Analysis of Branch Misses in Quicksort

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based on joint work with **Conrado Martínez** and **Markus E. Nebel**



04 January 2015

Meeting on Analytic Algorithmics and Combinatorics

Instruction Pipelines

- Computers do **not** execute instructions fully **sequentially**
- Instead they use an *"assembly line"*
- **Example:**

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41          ;
42  i := i + 1
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- Pipeline stalls are costly

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⇝ *Pipeline stalls are **costly***

- **Pipeline:** each instruction broken in 4 stages
- **Pipeline:** simpler steps ⇝ shorter CPU cycles
- **Pipeline:** one instruction per cycle (ideal)
- **Pipeline:** ⇝ 4 instructions per cycle

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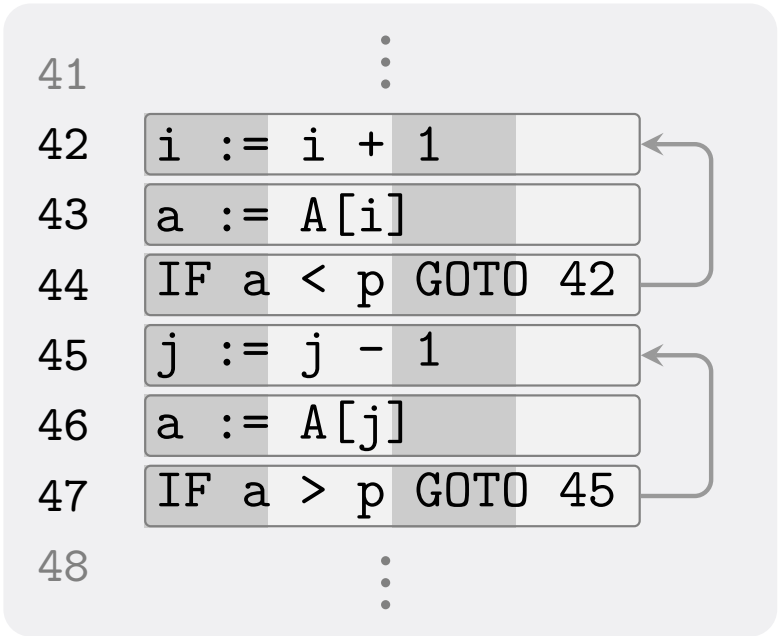



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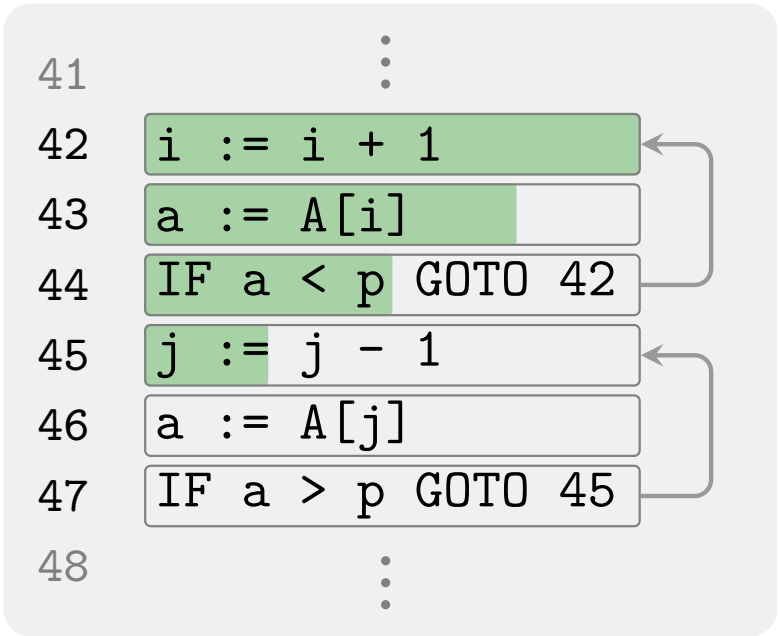



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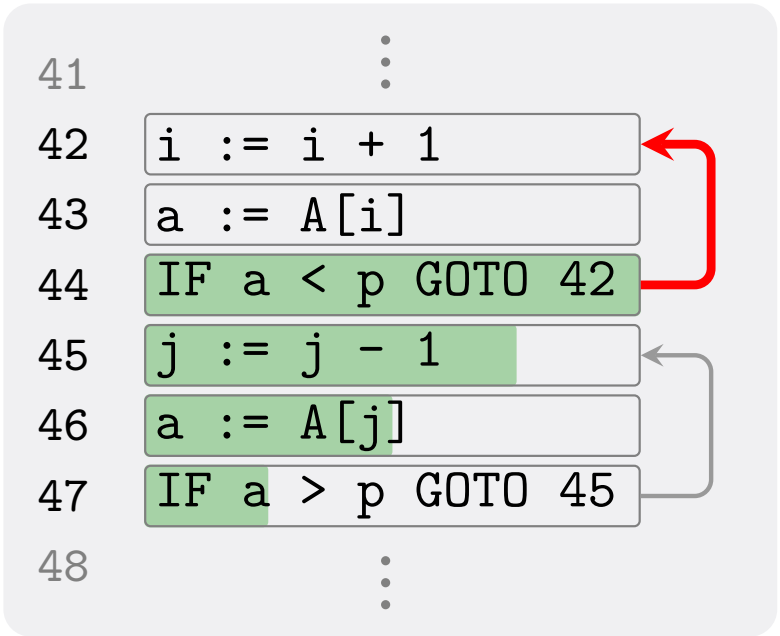

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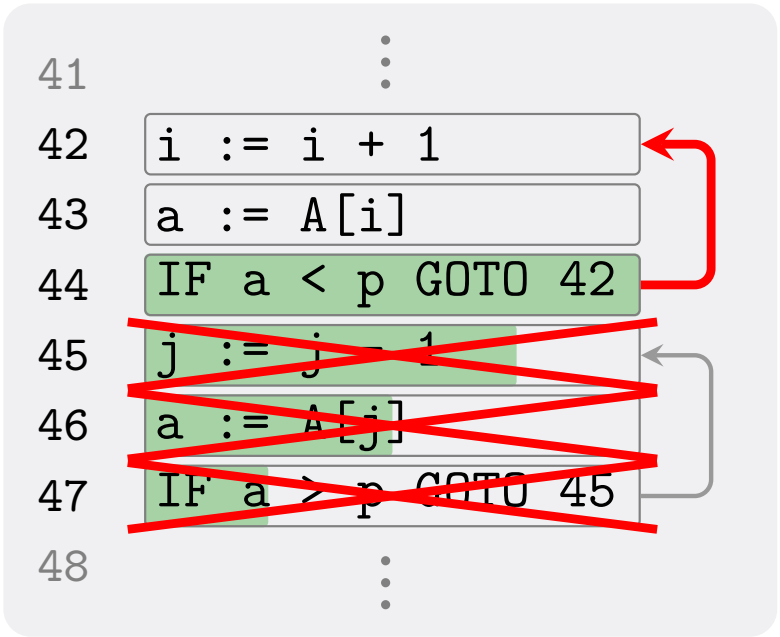

