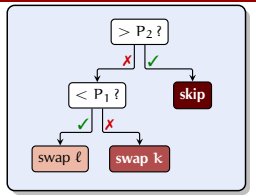

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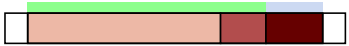
- 

many small elements $\rightsquigarrow$ k's range big

Comparisons

How many comparisons on average?

- Consider **first** partitioning step.
- How many #cmps per element?
  Well, either 2 or 1.
- How many **on average**?

Analysis Sketch:

- Have to compare with one pivot first, say P_1.
  (Similar if P_2 used first.)

$\rightsquigarrow$ 1 cmp for small elements,
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- On average, $\sim \frac{1}{3}n$ elements each kind.

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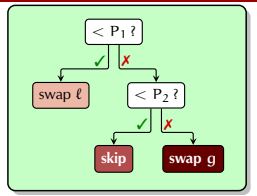

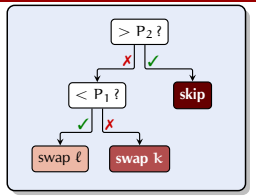

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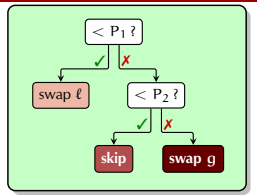

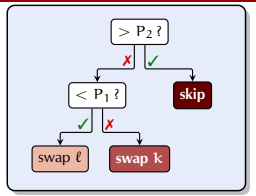

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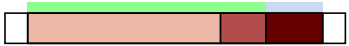
- 

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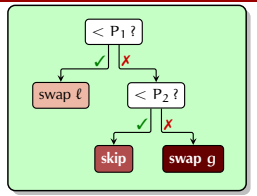

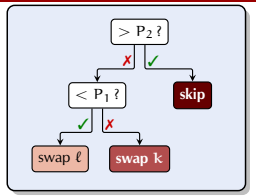

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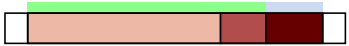
- 

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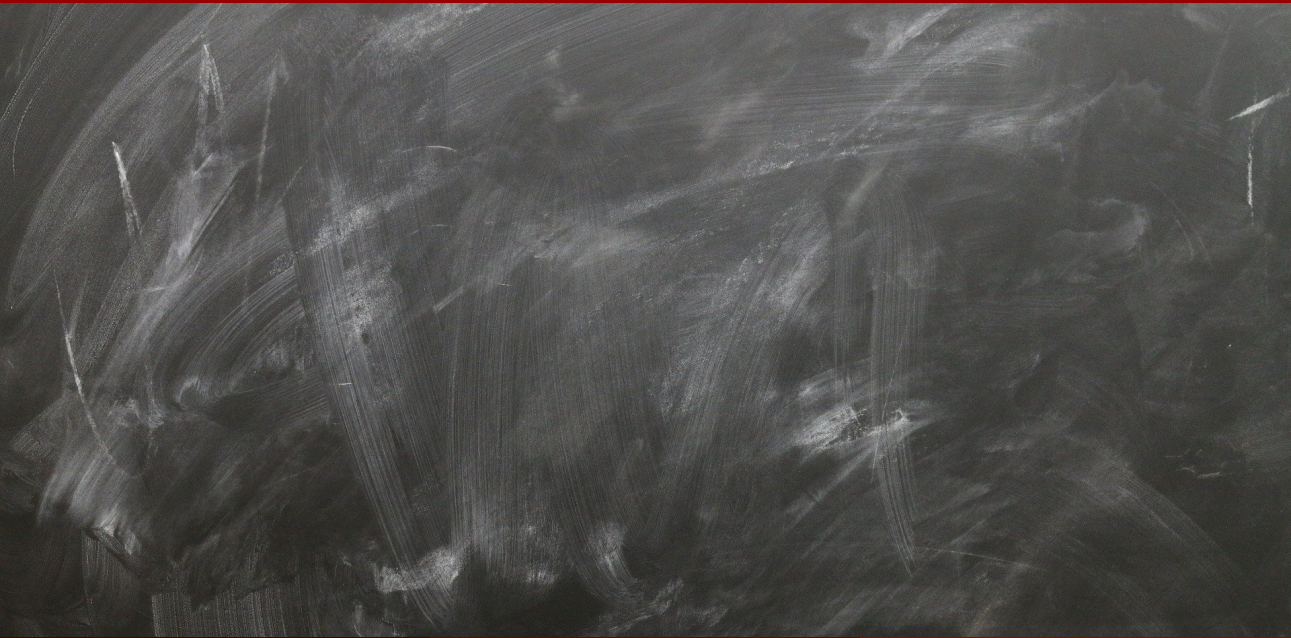
How many? . . . stay tuned.

Mathematical Analysis of Algorithms



Mathematical Analysis of Algorithms

Random Model

- π **i.i.d.** elements chosen **uniformly** in $(0,1)$

- pairwise **distinct** almost surely
- relative ranking is a **random permutation**
- $\rightsquigarrow$ equivalent to classic model

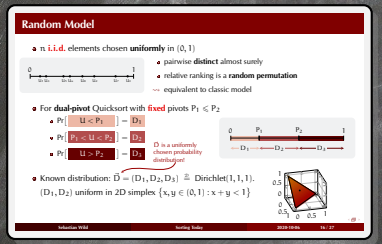
- For **dual-pivot** Quicksort with **fixed** pivots $P_1 \leqslant P_2$
  - $\Pr[\ U_i < P_1\ ] = D_1$
  - $\Pr[\ P_1 < U_i < P_2\ ] = D_2$
  - $\Pr[\ U_i > P_2\ ] = D_3$

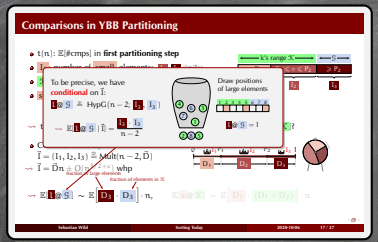
D is a uniformly chosen probability distribution!

- Known distribution: $\vec{D} = (D_1, D_2, D_3) \overset{\mathcal{D}}{=} \text{Dirichlet}(1,1,1)$.
  (D_1, D_2) uniform in 2D simplex $\{x, y \in (0,1) : x + y < 1\}$

Sebastian Wild | Sorting Today | 2019-10-04 | 9c / 27

Mathematical Analysis of Algorithms





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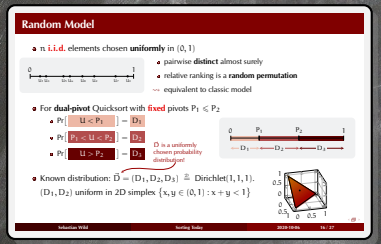

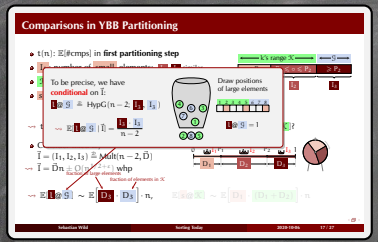

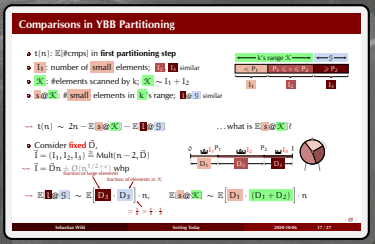

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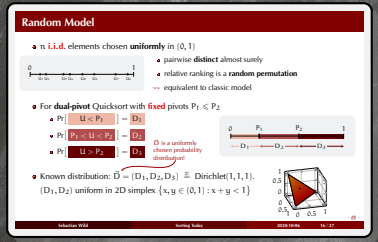

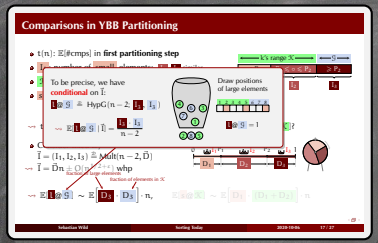

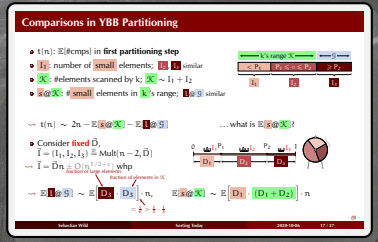

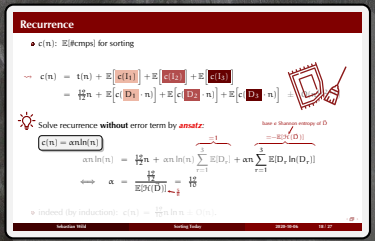

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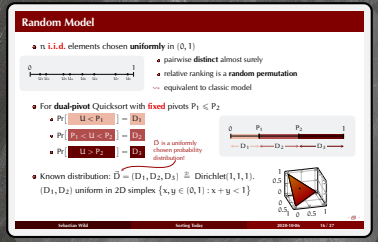

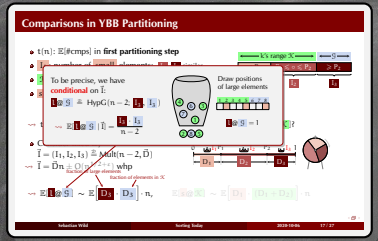

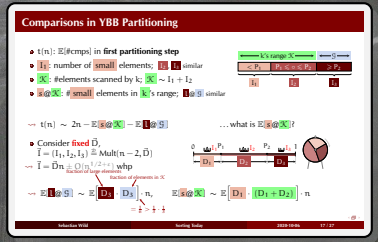

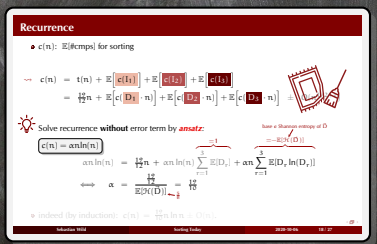

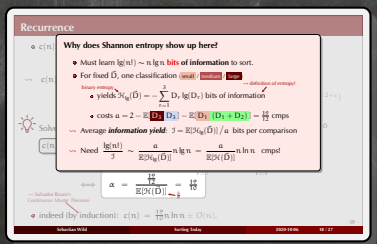

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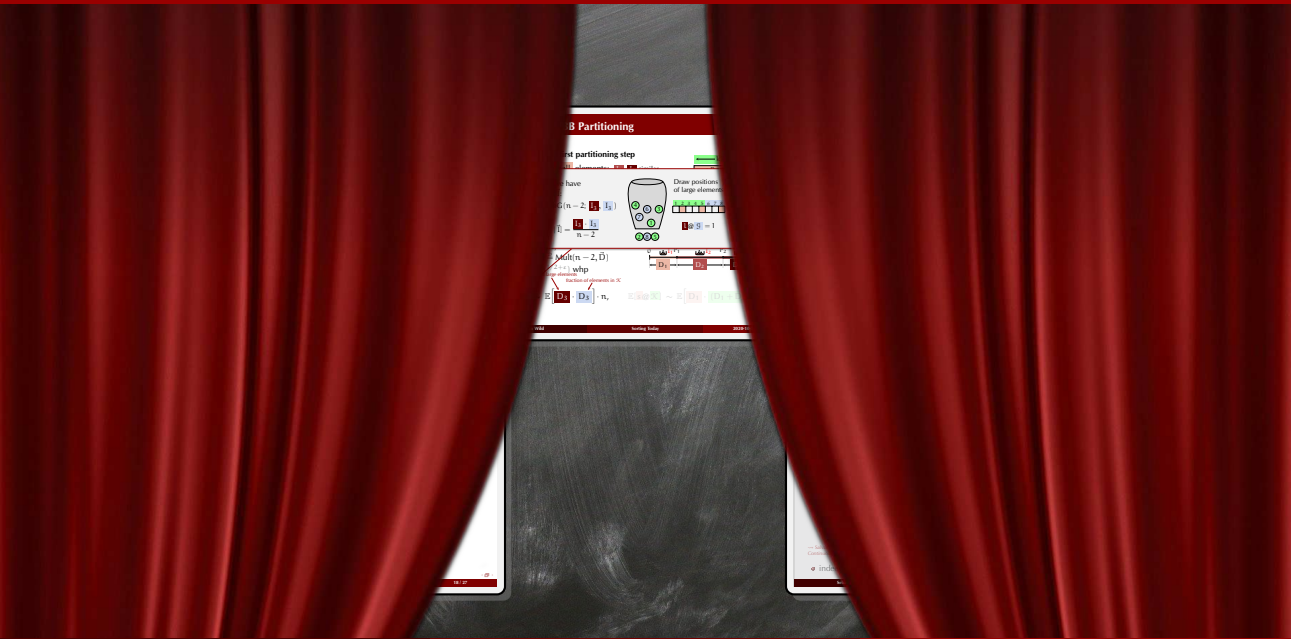
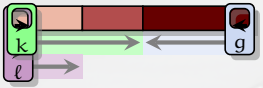
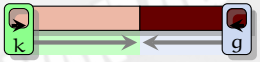
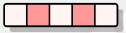


What changes with 2 pivots?

	Dual-Pivot Quicksort	Standard Quicksort	Relative Difference
		 (corrected for pivot Sampling)	
Pivots	 (Tertiles of 5)		

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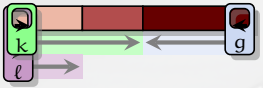
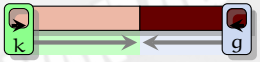
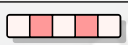
		 (corrected for pivot Sampling)	
Pivots	 (Tertiles of 5)		
comparisons	$\frac{680}{399} \approx 1.7043$	$\frac{100}{57} \approx 1.7544$	−2.9%
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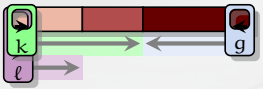
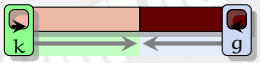
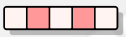
		 (corrected for pivot Sampling)	
Pivots	 (Tertiles of 5)		
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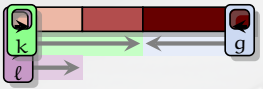
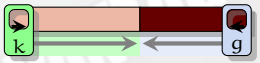
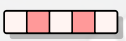
		 (corrected for pivot Sampling)	
Pivots	 (Tertiles of 5)		
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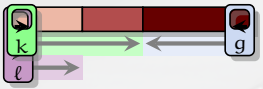
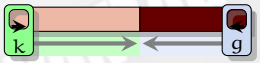
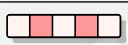
Java Bytecode instructions	$\frac{1100}{57} \approx 19.298$	$\frac{2216}{133} \approx 16.662$	+15.8%

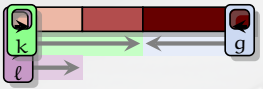
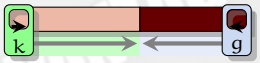
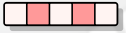
$\cdot n \ln n + O(n)$ *average case results*

What changes with 2 pivots?

	Dual-Pivot Quicksort	Standard Quicksort	Relative Difference
		 (corrected for pivot Sampling)	
Pivots	 (Tertiles of 5)		
comparisons	$\frac{680}{399} \approx 1.7043$	$\frac{100}{57} \approx 1.7544$	−2.9%
write accesses	$\frac{400}{399} \approx 1.0025$	$\frac{272}{399} \approx 0.6817$	+47.1%
Java Bytecode instructions	$\frac{1100}{57} \approx 19.298$	$\frac{2216}{133} \approx 16.662$	+15.8%
branch mispredictions (2-bit saturating counter)	$\frac{785\pi}{532} - \frac{535}{133} \approx 0.6131$	$\frac{10\pi}{19} - \frac{20}{19} \approx 0.6008$	+2.0%

$\cdot n \ln n + O(n)$ *average case results*

What changes with 2 pivots?

	Dual-Pivot Quicksort	Standard Quicksort	Relative Difference
		 (corrected for pivot Sampling)	
Pivots	 (Tertiles of 5)		
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2. Why was dual-pivot Quicksort not used earlier?

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What changes with 2 pivots?

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2. Why was dual-pivot Quicksort not used earlier?
   Because it was **not faster** then!

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Breaking News

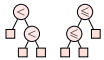
- OpenJDK 23 switched partitioning loop again (to “backwards partitioning”)!
  `https://mail.openjdk.org/pipermail/core-libs-dev/2019-June/060915.html`
- Analytical results (Annika Sabel, 2026) show **worse** comparisons and scanned elements
- better write accesses

⇝ *Java artefact or foundational algorithmic benefit?*

Multiway Quicksort Summary

Multiway Quicksort

1. tricky **asymmetries** in *YBB Quicksort*
   ⇝ saves a few comparisons



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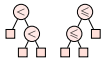

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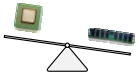

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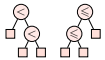

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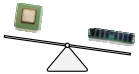

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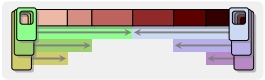

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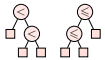

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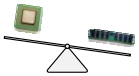

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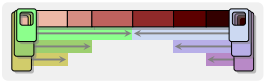

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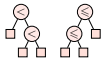

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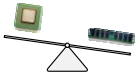

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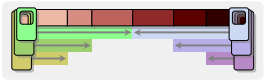

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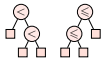

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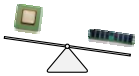

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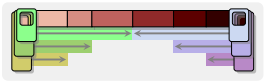

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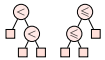

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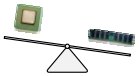

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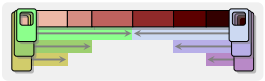

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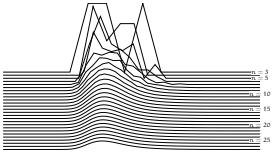

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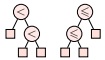

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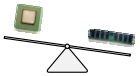

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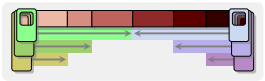

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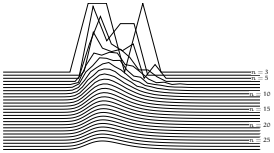

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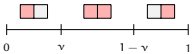

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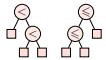

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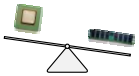

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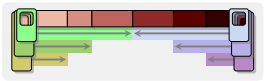

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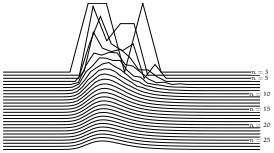

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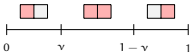

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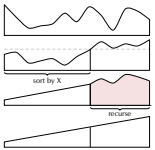
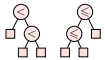


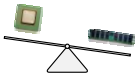
Multiway Quicksort Summary

Multiway Quicksort

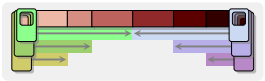
1. tricky **asymmetries** in *YBB Quicksort*
   ⇝ saves a few comparisons



2. evidence for growing weight of memory transfer costs



3. **Multiway Quicksort** reduces **memory transfer**

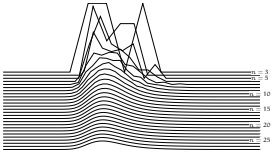


   - No other optimization of Quicksort achieves that

4. *AofA for Algorithm Science!*

Much more to tell …

- Limiting **distributions** of costs



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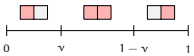

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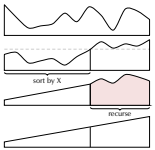
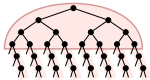


- Intriguing interplay of parameters

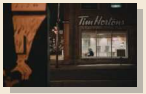


Outline

1 Context

2 Java's Sorting Methods

3 Dual-Pivot Quicksort



4 Timsort

5 Powersort

4 Timsort

Sorting

- We use sorting to
  - to make searching faster (binary search!)
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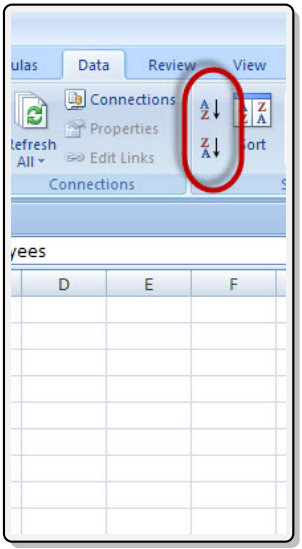
  - to present data neatly for users
  - as building block in algorithms (database join, sweepline, ...)
  - ...