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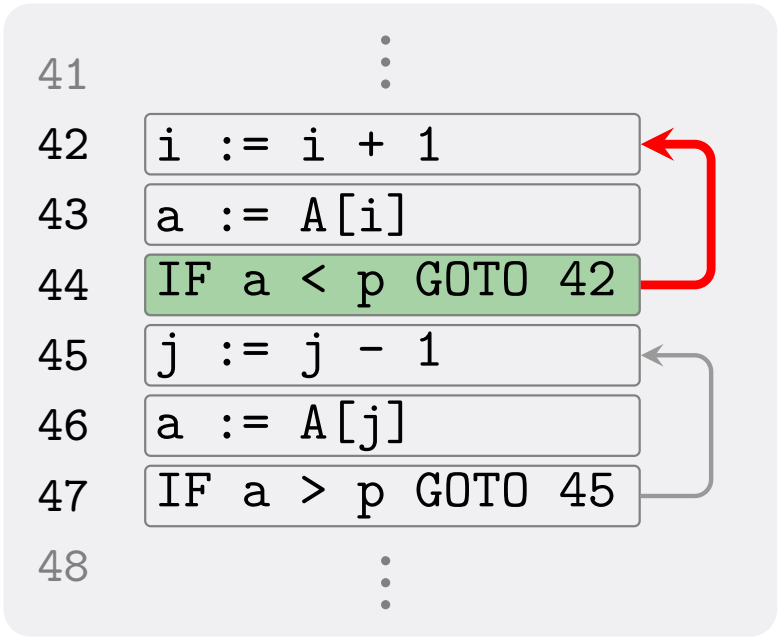

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- We could avoid stalls if we knew whether a branch will be taken or not
- in general not possible $\rightsquigarrow$ prediction with heuristics:
  - Predict same outcome as last time. (**1-bit predictor**)
  - Predict most frequent outcome with finite memory (**2-bit saturating counter**)
  - Flip prediction only after two consecutive errors (**2-bit flip-consecutive**)
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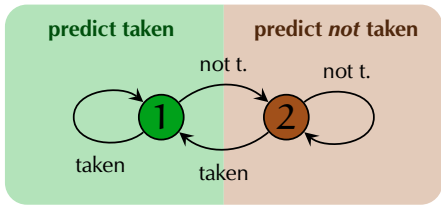

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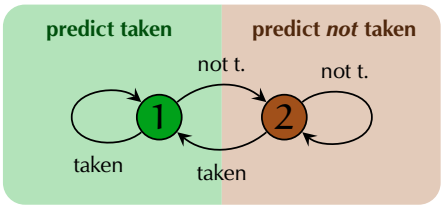

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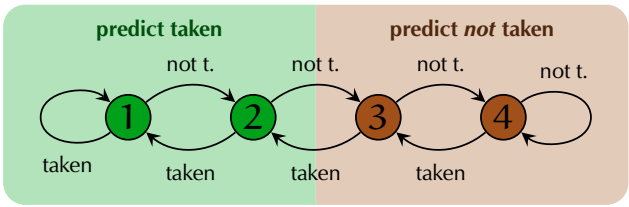

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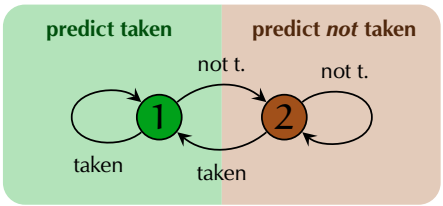

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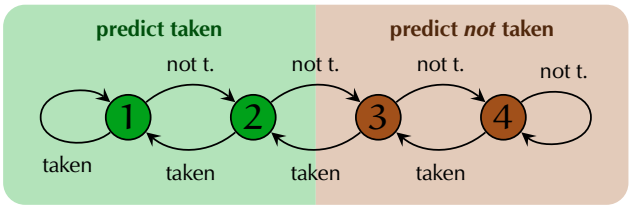

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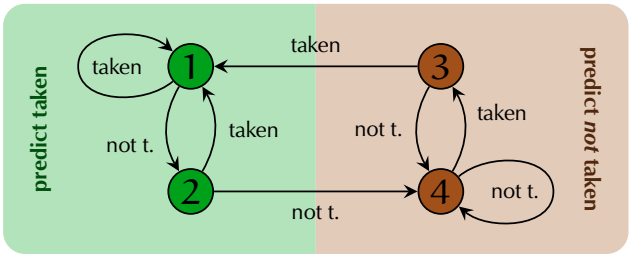

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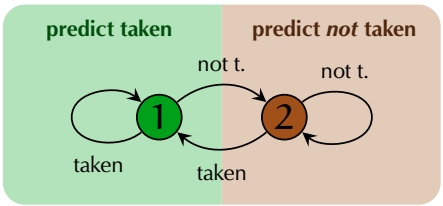

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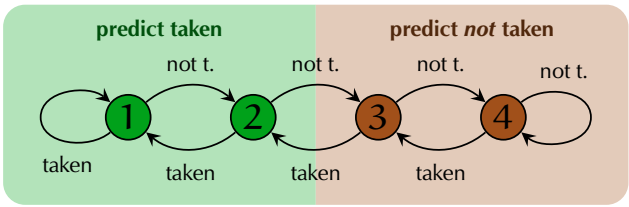

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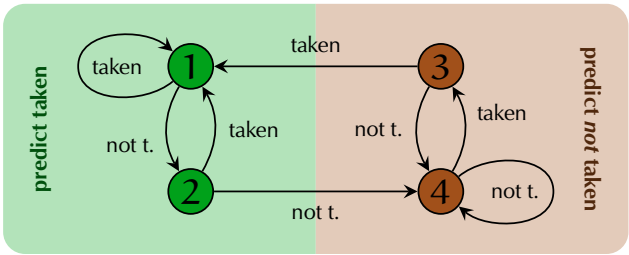

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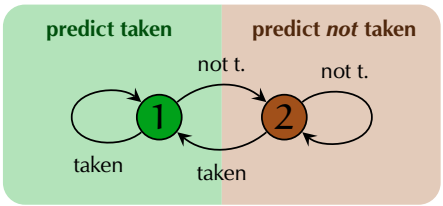

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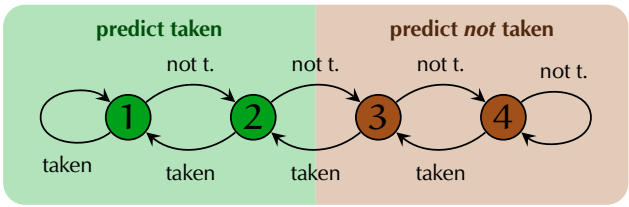