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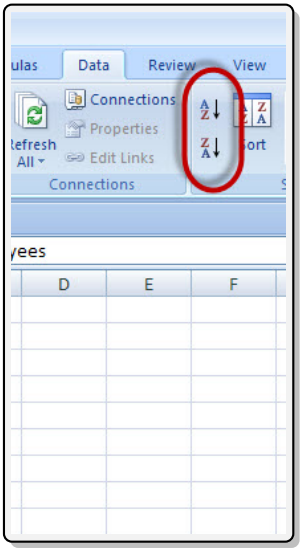

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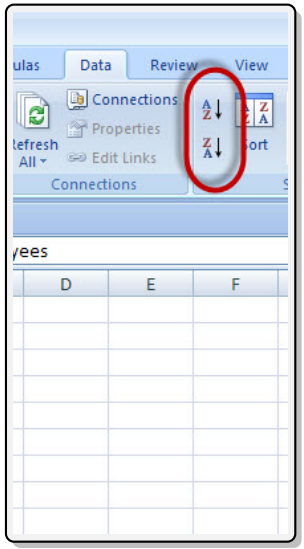

Sorting in Python

- built-in functions: `my_list.sort()` and `sorted(my_list)`
  - `key` parameter to specify sorting criterion
  - prefer pairwise comparator function? ⇝ `functools.cmp_to_key`

CPython Sorting History

Python Version (Year)	Sorting method	Remarks	Stable?
0.9 – 1.4 1991	*qsort*	call to C library, general purpose Quicksort, [BM93]	✗

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3.11.1 – ∞?	2022	***Powersort*** NEW	Timsort with Powersort merge policy [MW18]	✓

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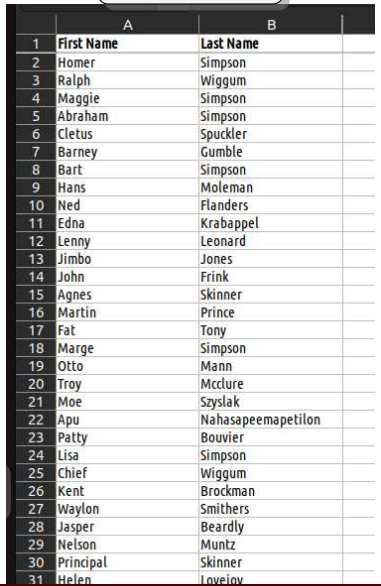
[MW18] **Munro & Wild**: *Nearly-Optimal Mergesorts: Fast, Practical Sorting Methods That Optimally Adapt to Existing Runs*, ESA **2018**

Stable Sorting

unsorted input



	A	B
1	First Name	Last Name
2	Homer	Simpson
3	Ralph	Wiggum
4	Maggie	Simpson
5	Abraham	Simpson
6	Cletus	Spuckler
7	Barney	Gumble
8	Bart	Simpson
9	Hans	Moleman
10	Ned	Flanders
11	Edna	Krabappel
12	Lenny	Leonard
13	Jimbo	Jones
14	John	Frink
15	Agnes	Skinner
16	Martin	Prince
17	Fat	Tony
18	Marge	Simpson
19	Otto	Mann
20	Troy	Mcclure
21	Moe	Szyslak
22	Apu	Nahasapeemapetilon
23	Patty	Bouvier
24	Lisa	Simpson
25	Chief	Wiggum
26	Kent	Brockman
27	Waylon	Smithers
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sorted by First Name

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sorted by Last Name

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3	Selma	Bouvier
4	Kent	Brockman
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6	Carl	Carlson
7	Ned	Flanders
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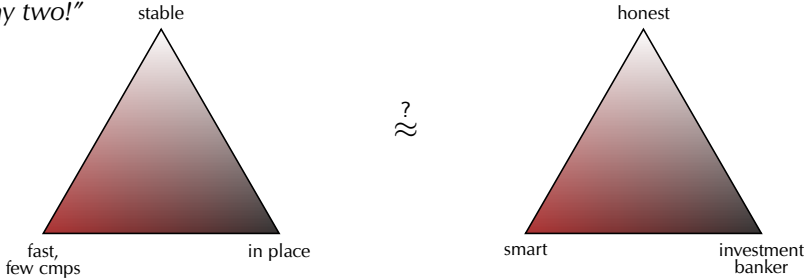

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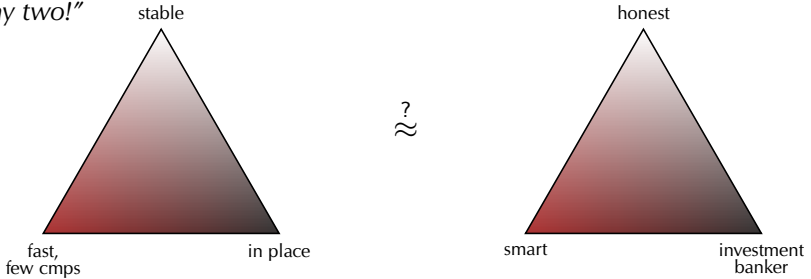

- Python has lots of objects and pointers anyways … "in-place" property least painful to sacrifice

⇝ Mergesort!

Timsort = Adaptive Mergesort++

Various modifications

1. detect existing **runs**
   run = any (weakly) increasing or
   (strictly) decreasing range
2. **merge policy** decides when to merge which runs
3. keeping runs on a **fixed-size** stack
4. **galloping** merges
   use exponential searches
   to find position in other run

5. **minimum run lengths** with binary insertion sort
   choose $32 \leqslant \texttt{minrun} \leqslant 64$, so that $\lceil \frac{n}{\texttt{minrun}} \rceil$ is 2^k or slightly smaller
   fill runs to `minrun` elements

Tim Peters et al.: *listsort.txt,* CPython sources

Timsort merge policy (original)

```
def timsort(lst):
    i = 0; runs = []
    while i < len(lst):
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        while Rule A/B/C applicable
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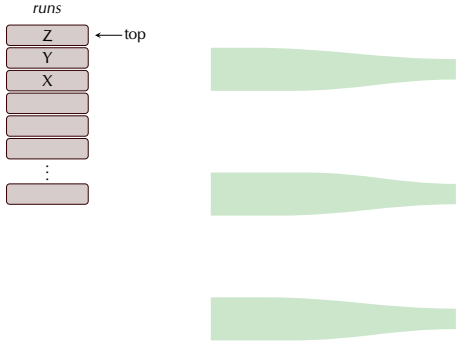

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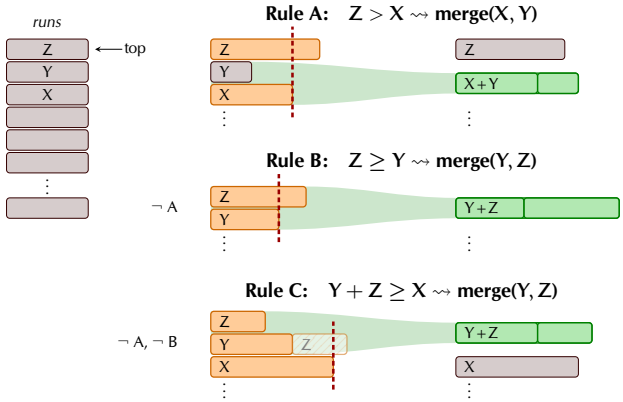

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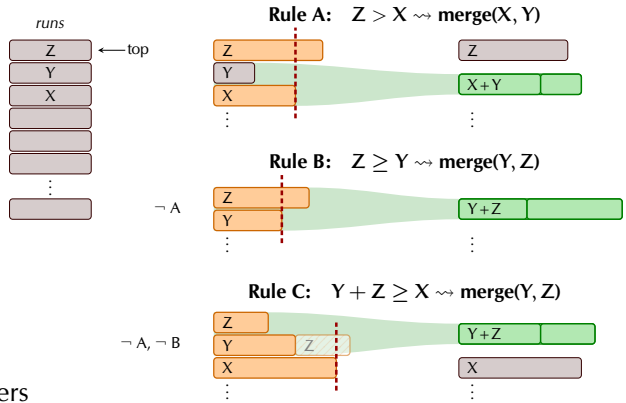

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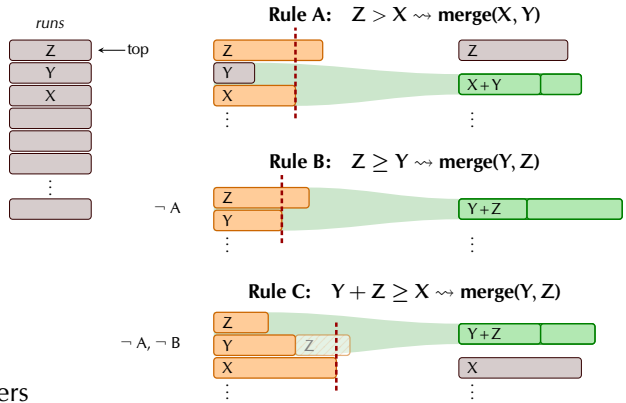

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  $\rightsquigarrow$ max stack height $\approx \log_\phi(n)$
- Why exactly these rules?
  - Tim Peters: *"first thing I tried that 'worked well'"*

(a) small runs stack,
(b) balanced on equal runs,
(c) $O(n \log n)$ worst case*

Invariant trouble

Recall this?

- **Goal:** runs on stack grow like Fibonacci:

  Invariant: $\forall \mathtt{j} = 0, \ldots, \mathtt{len(runs)} - 3 :$
  $\mathtt{runs}[\mathtt{j}] \geqslant \mathtt{runs}[\mathtt{j}+1] + \mathtt{runs}[\mathtt{j}+2]$

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- In Java: `Arrays.sort(Object[])`
  could throw `ArrayIndexOutOfBoundException`
  for specific input of size 67,108,864
  - Timsort in Java since 2009, but issue never reported?
  - first (incorrectly!) patched in 2013, then a second time in 2015 ...
- for CPython: patched in 2015
  - mostly a theoretical issue (needs $\geqslant 2^{49}$ elements $\approx 2\,\text{PB}$ ...)

Timsort merge policy (patched)

```
def timsort(lst):
    i = 0; runs = []
    while i < len(lst):
        j = extend_run(lst, i)
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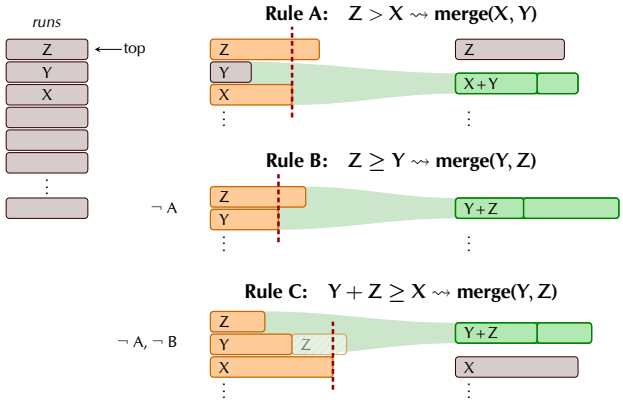

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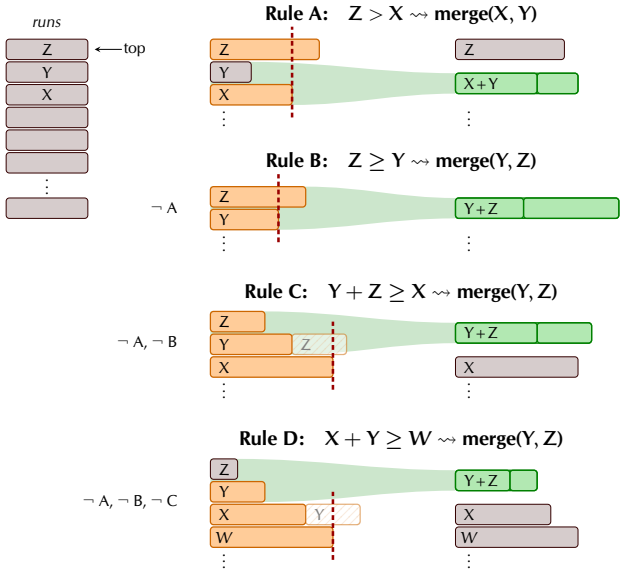

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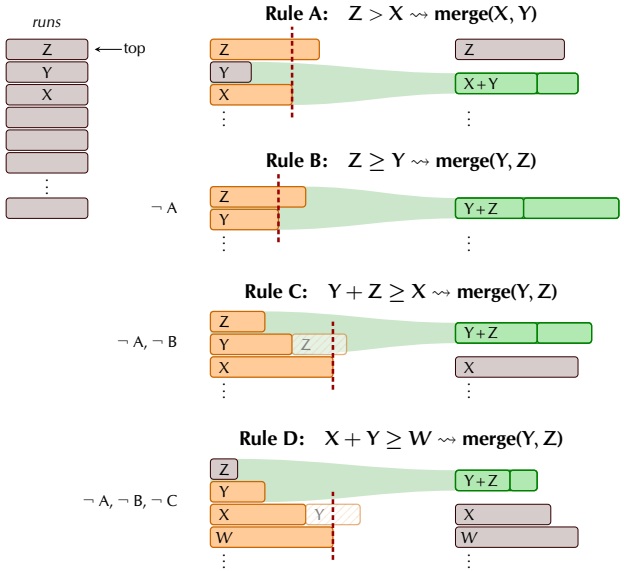

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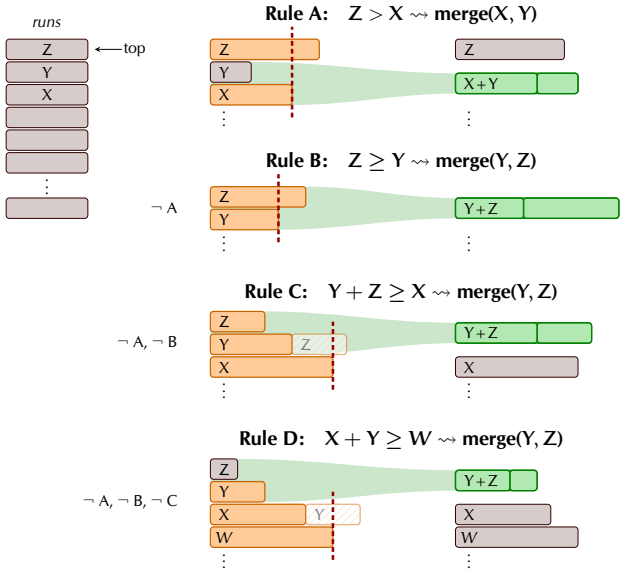
- running time indeed $O(n \log n)$ … *very* complicated to prove

Auger, Jugé, Nicaud, Pivoteau: *On the Worst-Case Complexity of TimSort,* ESA **2018**



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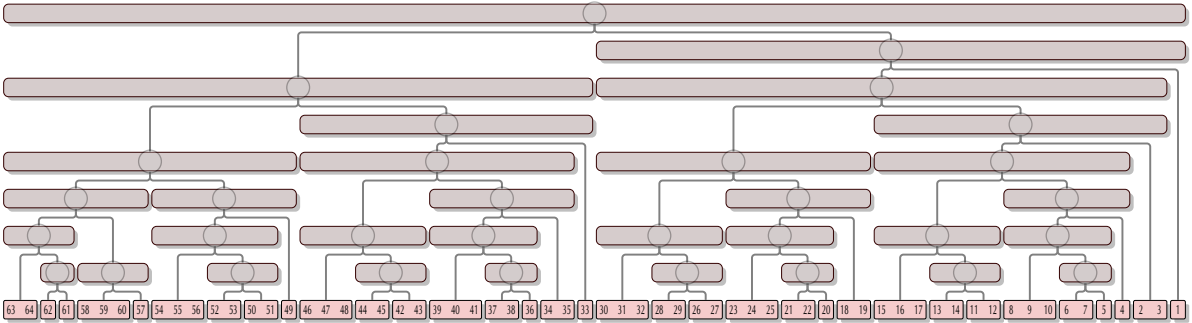

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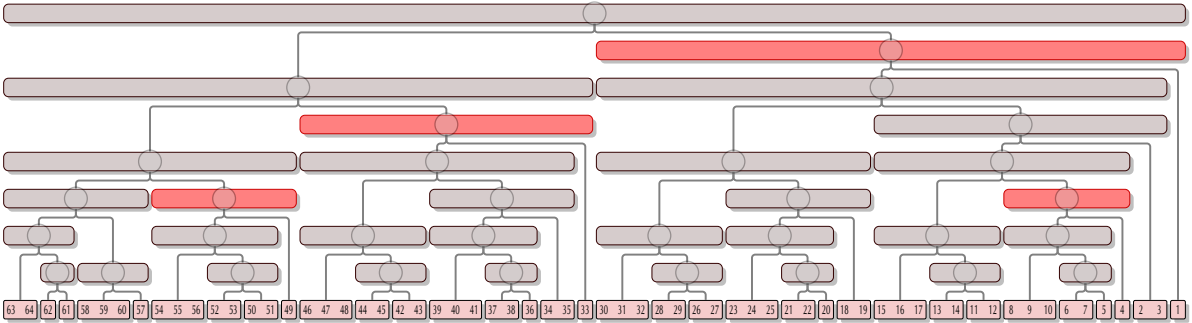

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Buss, Knop: *Strategies for Stable Merge Sorting*, SODA **2019**

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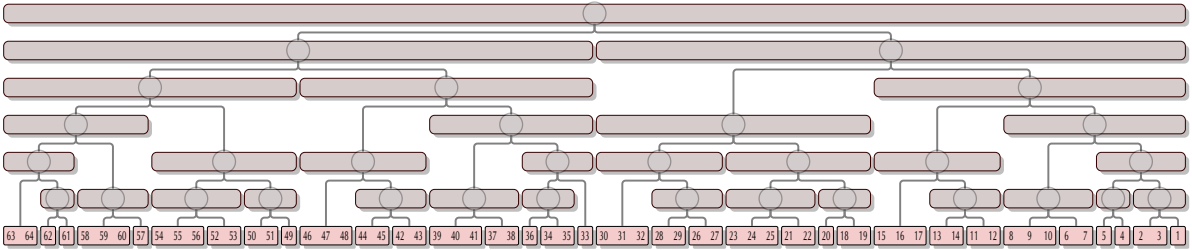

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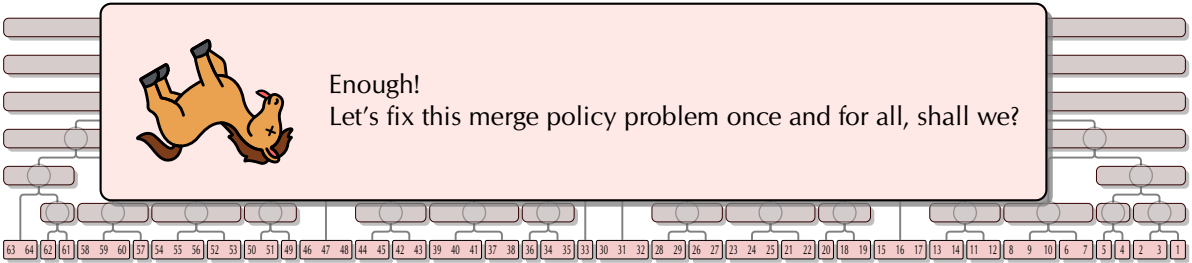

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Outline

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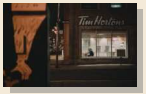
2 Java's Sorting Methods

3 Dual-Pivot Quicksort



4 Timsort

5 Powersort


POWER
sort
5
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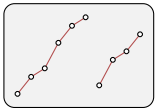
Merge policies from first principles

Goal: ***Simple*** *stable sorting method that optimally exploits* ***sorted runs***



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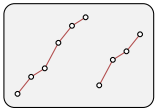

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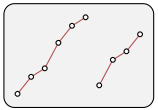

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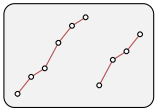

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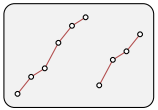

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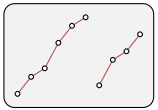

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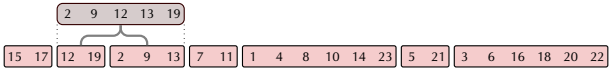

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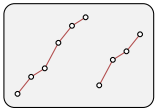

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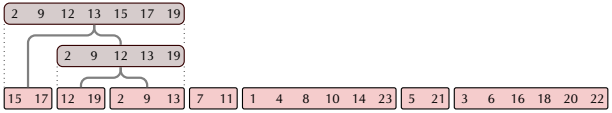

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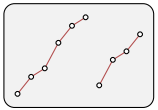

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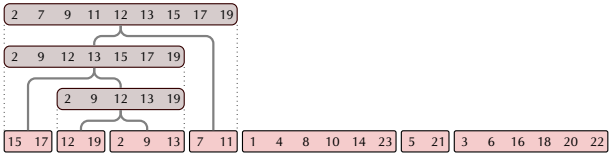

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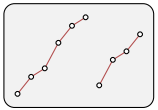

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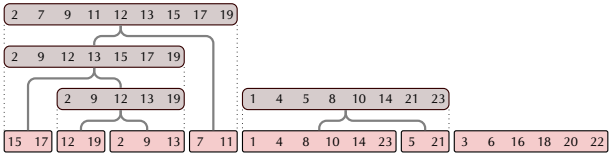

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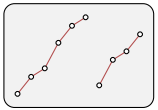

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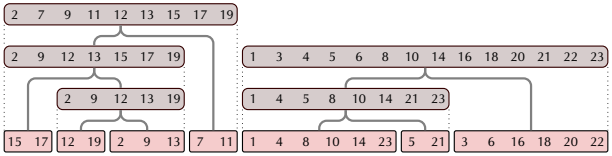

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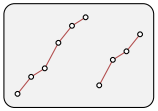

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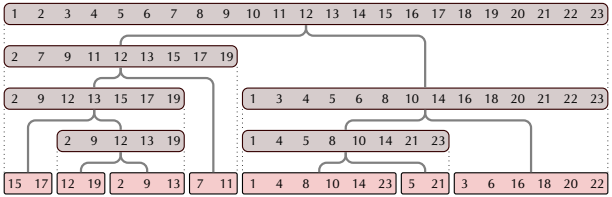

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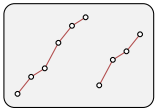

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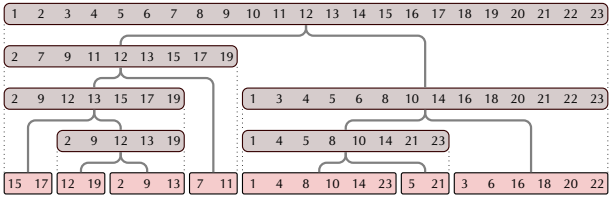

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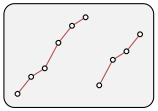

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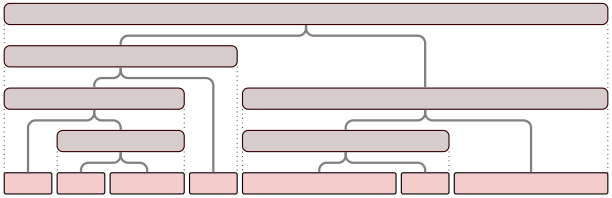
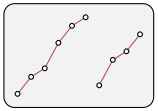


⇝ **Merge policy** = algorithm to choose merge tree

- **Merge cost** of one merge := size of output
  - ≈ memory transfers
  - ⩾ # comparisons
- total cost = total area of 

Merge policies from first principles

Goal: ***Simple*** *stable sorting method that optimally exploits* ***sorted runs***



Run-adaptive mergesort

interleaved in code

Conceptually two steps:

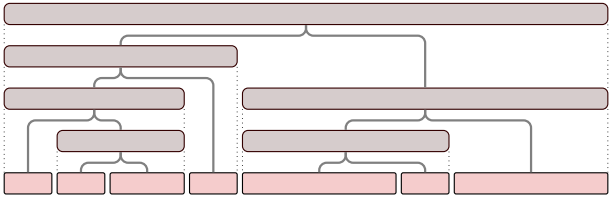
1. *Find runs in input.*
2. *Merge them* ***in some order****.*

determined by **merge policy**

- for stable sort
  - ⇝ *merge 2* ***adjacent*** *runs*

Find good tree with little overhead?

⇝ Merge order = **merge tree**:



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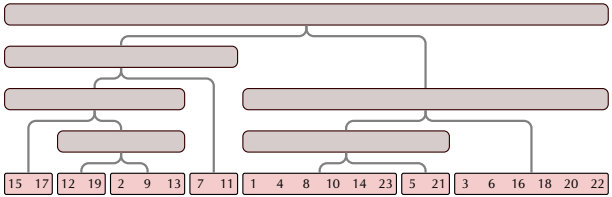
Mergesort meets Binary Search Trees



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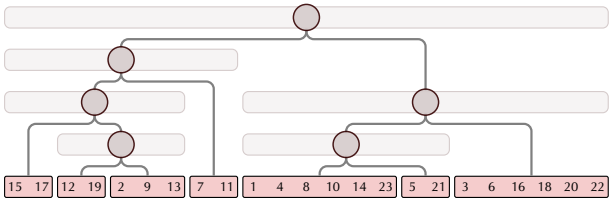

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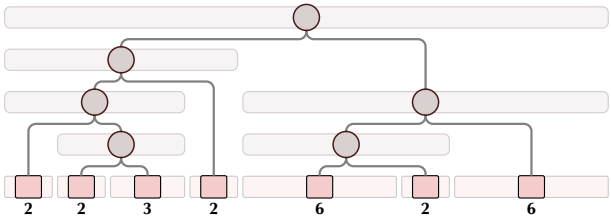

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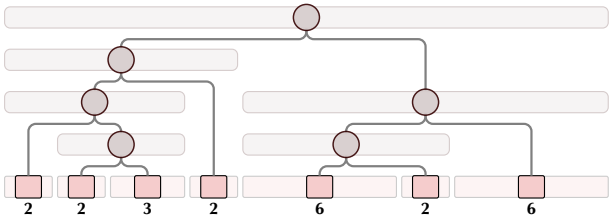

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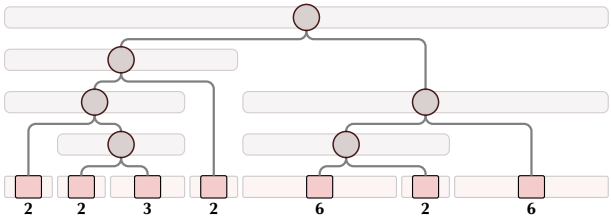

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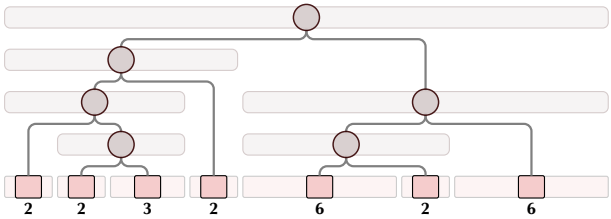

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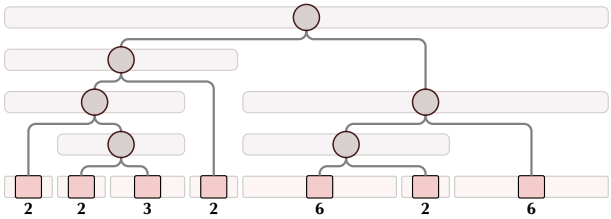

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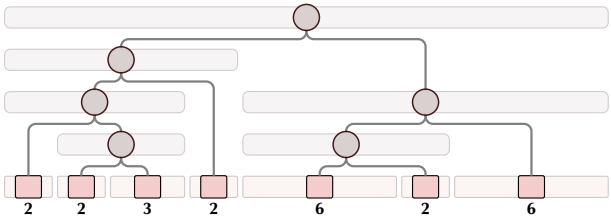

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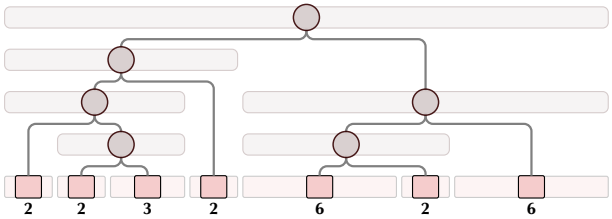

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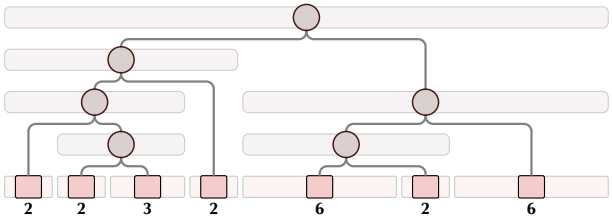

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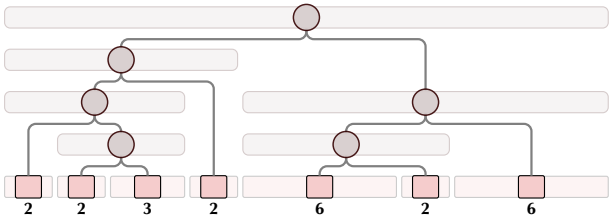

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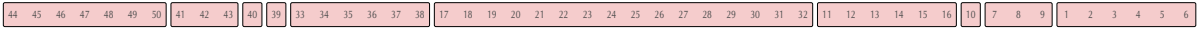
- **But:** Can't use recursion!

Run-Boundary Powers



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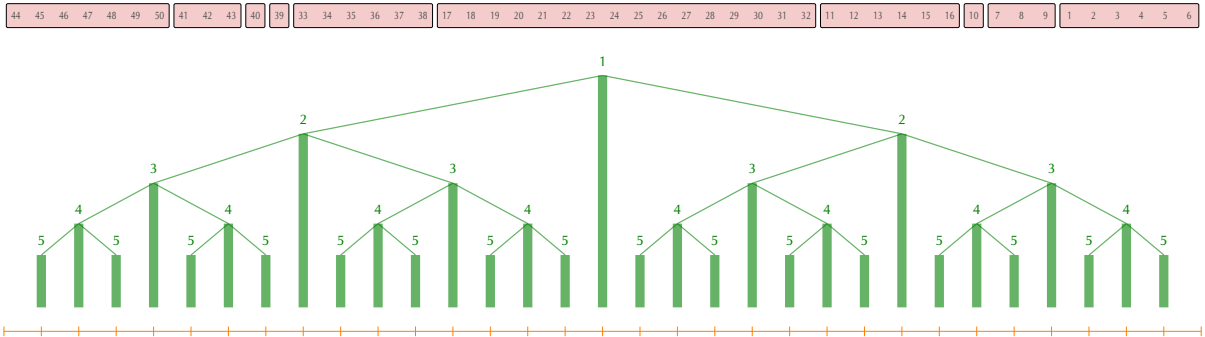
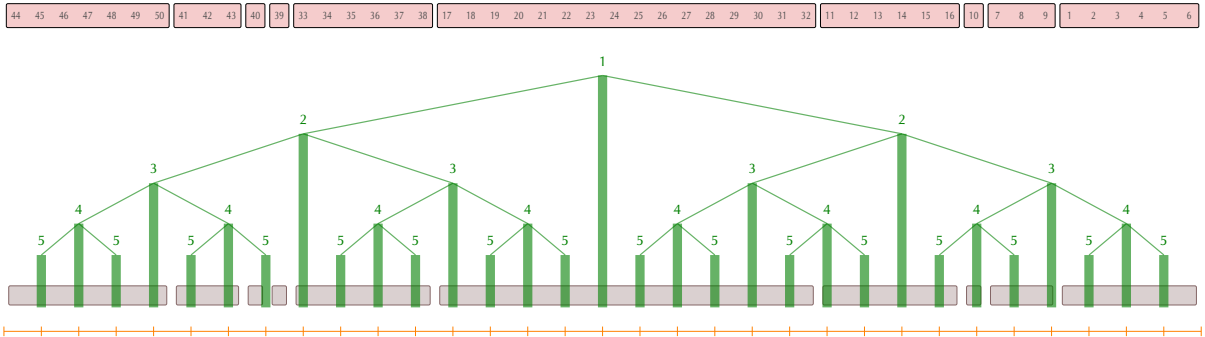


- (virtual) perfect balanced binary tree

Run-Boundary Powers



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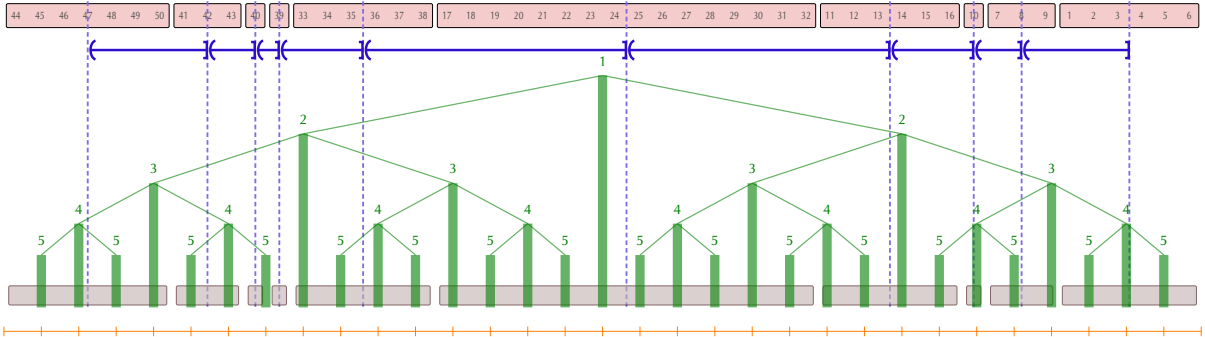