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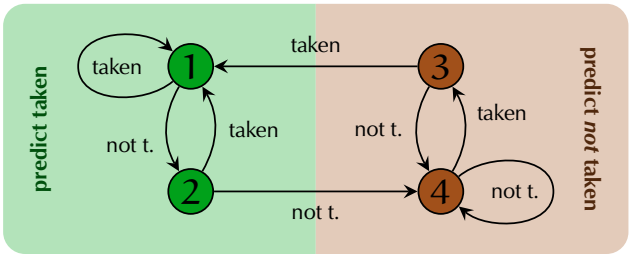

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- (Comparison-based) **sorting** is **different**!
  - Branch based on **comparison** result
  - Comparisons reduce **entropy** (uncertainty about input)

$\rightsquigarrow$ The **less** comparisons we use, the **less predictable** they become
  - for classic Quicksort: misprediction rate $\geqslant$ **25 %**
  - with *median-of-3*: $\geqslant$ **31.25 %**

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- **faster** than previously used *classic Quicksort (CQS)* in practice
- **traditional** cost measures do **not** explain this!

	CQS	YQS	Relative
Running Time	(from various experiments)		−10±2%
Comparisons	2	1.9	−5%
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Bytecode Instructions	18	21.7	+20.6%
MMIX oops υ	11	13.1	+19.1%
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- Since 2009, Java uses **YAROSLAVSKIY's *dual-pivot Quicksort (YQS)***
- **faster** than previously used *classic Quicksort (CQS)* in practice
- **traditional** cost measures do **not** explain this!

	CQS	YQS	Relative
Running Time	(from various experiments)		**−10±2%**
Comparisons	2	1.9	**−5%**
Swaps	$0.\overline{3}$	0.6	+80%
Bytecode Instructions	18	21.7	+20.6%
MMIX oops υ	11	13.1	+19.1%
MMIX mems μ	$2.\overline{6}$	2.8	+5%
scanned elements[1] *(≈ cache misses)*	2	1.6	**−20%**

$\cdot n \ln n + O(n)$, *average case results*

*What about **branch misses**? Can they **explain** YQS's success? . . . stay tuned.*

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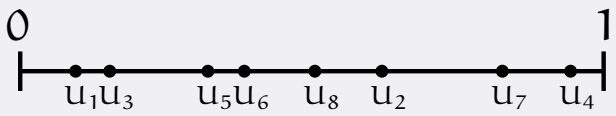
- n **i. i. d.** elements chosen **uniformly** in $[0, 1]$



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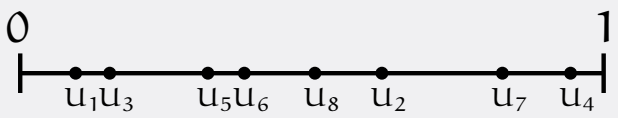

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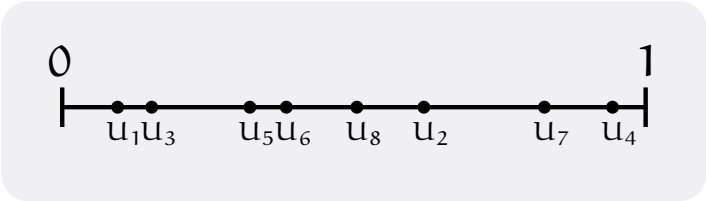

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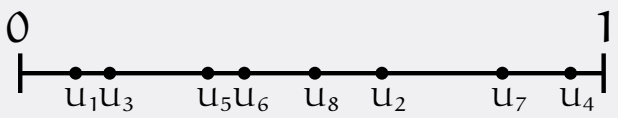

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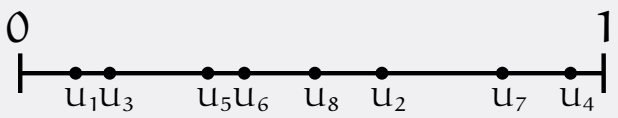

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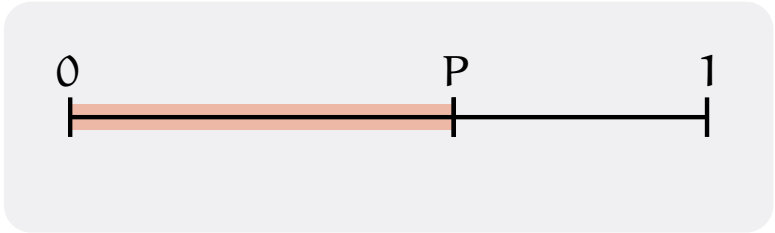

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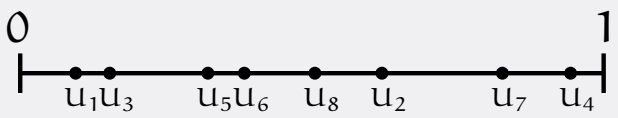

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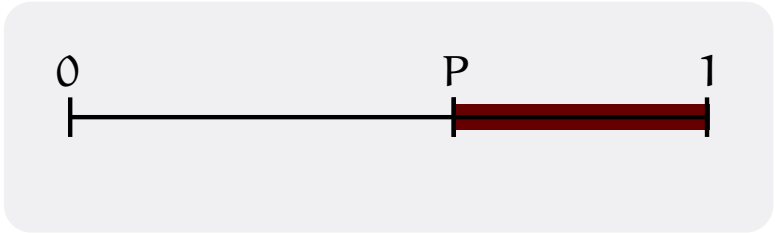

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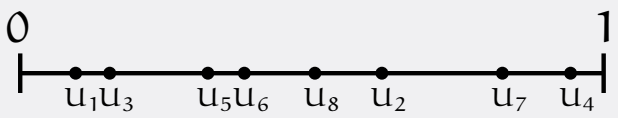

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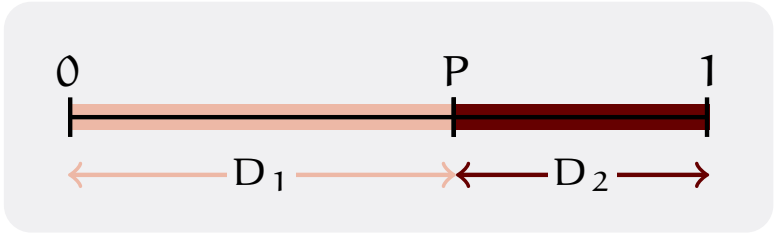

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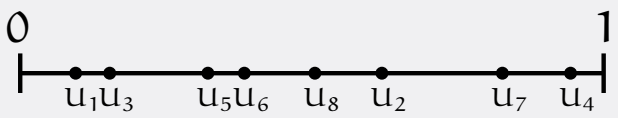