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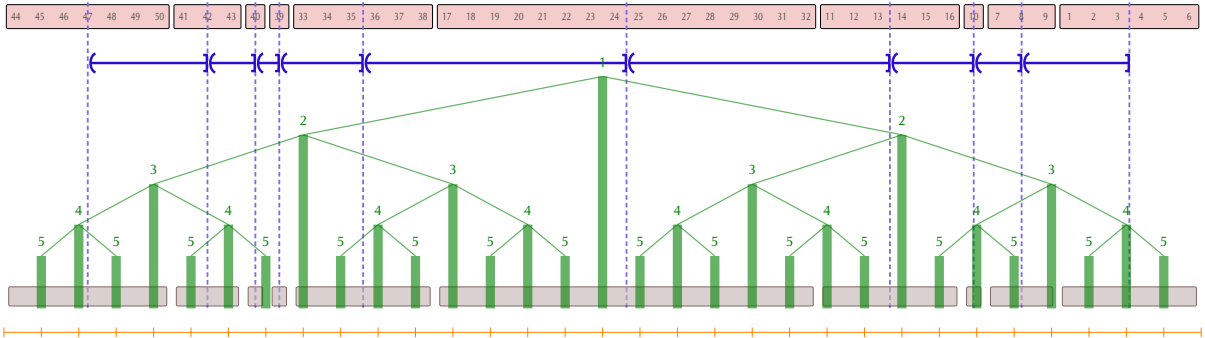

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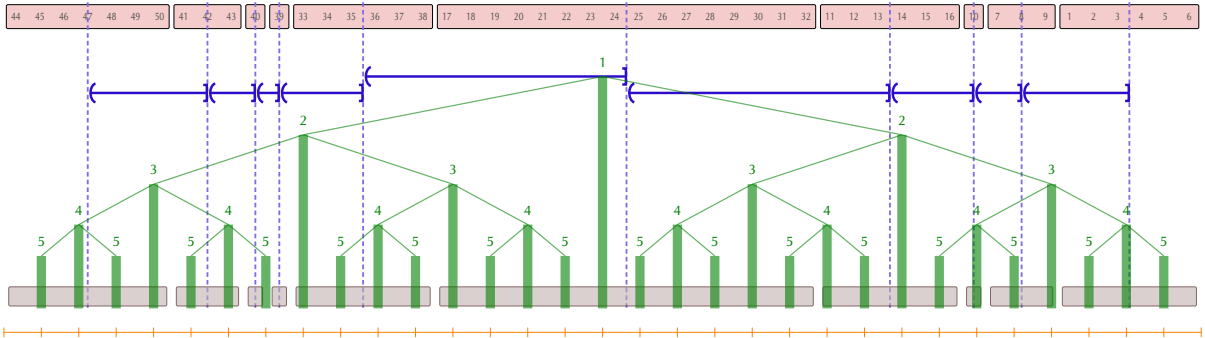

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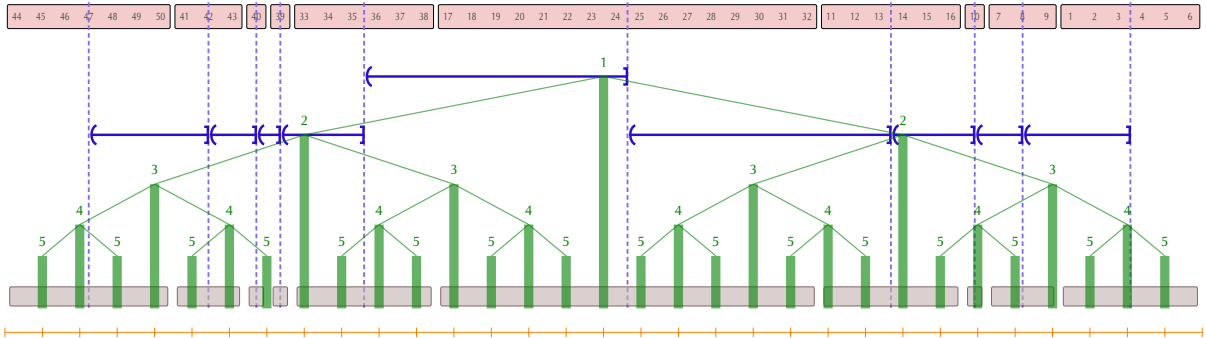

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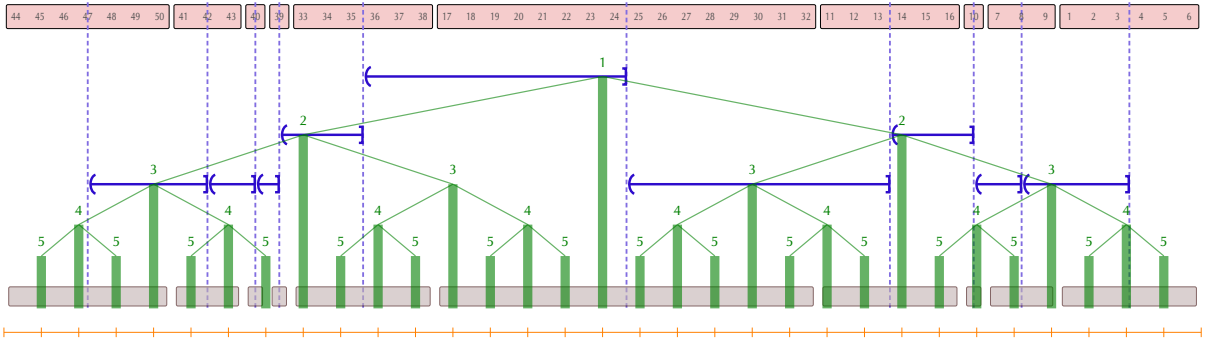

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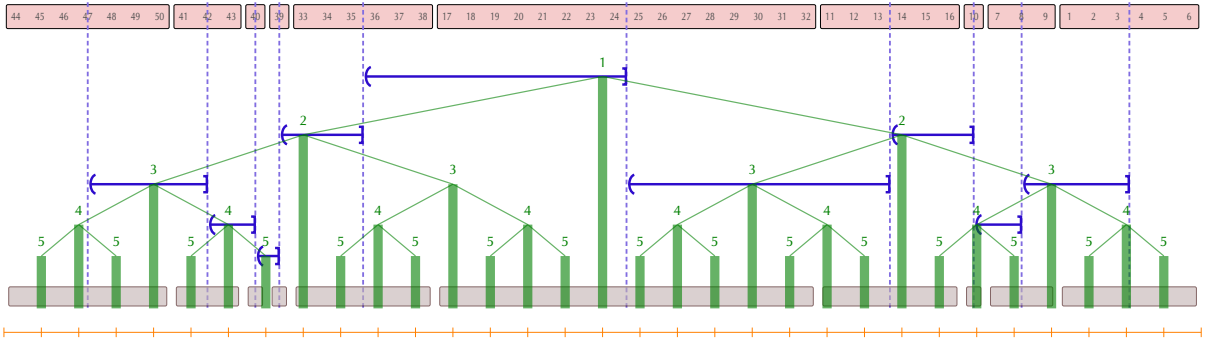

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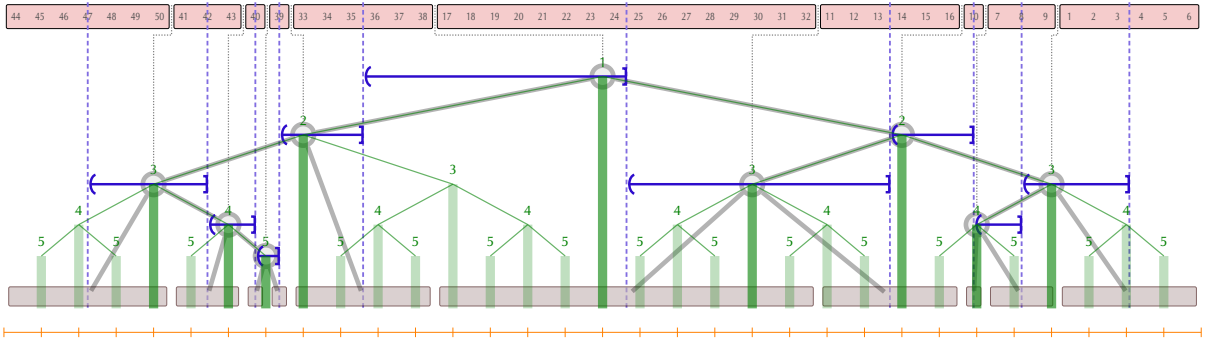

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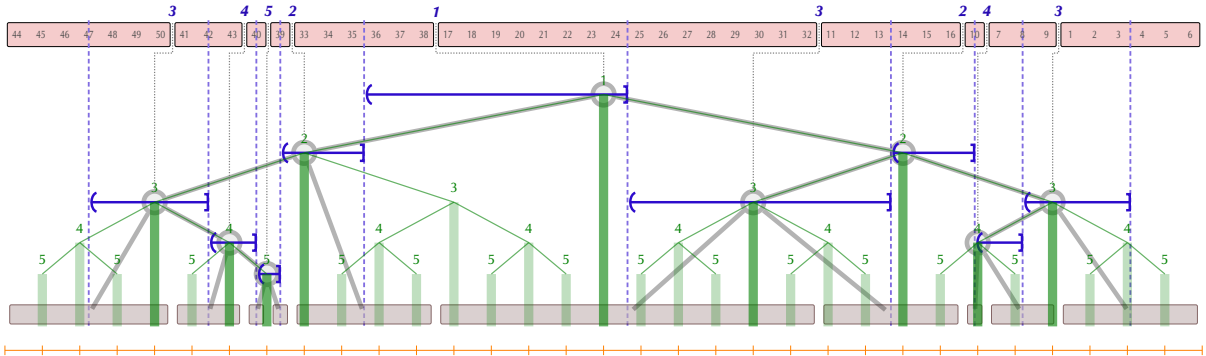

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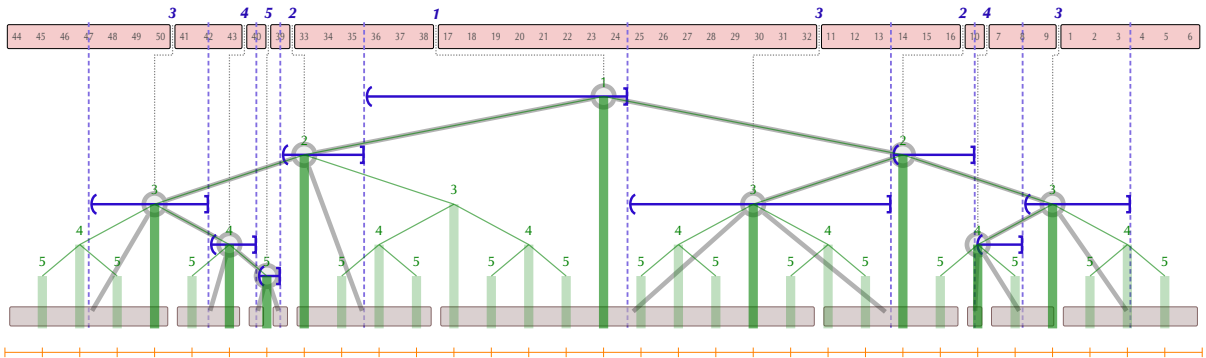

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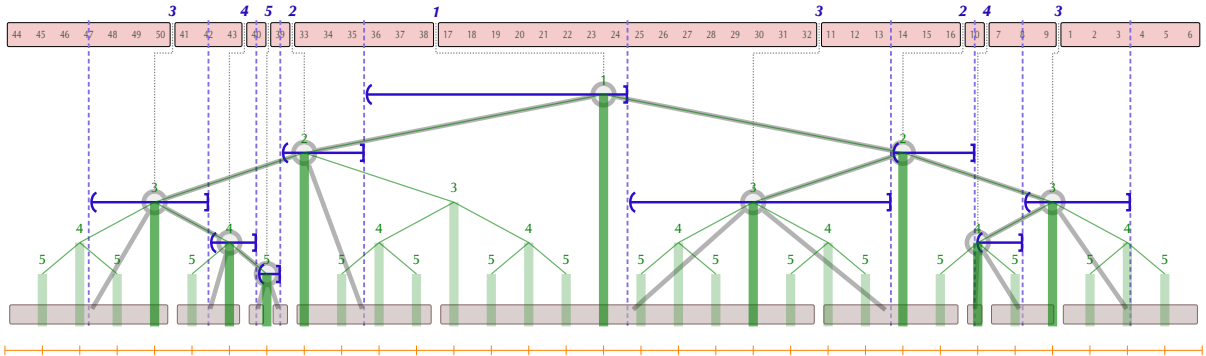

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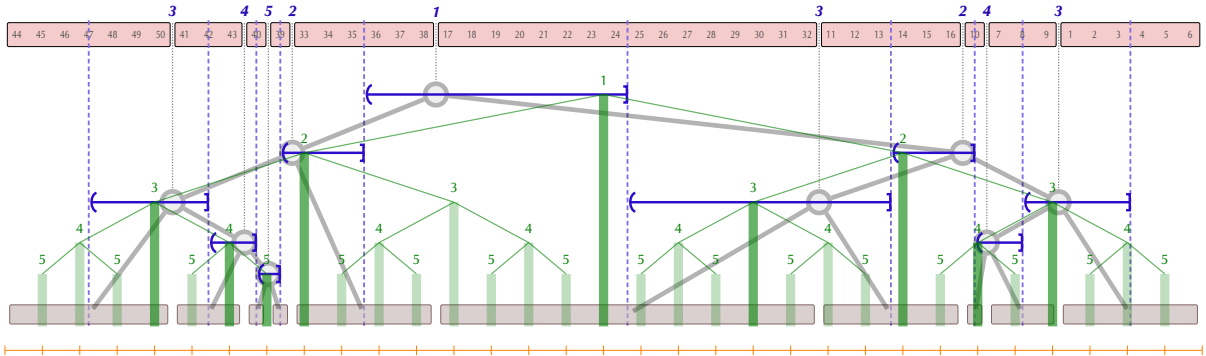

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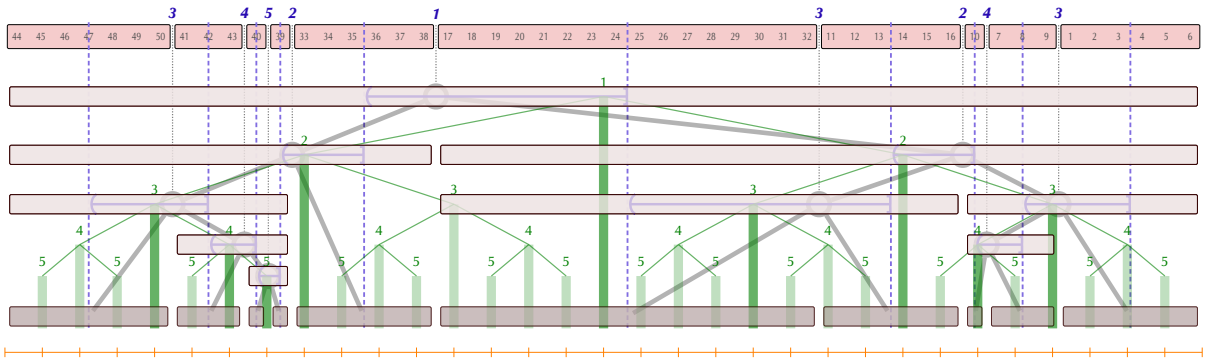

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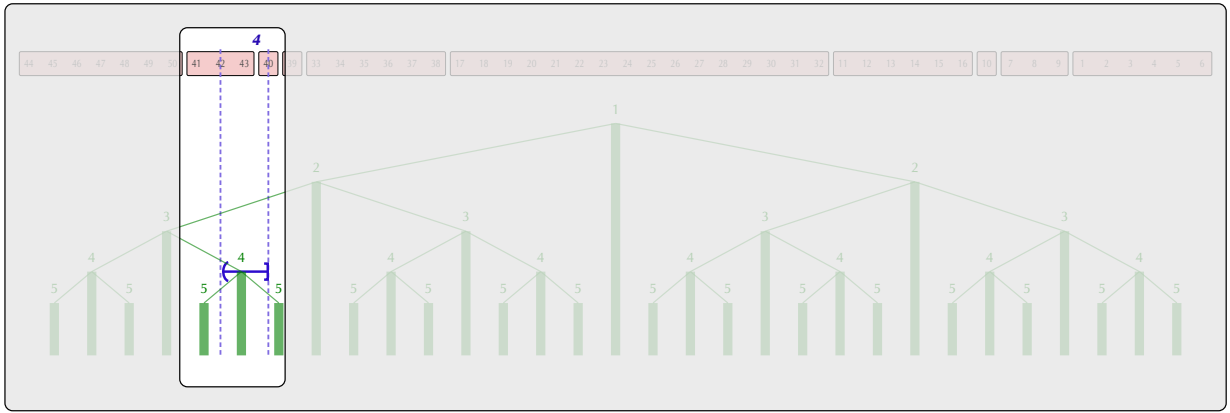
Run-Boundary Powers are Local



Computation of powers only depends on two adjacent runs.

Powersort

```
def powersort(lst)
    i = 0; n = len(lst)
    runs = [];
    j = extend_run(lst, i)
    runs.append((i,j-i,0)); i = j
    while i < n:
        j = extend_run(lst, i)
        p = power(runs[-1], (i,j-i), n)
        while p <= runs[-1][2]
            merge_topmost_2(lst, runs)
        runs.append((i,j-i,p)); i = j
    while len(runs) >= 2:
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```
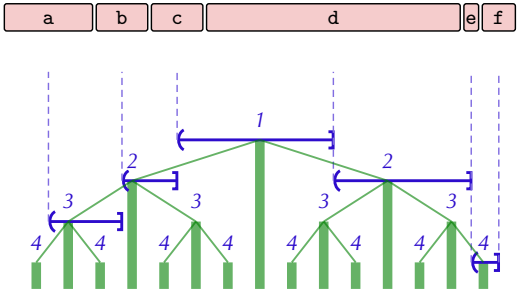
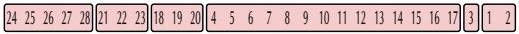

full code:
`https://power-sort.github.io`





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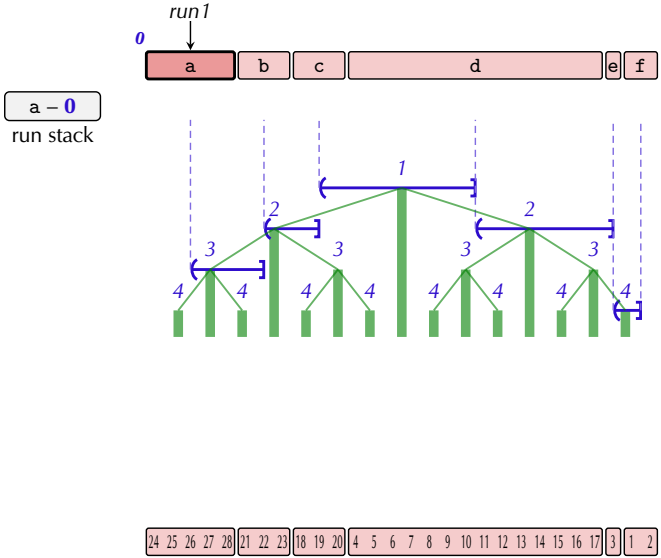

Powersort

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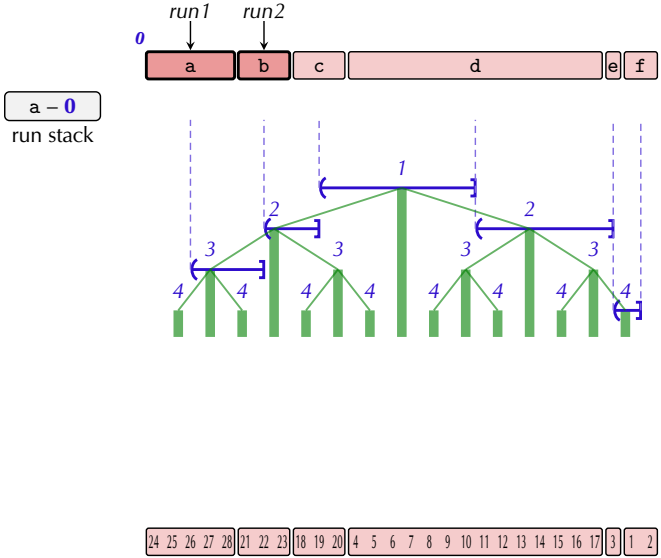

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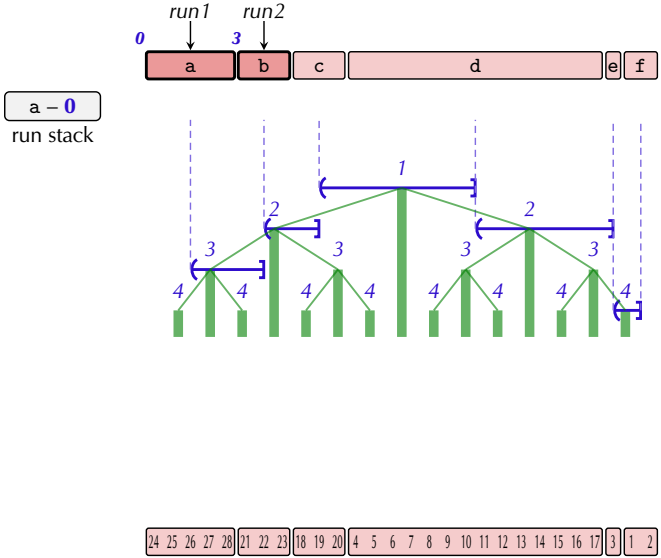

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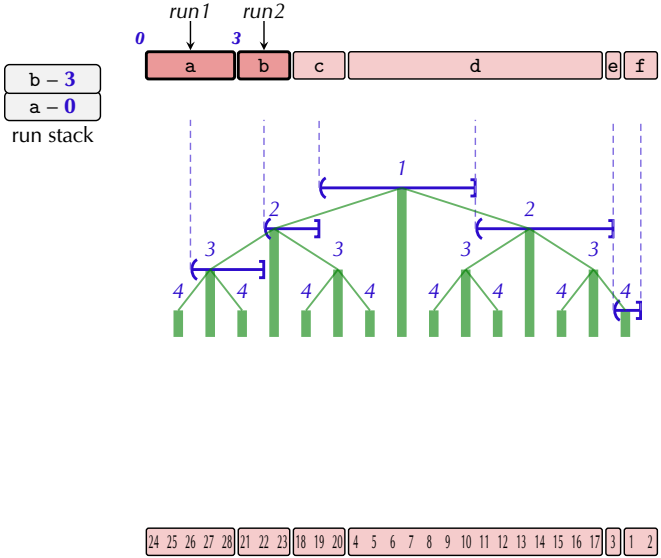

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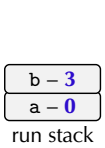

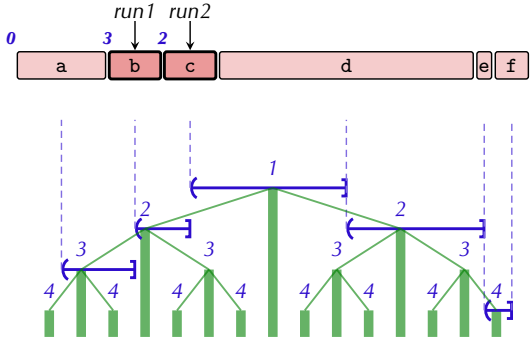

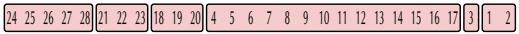

Powersort

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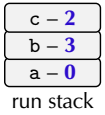

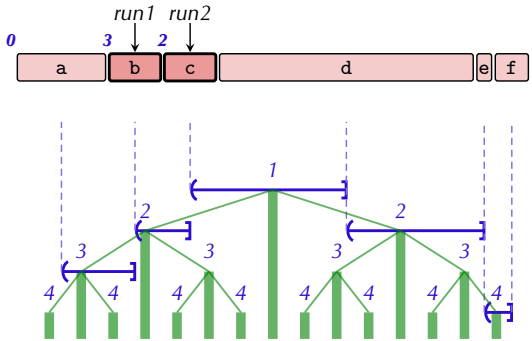

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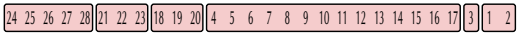

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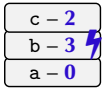

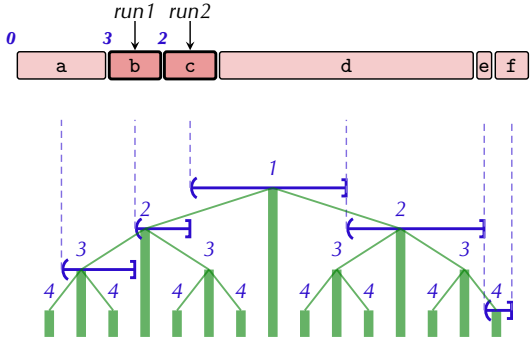

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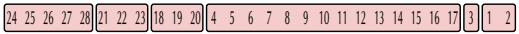

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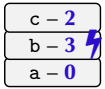

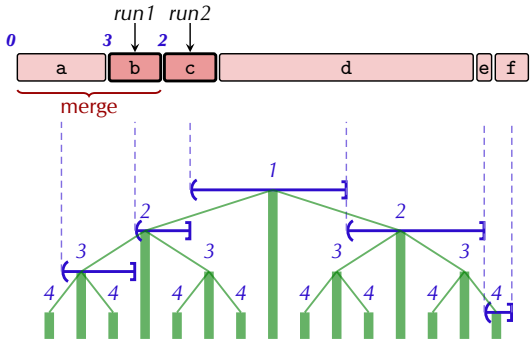

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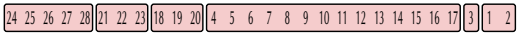

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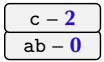

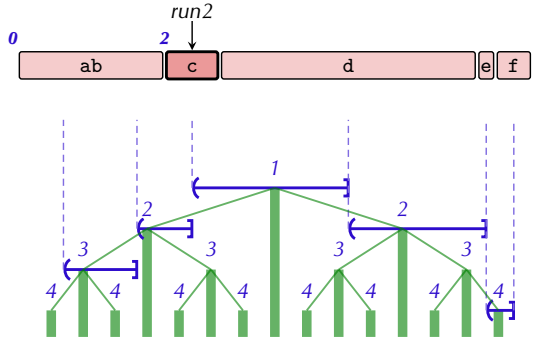

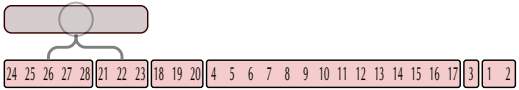

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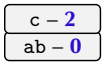

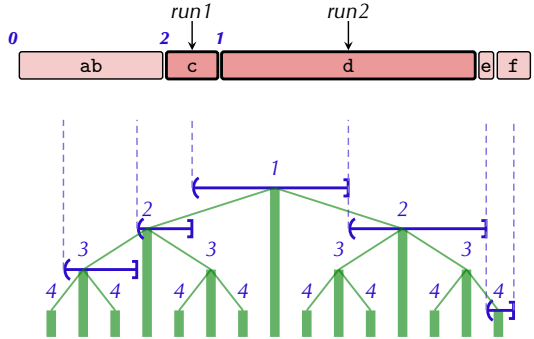

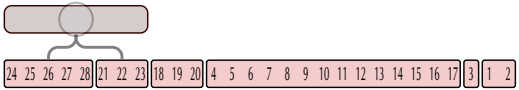

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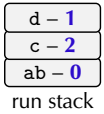

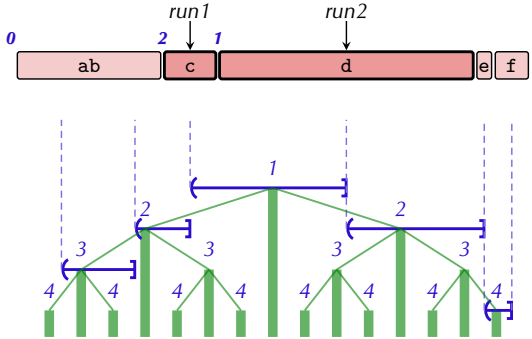

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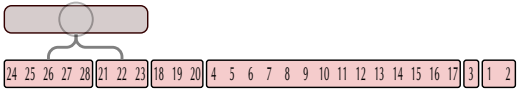