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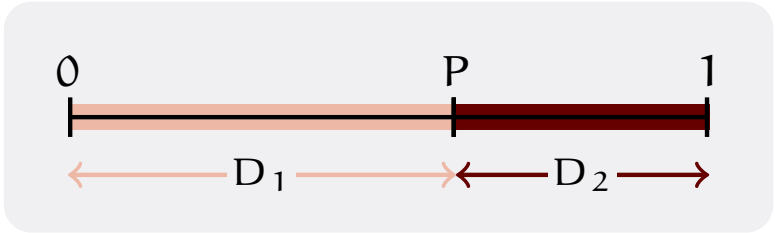

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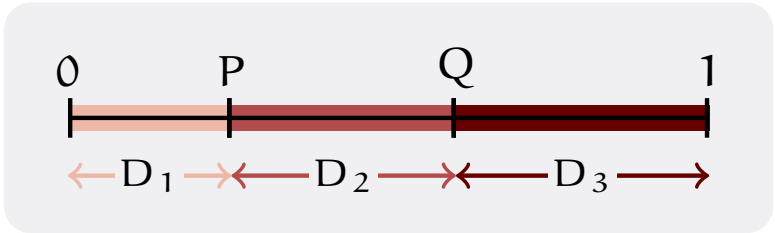

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These probabilities hold for all elements U, **independent** of all other elements!

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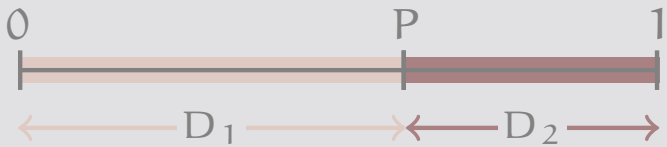

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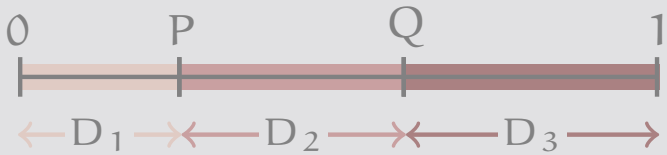

Branches in CQS

How many branches *in first partitioning step of CQS?*

- one **comparison branch** per element U:
  - $U[i] < P$ $\rightsquigarrow$ left partition
  - $U[i] \geq P$ $\rightsquigarrow$ right partition
- other branches (loop logic etc.)
  - easy to predict
  - only constant number of mispredictions

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Misprediction Rate for Memoryless Sources

Branches taken i. i. d. with probability p.

- **Information theoretic lower bound:**

  $$\frac{H(p)}{\log(\text{\#Branches})}$$

  $$H(p) = -p\log(p) - (1-p)\log(1-p)$$

- ⇝ **But: Actual predictors have very little memory!**

- **1-bit Predictor**

  - Always predict the same as last time
  - Counter-intuitive
  - Simple

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  - $\rightsquigarrow$ Miss rate: $f_{OPT}(p) = \min\{p, 1-p\}$
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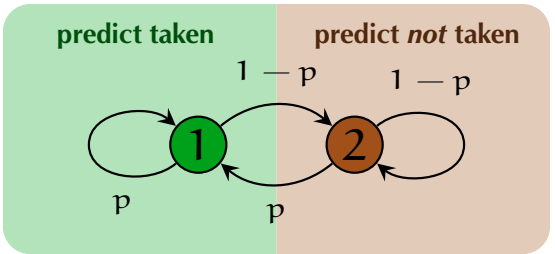

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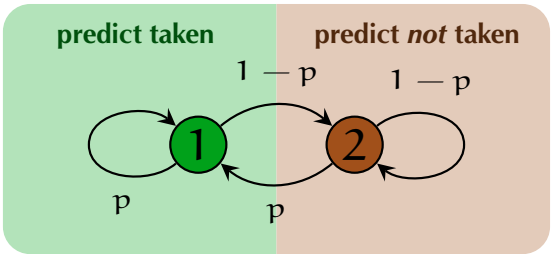

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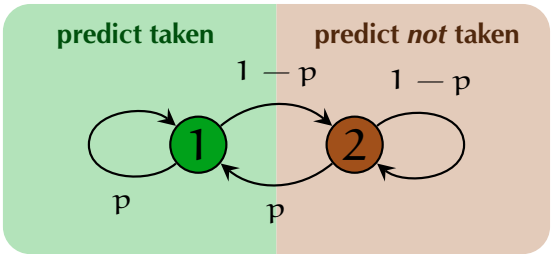

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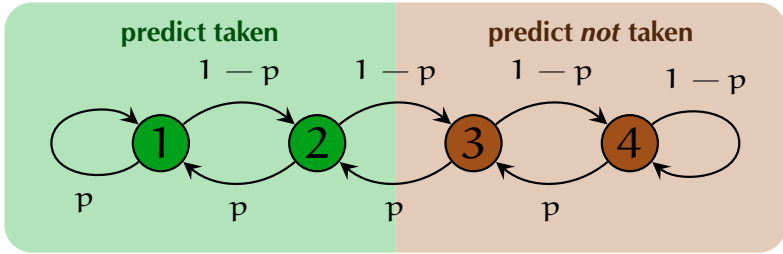

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  - different initial state distributions
  - 20 iterations for $p = \frac{2}{3}$

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expected miss rate over states in stationary distribution

- here: $f_{\text{2-bit-sc}}(p) = \dfrac{q}{1-2q}$ with $q = p(1-p)$.

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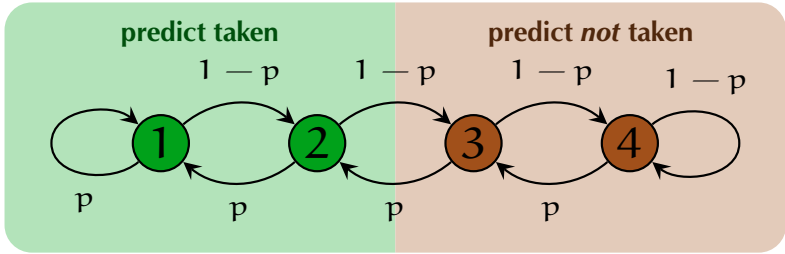

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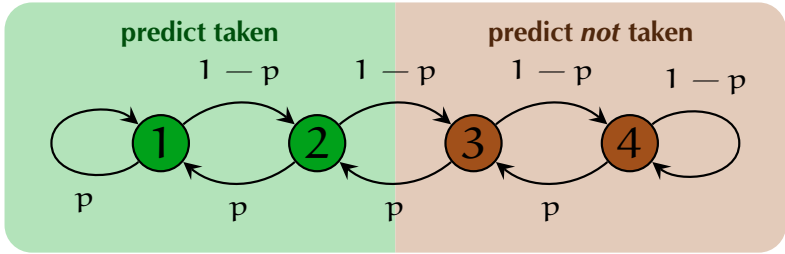

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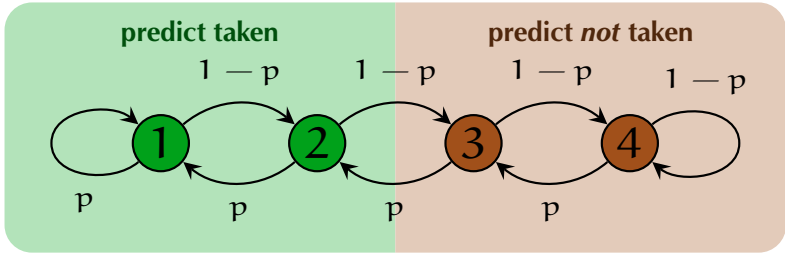

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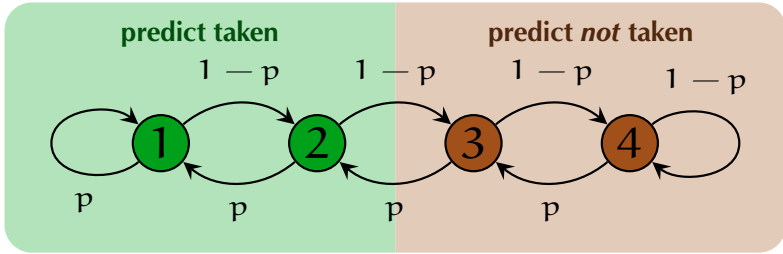

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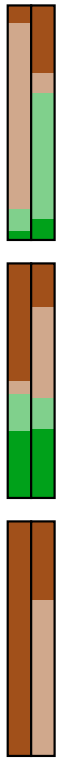

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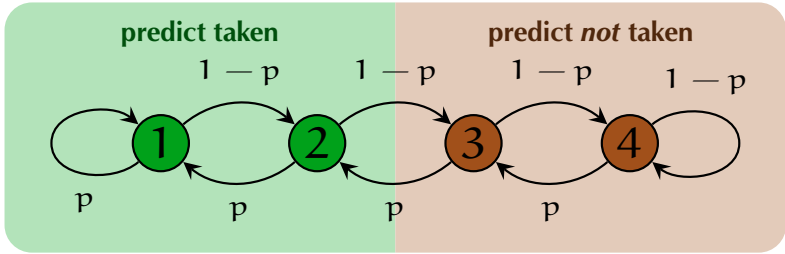

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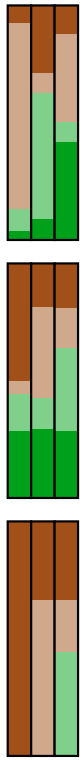

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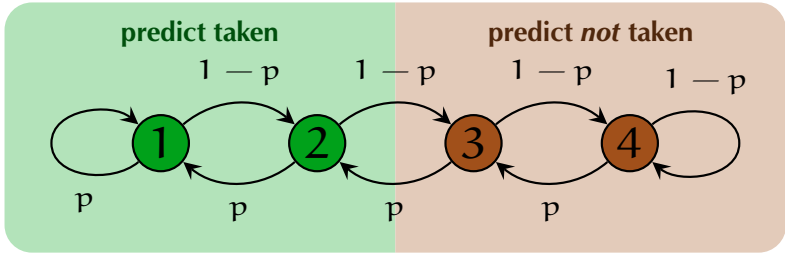

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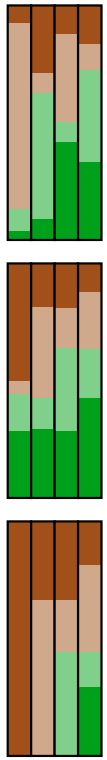

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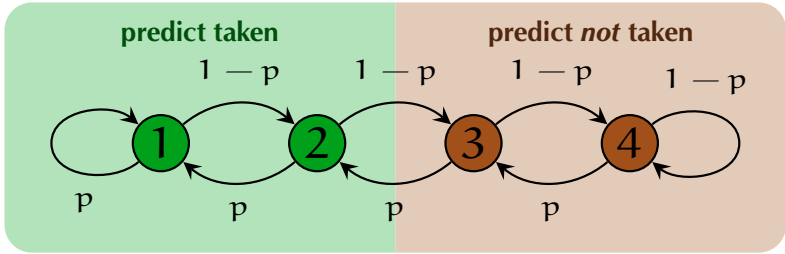

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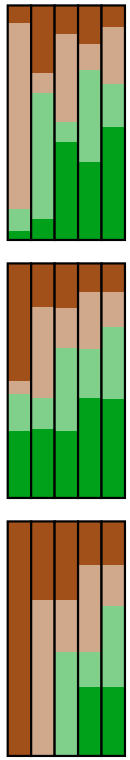

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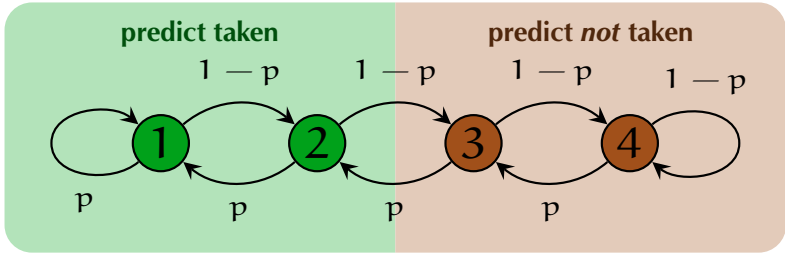

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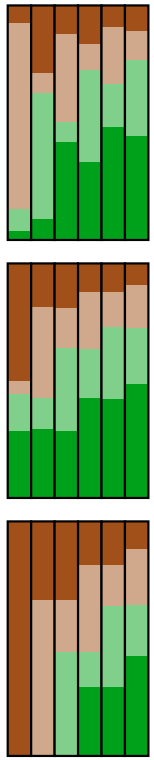

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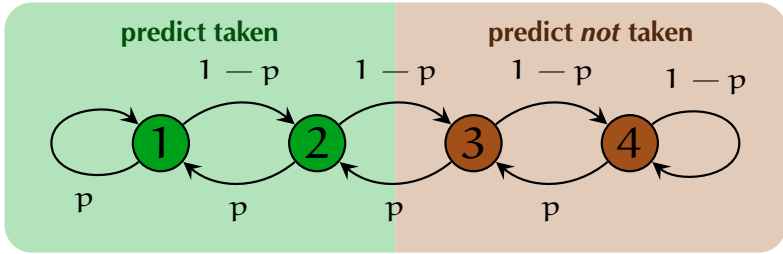