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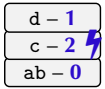
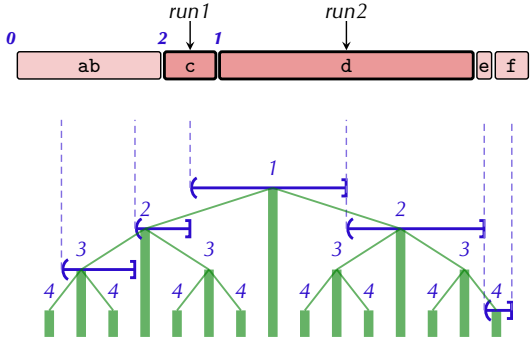


run stack



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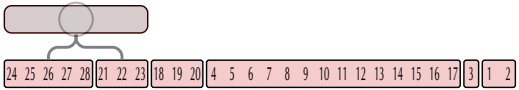

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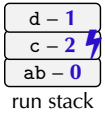

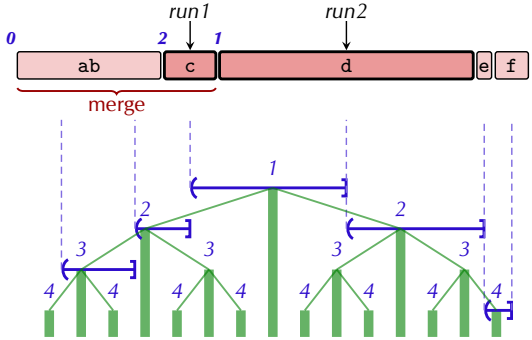

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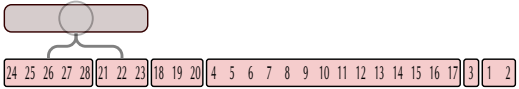

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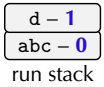

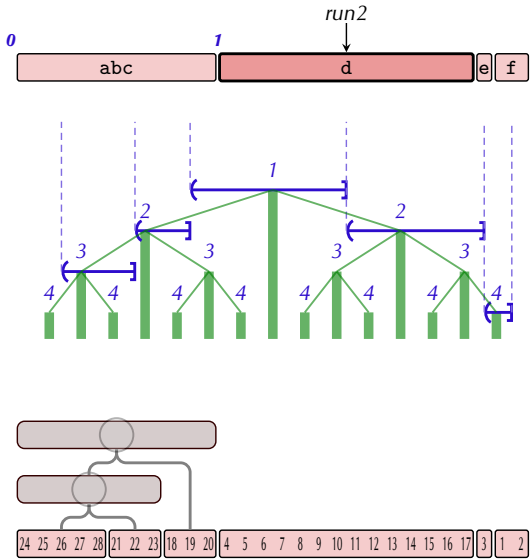

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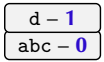

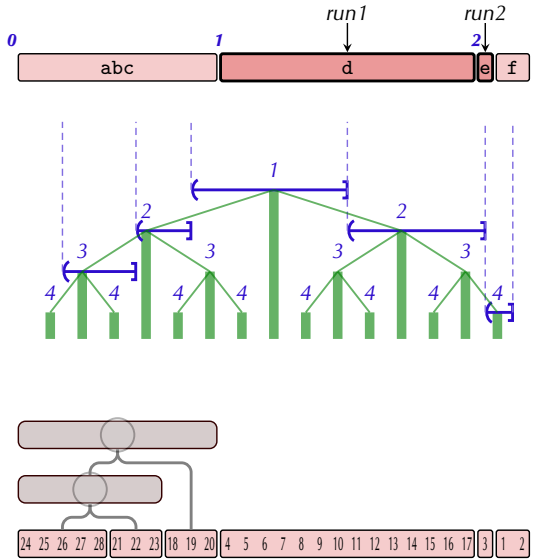

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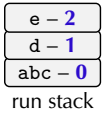

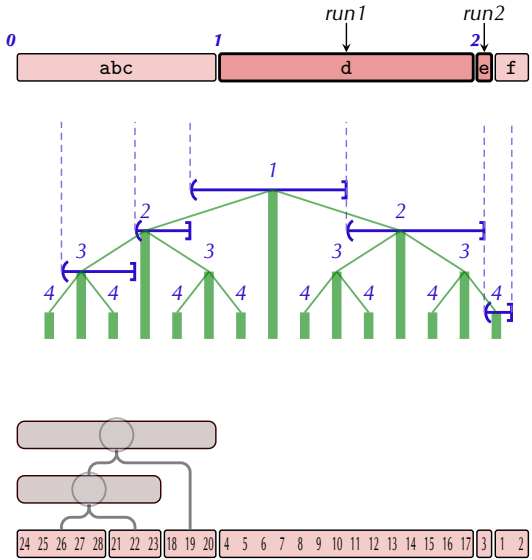

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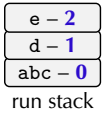

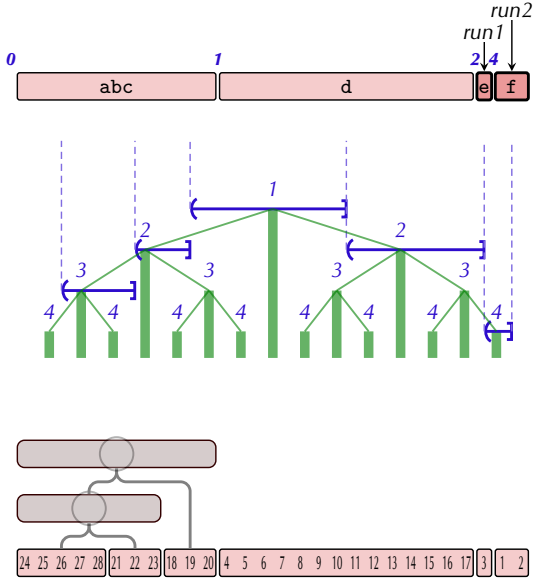

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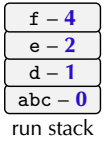

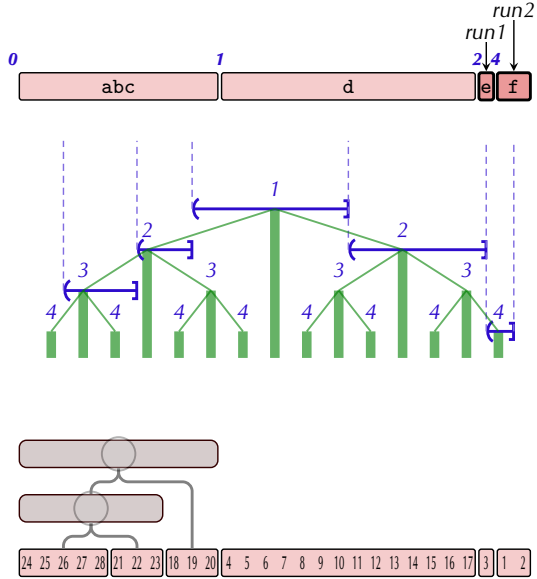

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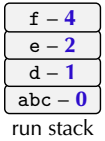

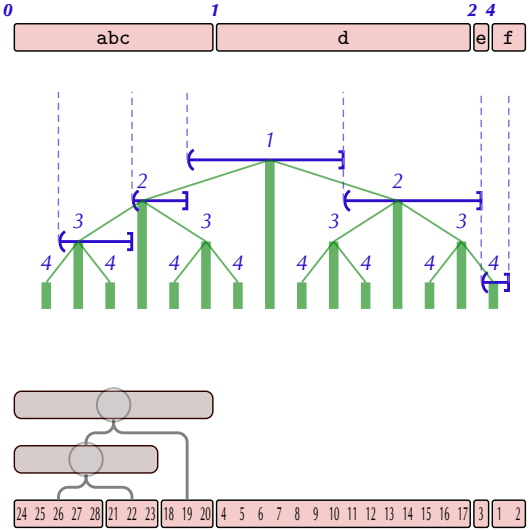

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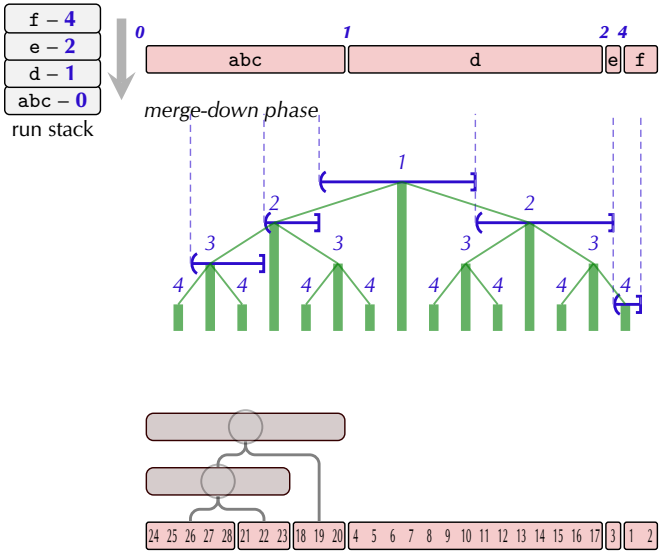

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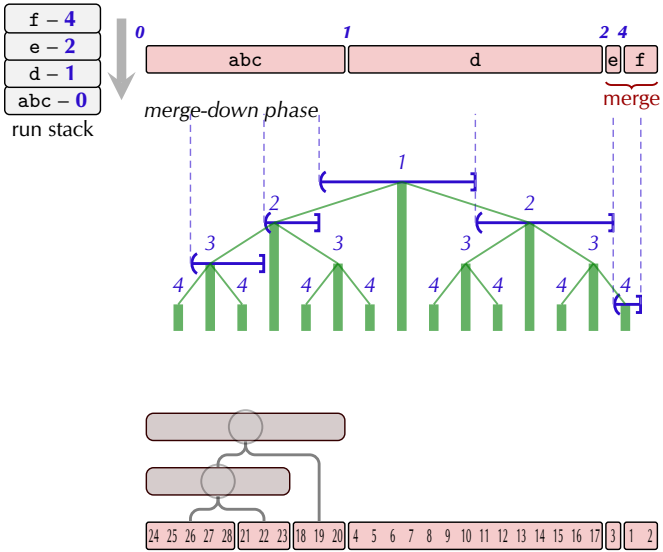

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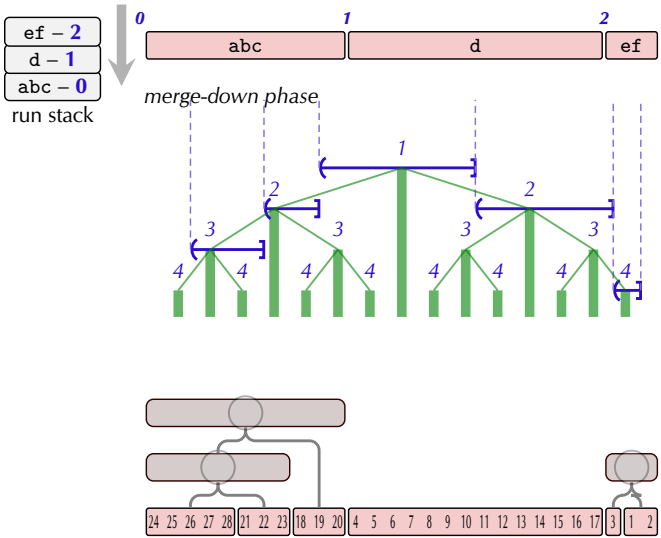

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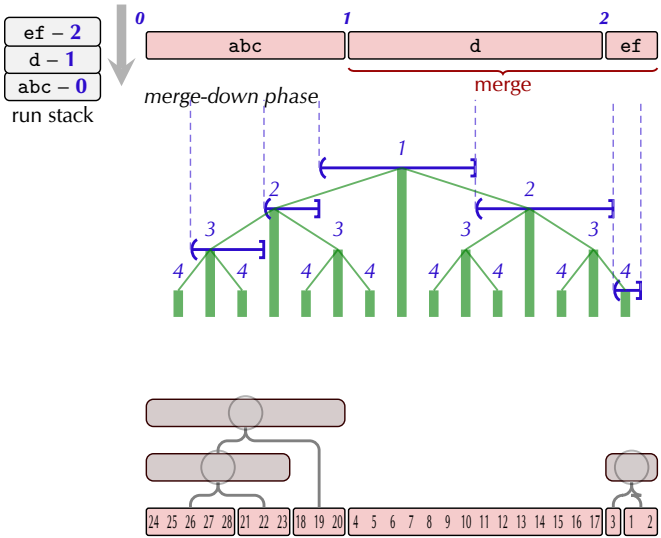

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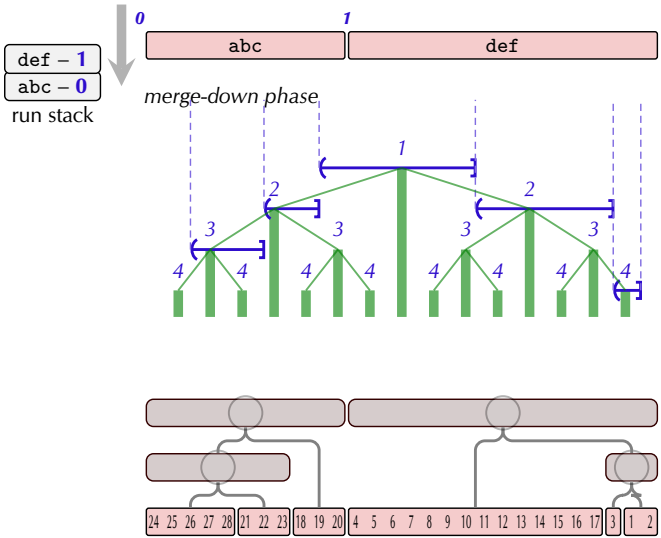

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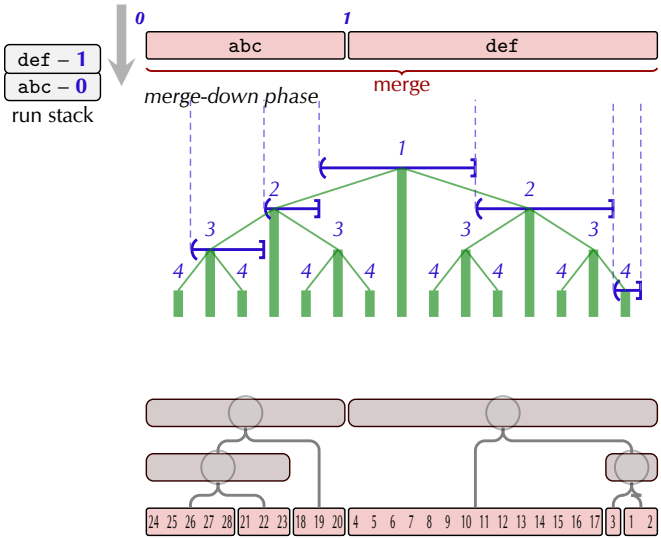

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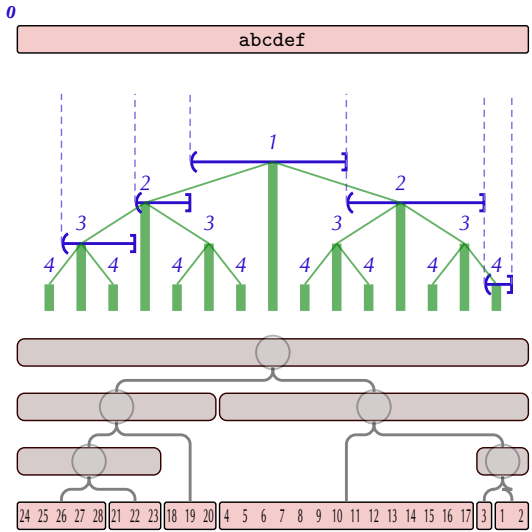
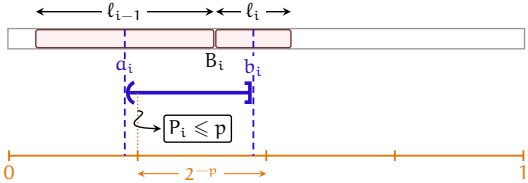
abcdef – 0
run stack



Computing powers

- Computing the power of (run boundary between) two runs
  - $\vdash\!\!\dashv$ = normalized midpoint interval
  - power = min ℓ s.t. $\vdash\!\!\dashv$ contains $c \cdot 2^{-\ell}$



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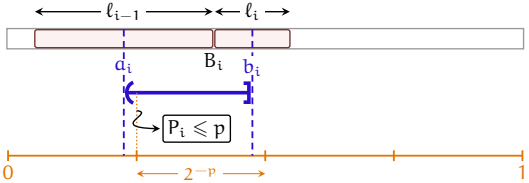