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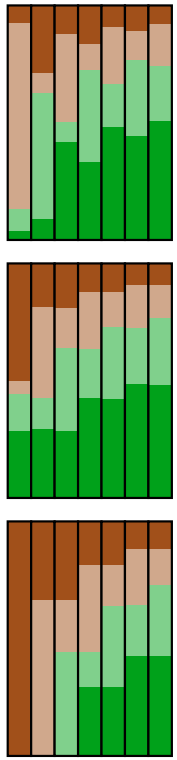

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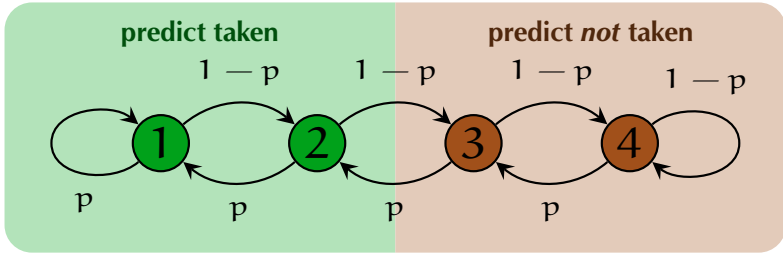

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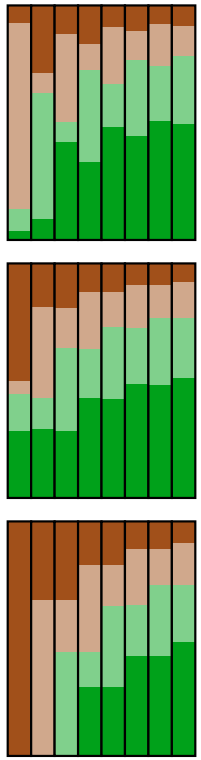

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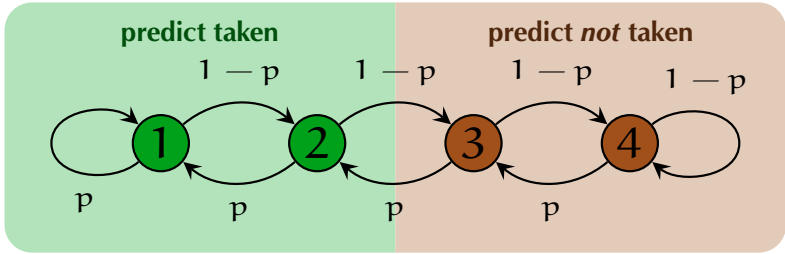

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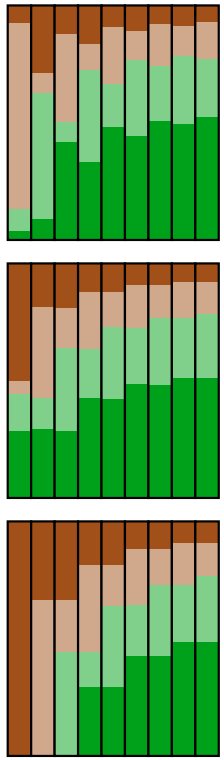

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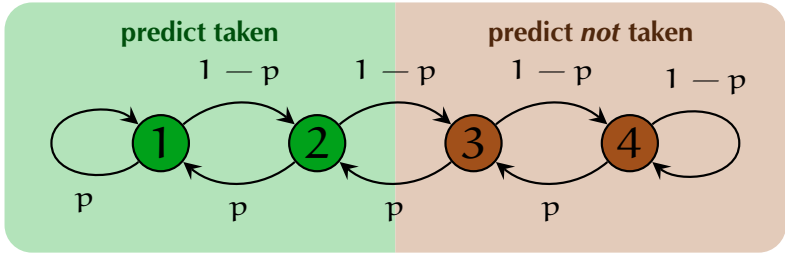

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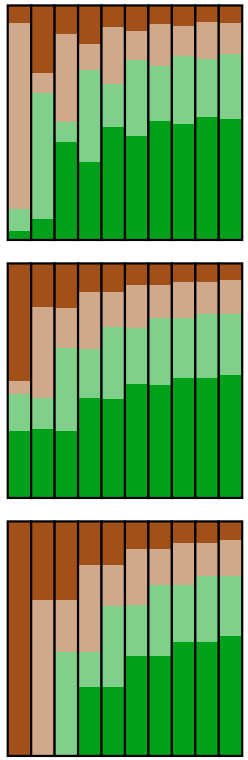

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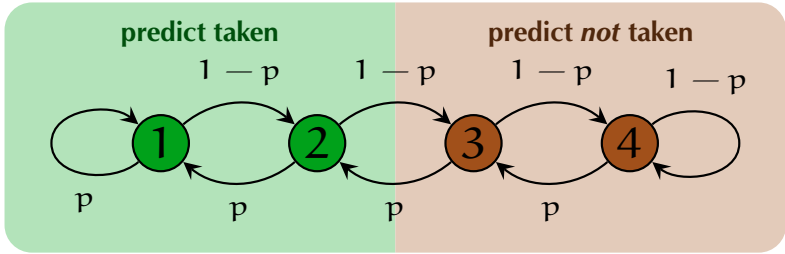

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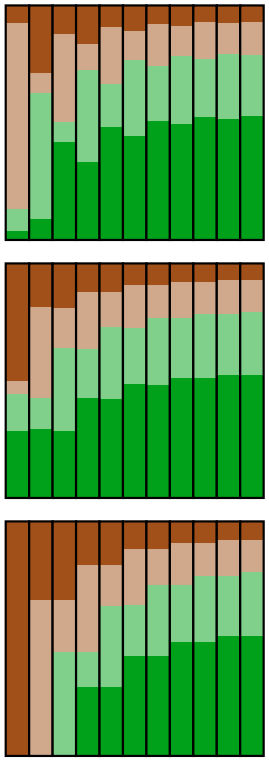

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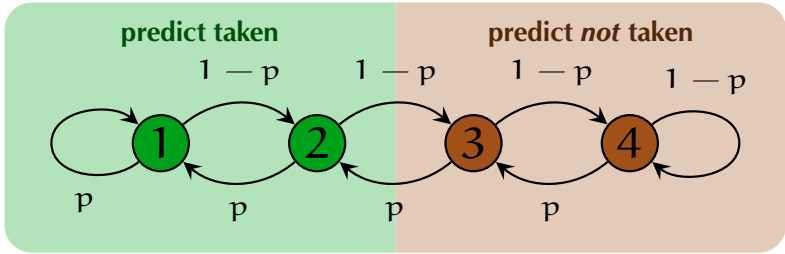

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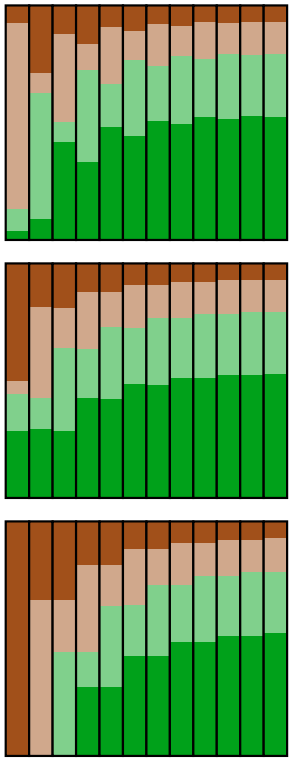

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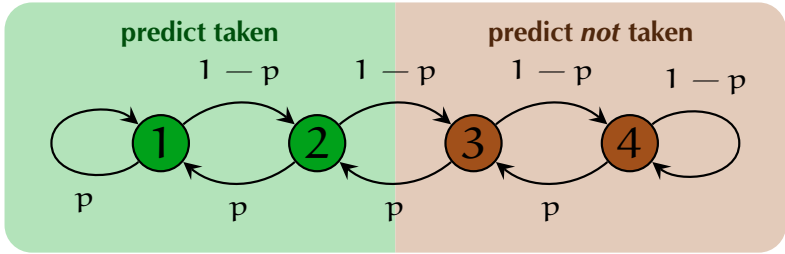

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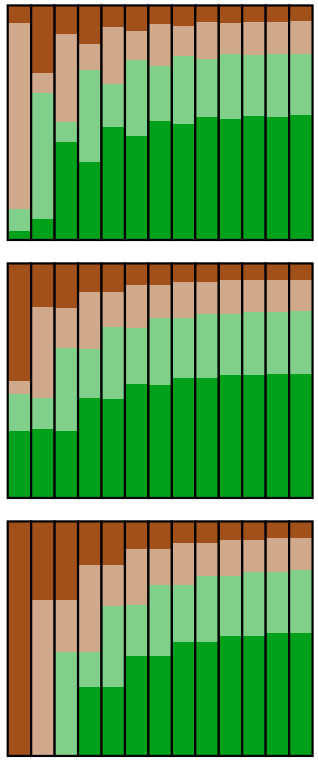

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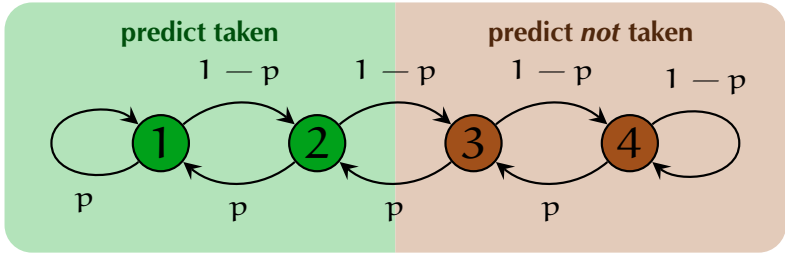

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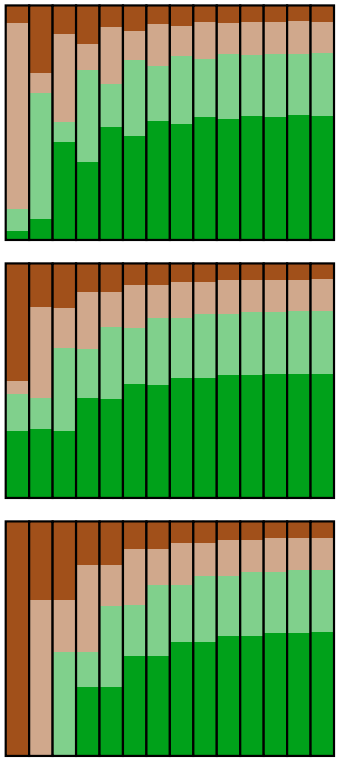

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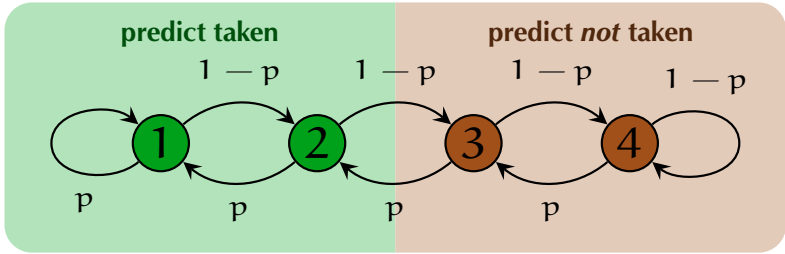

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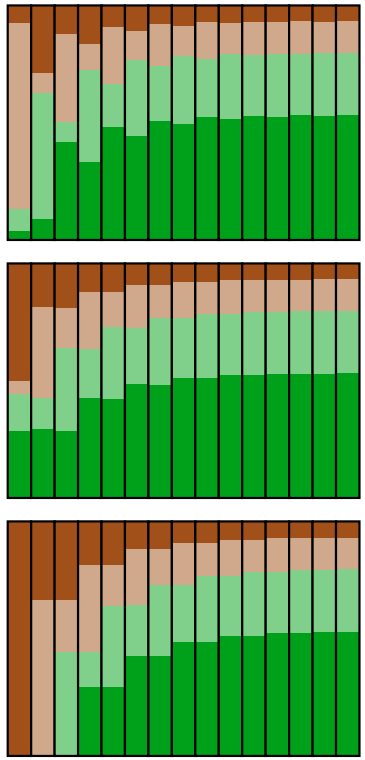

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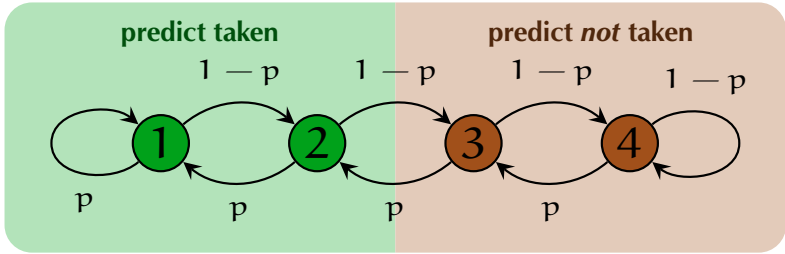