```
def power(run1, run2, n):
    i1, n1 = run1
    i2, n2 = run2
    A = 2 * i1 + n1   # A = 2na_i
    B = A + n1 + n2   # B = 2nb_i
    p = 0
    while True:
        p += 1
        if A >= n:
            A -= n; B -= n
        elif B >= n:
            break
        A <<= 1; B <<= 1
    return p
```

- with bitwise trickery O(1) time possible

Computing powers

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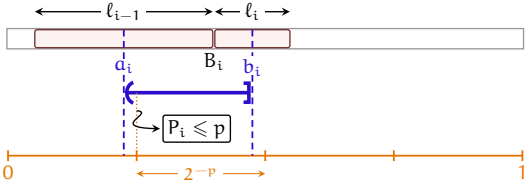


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Obvious analysis

- $0 \leqslant P_i \leqslant \lceil \lg n \rceil$
- powers on stack strictly increasing

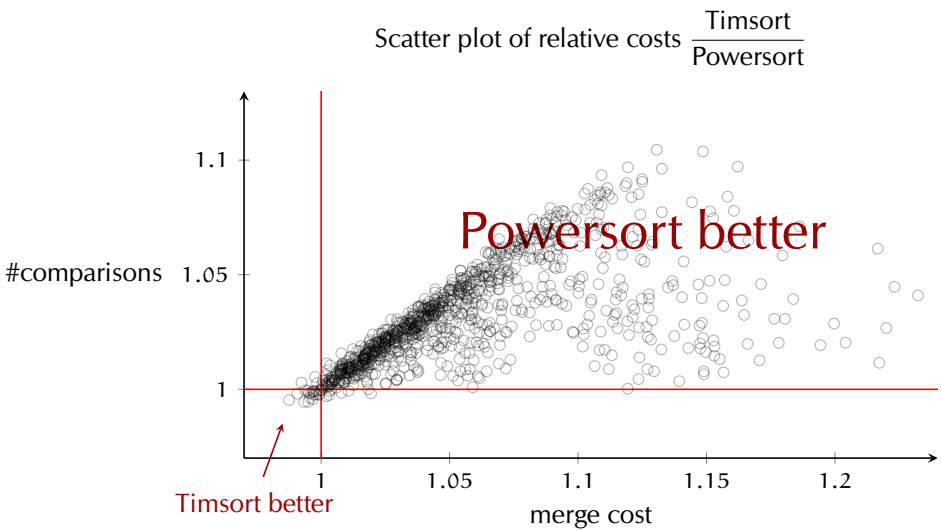
$\rightsquigarrow$ Stack height $\leqslant \lceil \lg n \rceil + 1$

Some performance data

Summary claims:

1. Typical improvement from Powersort:
   0-5% fewer comparisons; occasionally 20–60%. Data
   (One can contrive inputs where Powersort does worse; seems inevitable)
2. No running time regressions: never measurably slower in actual running time Data
3. Sometimes substantially faster (20–40%)

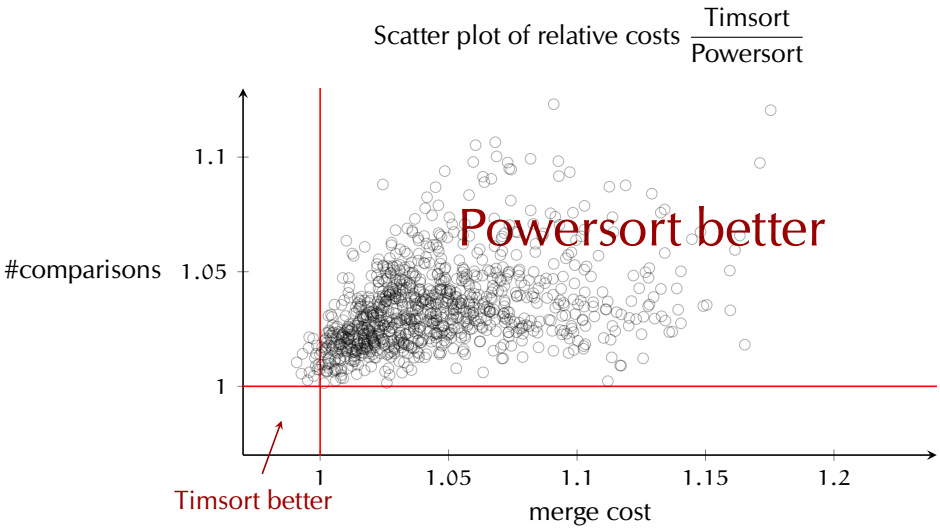
Abstract cost measures



- Worse on 2-3%, but if so, only slightly.
- On average, 3% fewer cmps and 5% less merge cost.

Input: Runs of expected length $\sqrt{n}$, $n = 10^5$

Abstract cost measures



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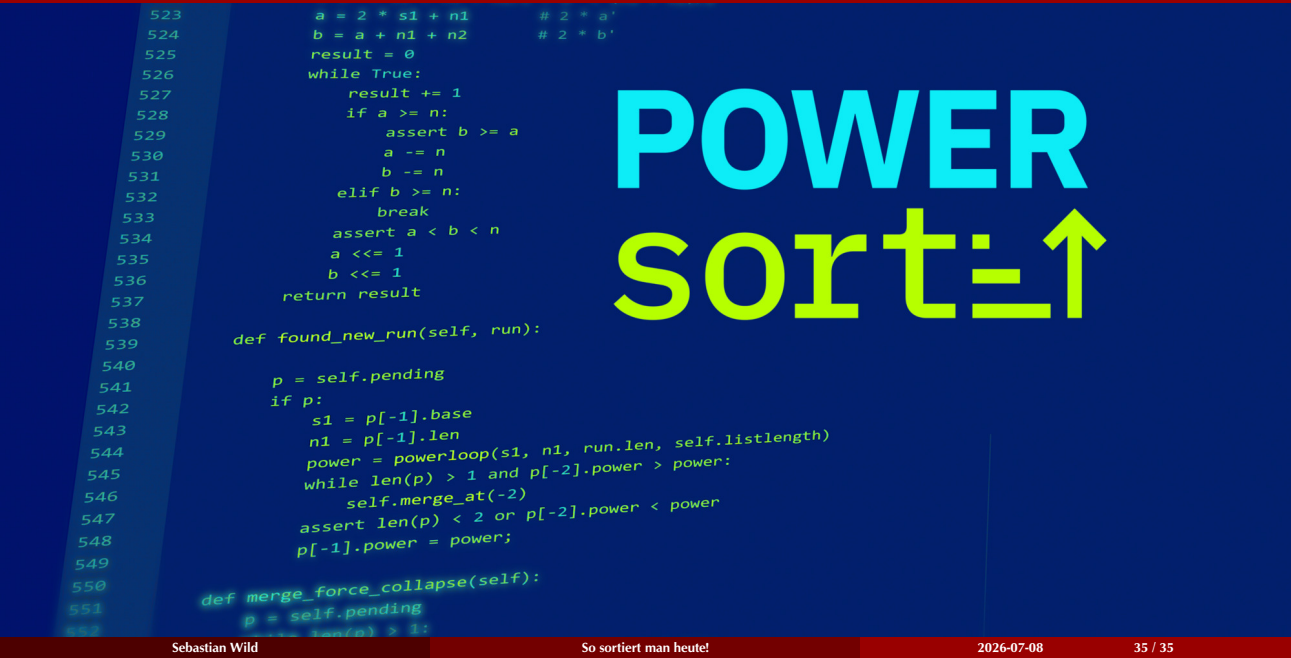
Input: Tim's mixture of long and short runs

Conclusion



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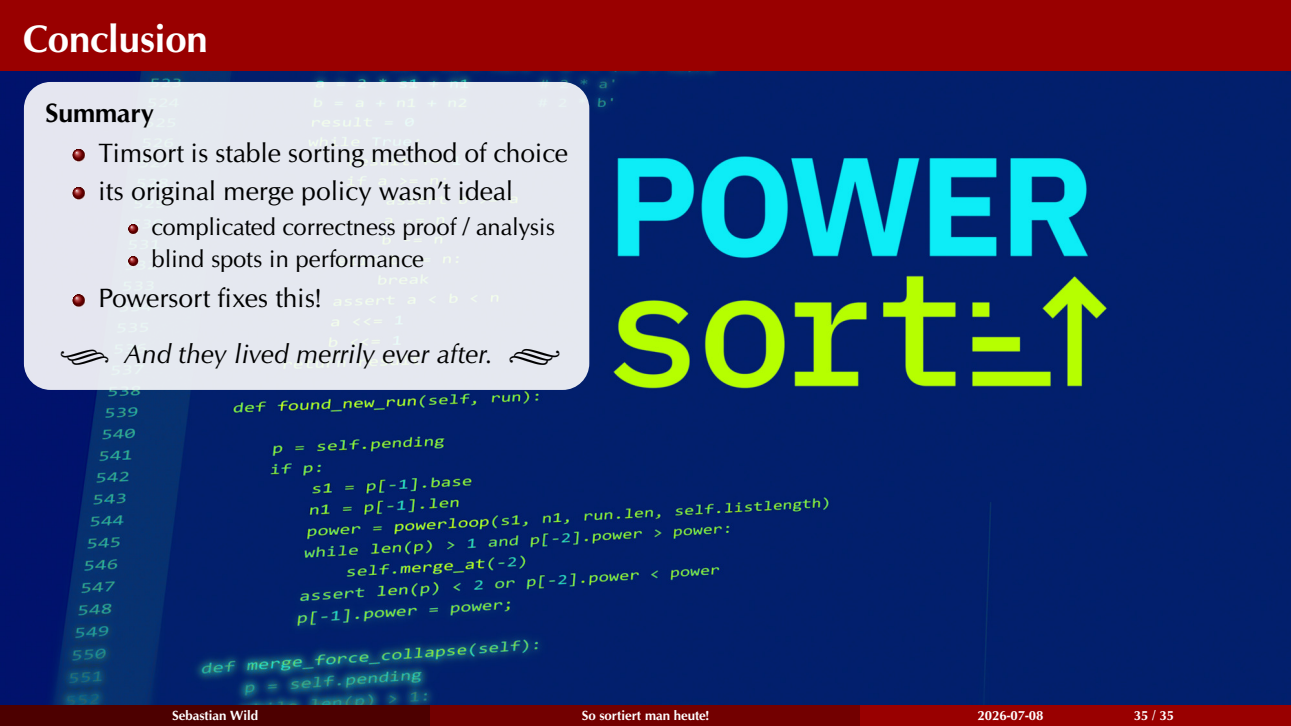
And they lived merrily ever after.



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POWER

Goals

1. Powersort in **other libraries** using Timsort?
   - numpy & pandas ✓
   - OpenJDK
   - Android Java runtime
   - V8 JavaScript engine
   - GNU STL `std::stable_sort`

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   - numpy & pandas ✓
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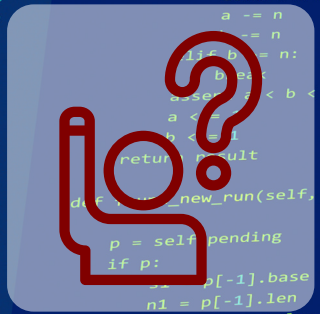
   - GNU STL `std::stable_sort`
2. Where else can adaptive-algorithms ideas spark practical improvements?



POWER sort

Powersort: An Algorithm Science Success Story

Observation: data often partially sorted

↓

Timsort in CPython 2001
ad-hoc rules for merge policy

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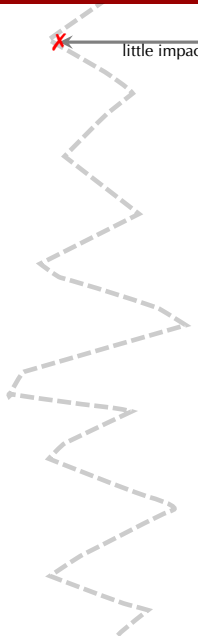

1990s: Theoretical literature on adaptive sorting

✗ little impact on practice

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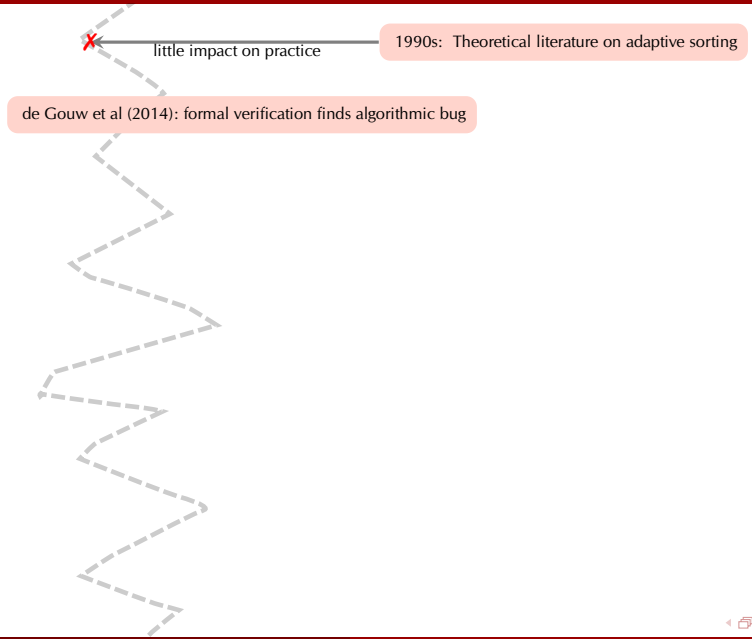

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2015 revised to fix stack overflow(!)

de Gouw et al (2014): formal verification finds algorithmic bug

little impact on practice

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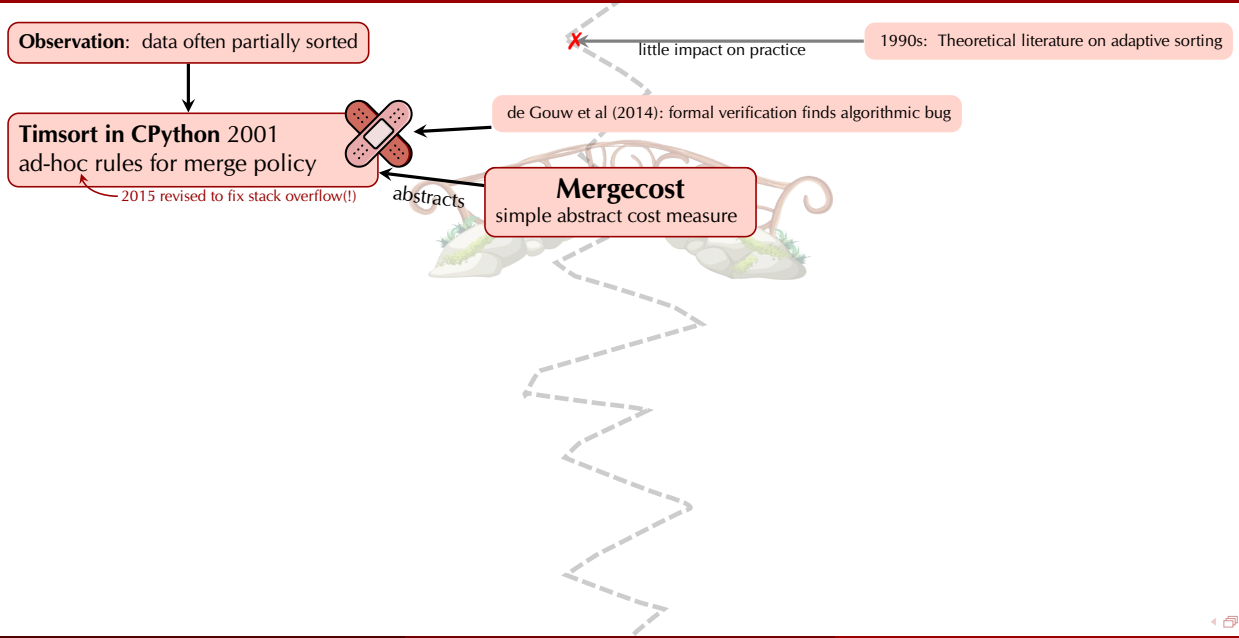
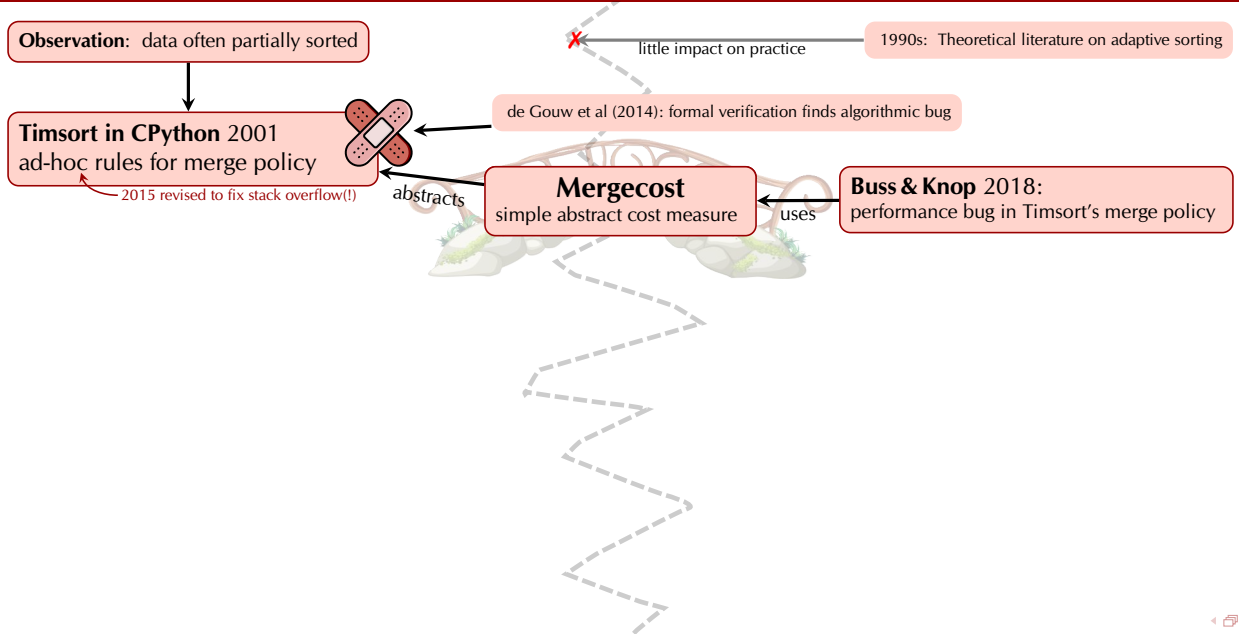


Powersort: An Algorithm Science Success Story



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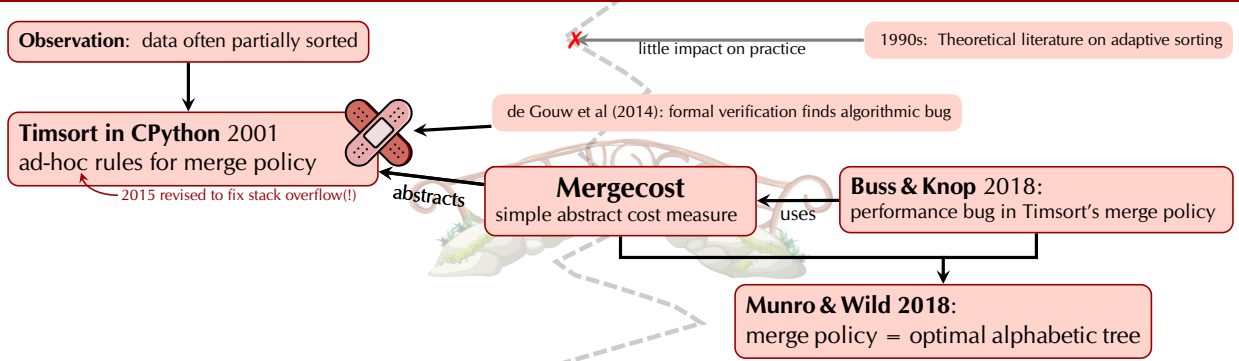

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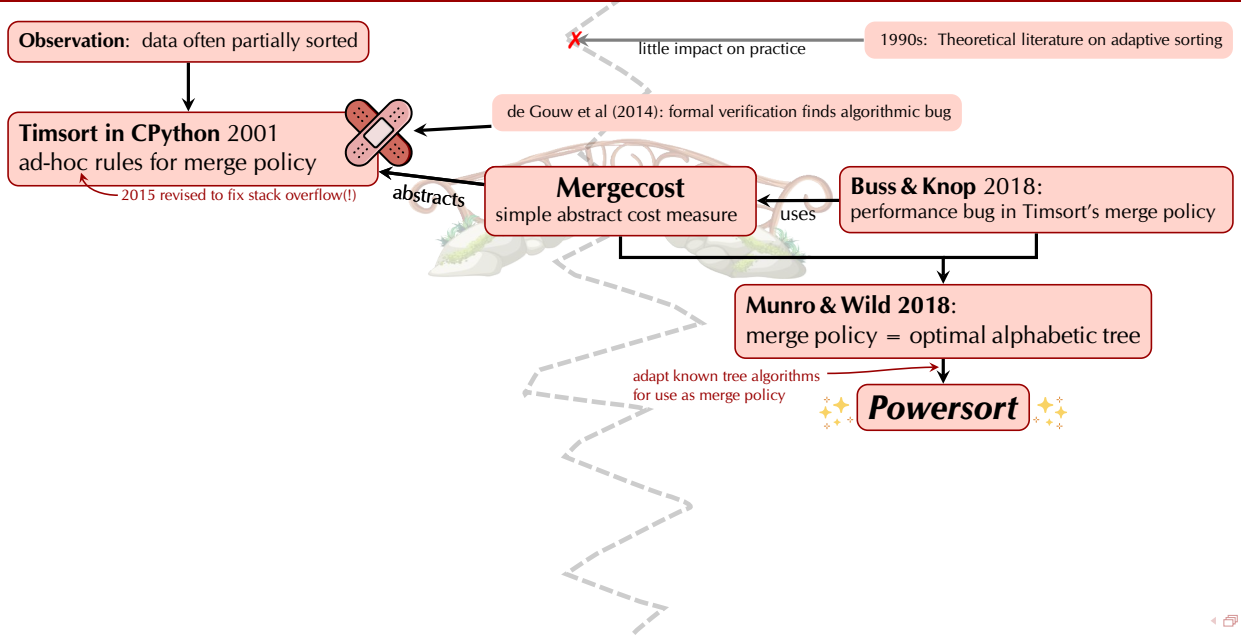

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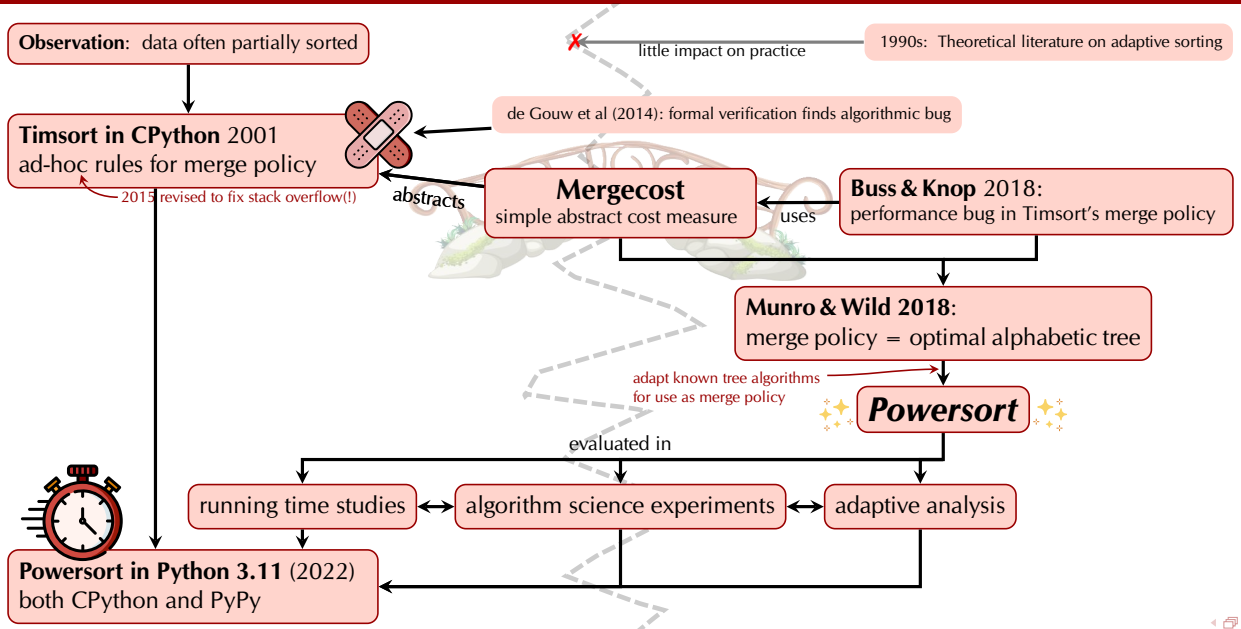

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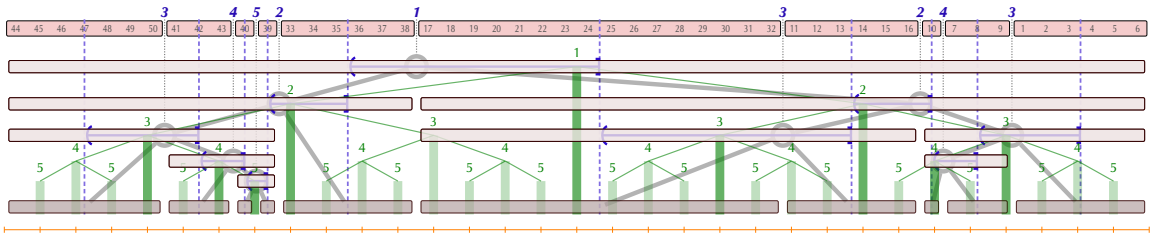

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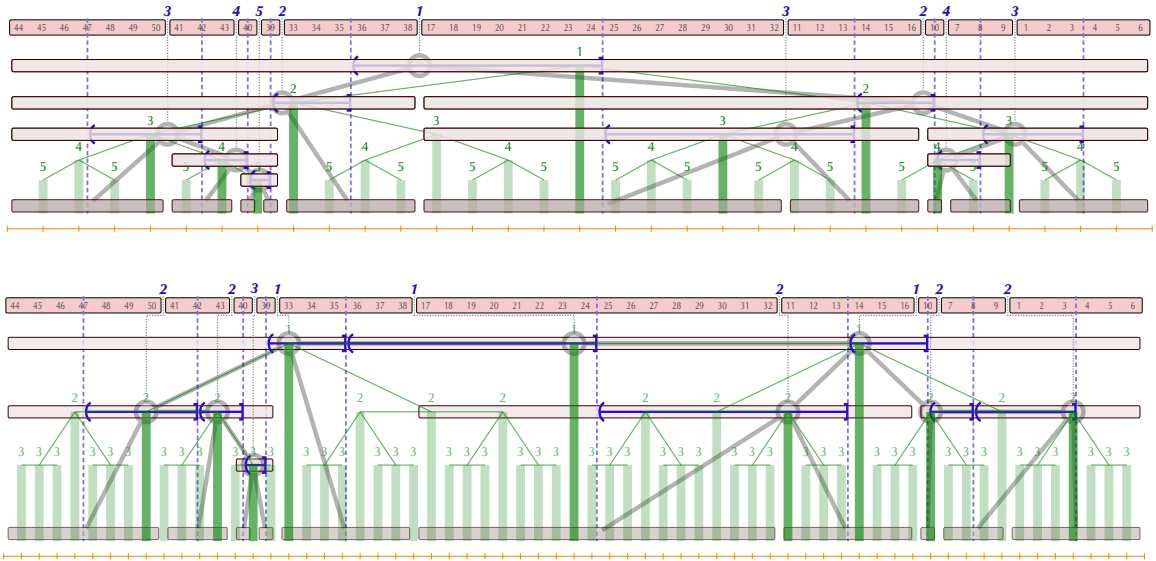
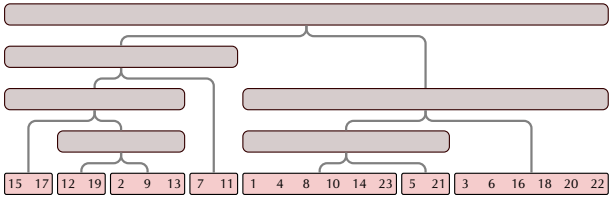


Mergesort meets Binary Search Trees



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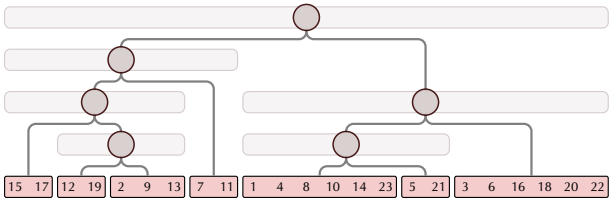

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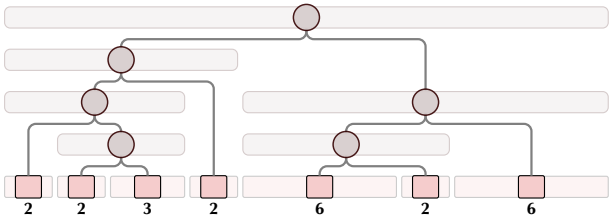

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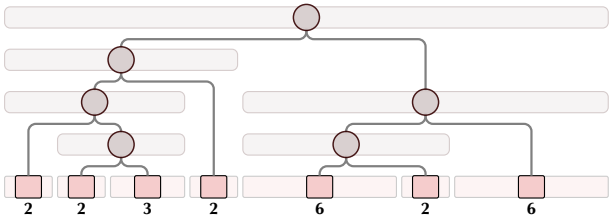

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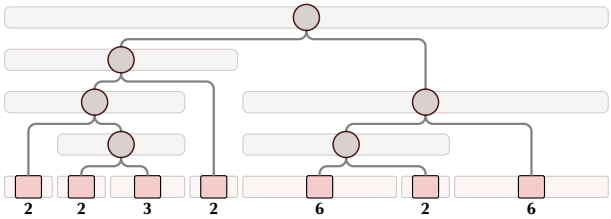

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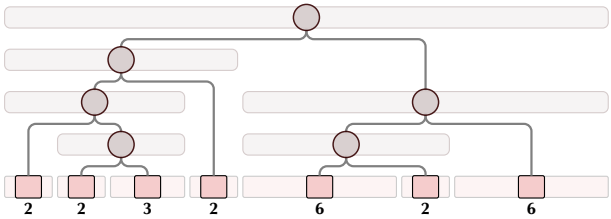

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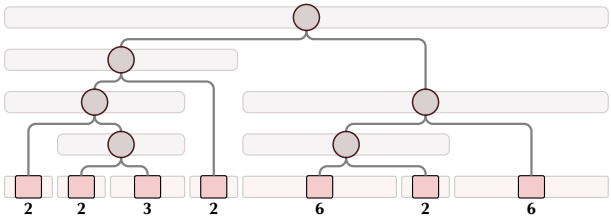

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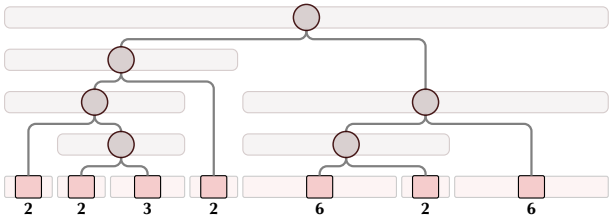

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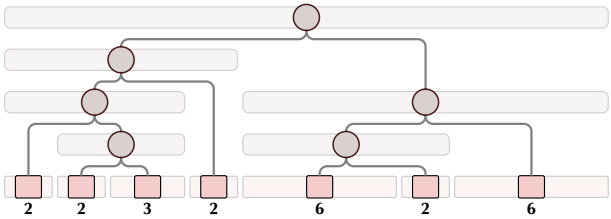

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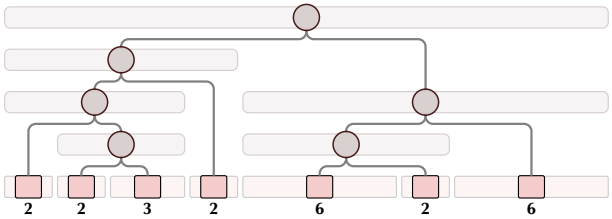

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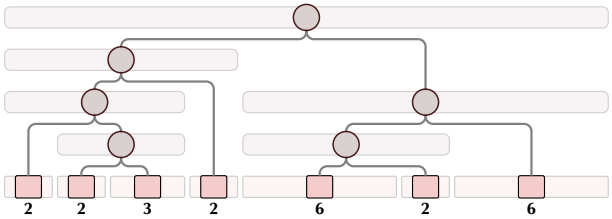

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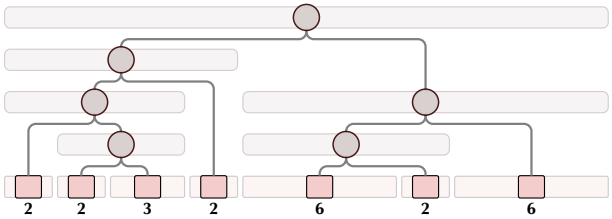

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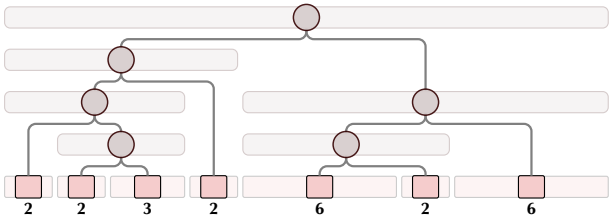

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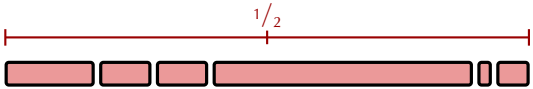

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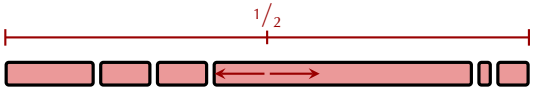

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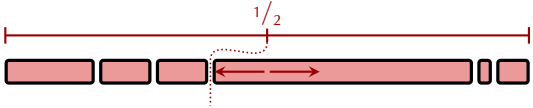

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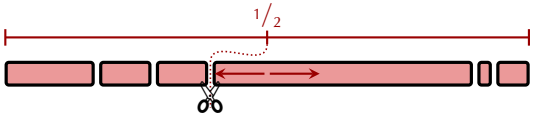

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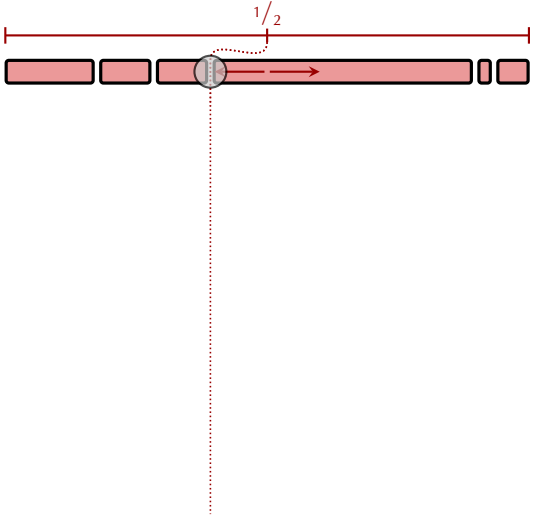

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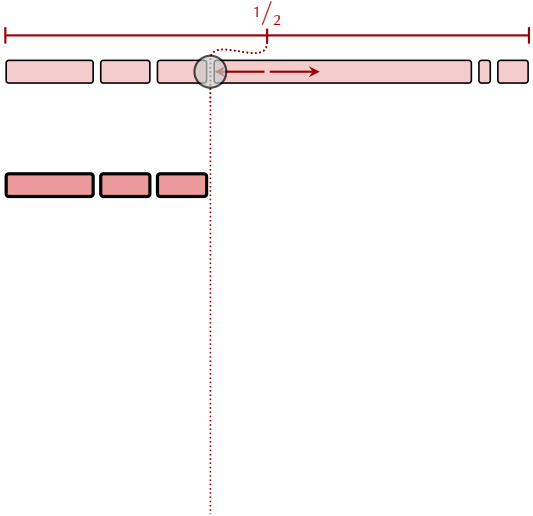

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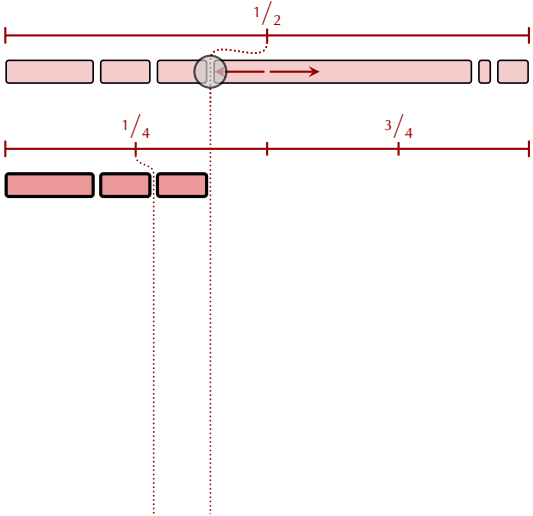

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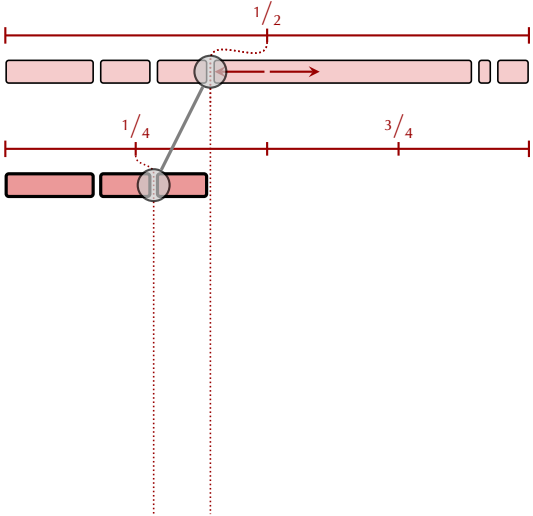

The Bisection Method

- Powersort is based on the **bisection method**

> **Mehlhorn**: *A best possible bound for the weighted path length of binary search trees,* SIAM J Comp **1977**

- **Intuition:**
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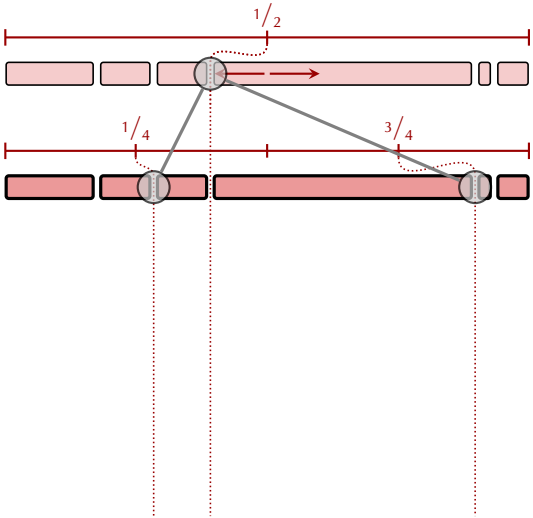

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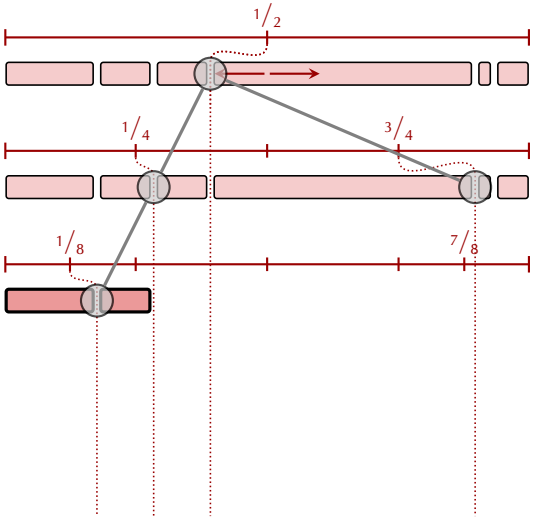

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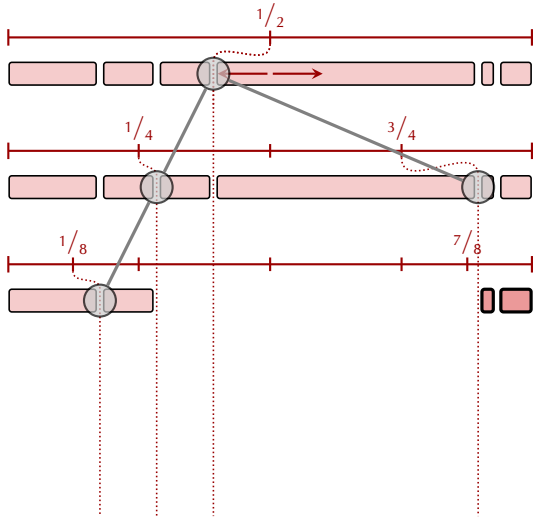

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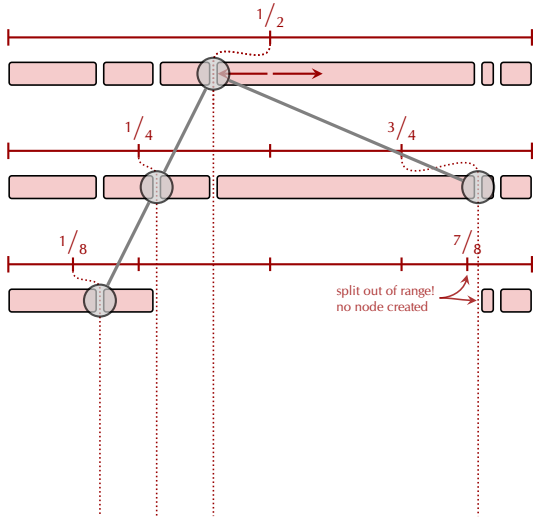

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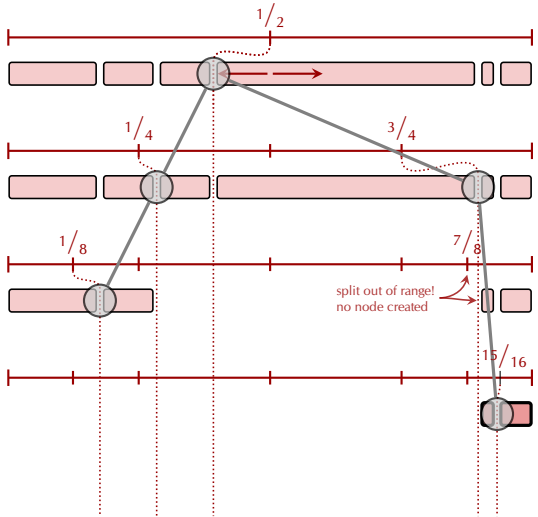

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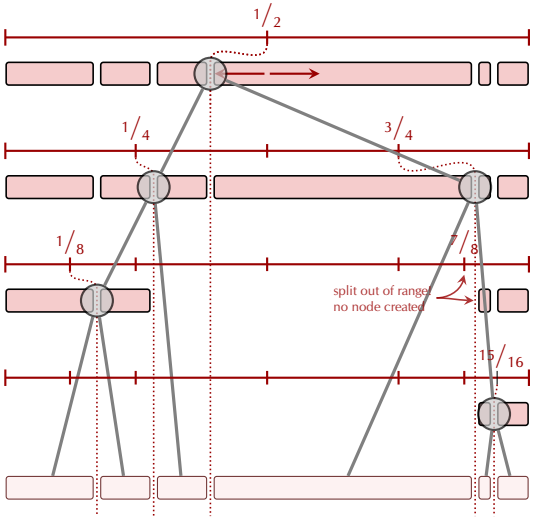

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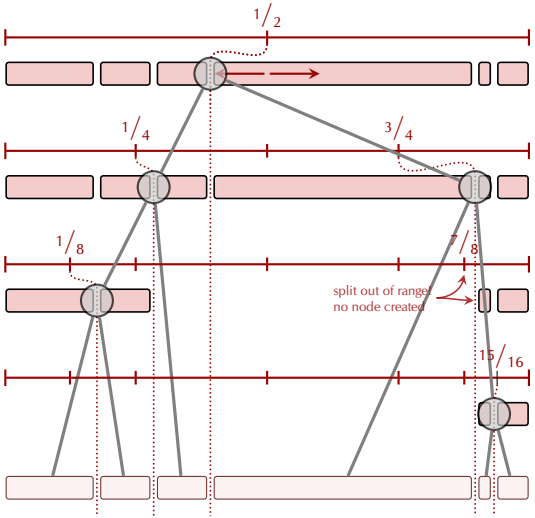
- *Up next:* Don't use recursion!



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