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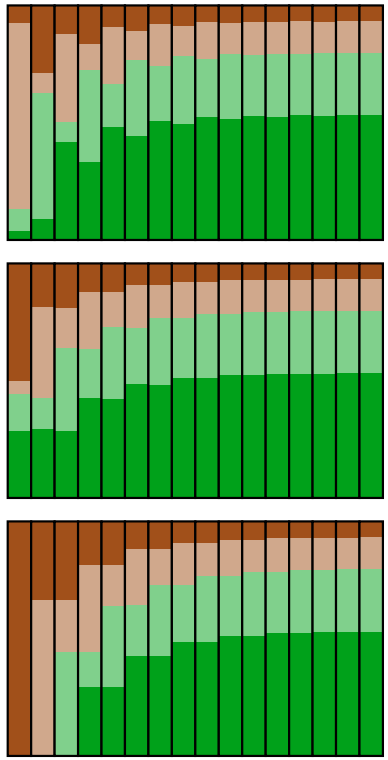

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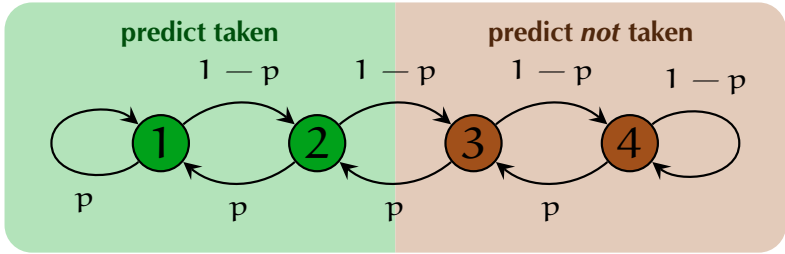

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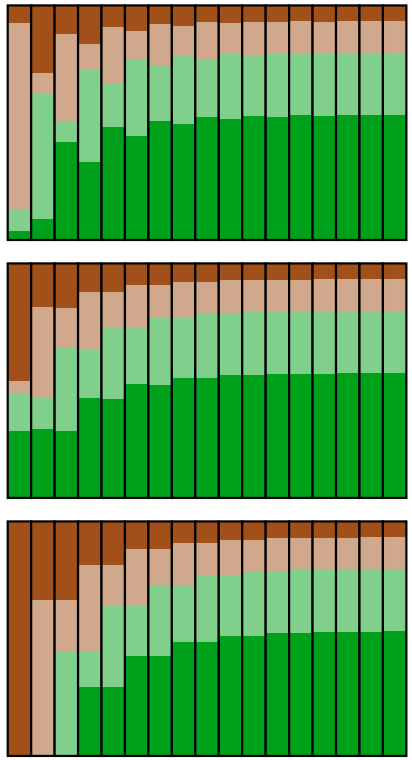

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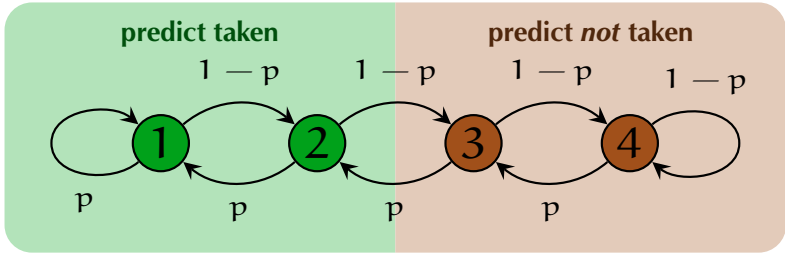

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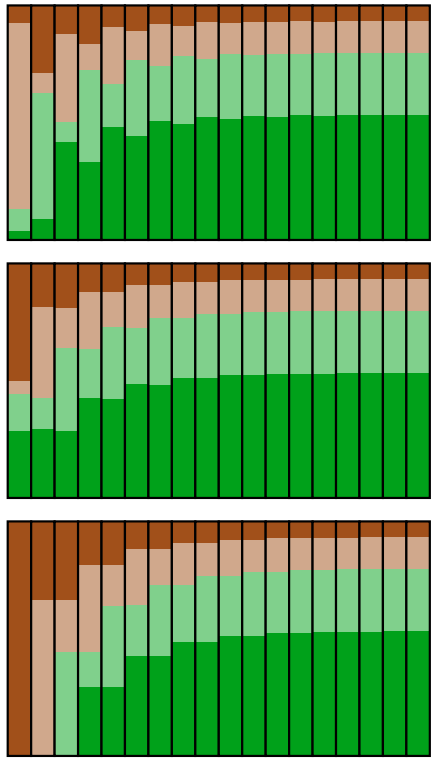

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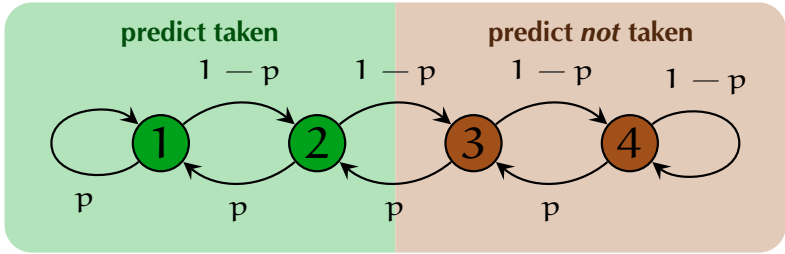

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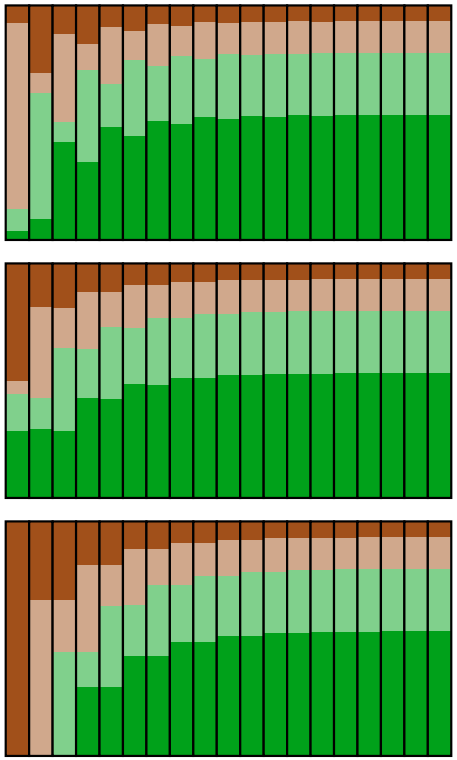

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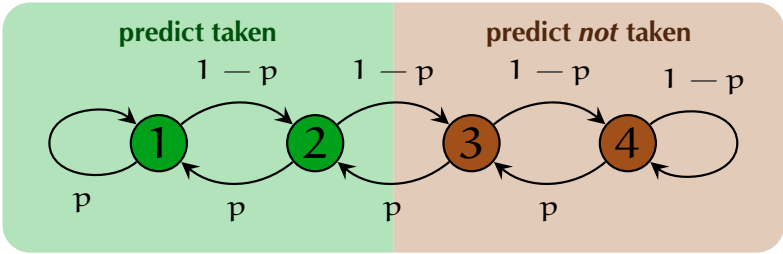

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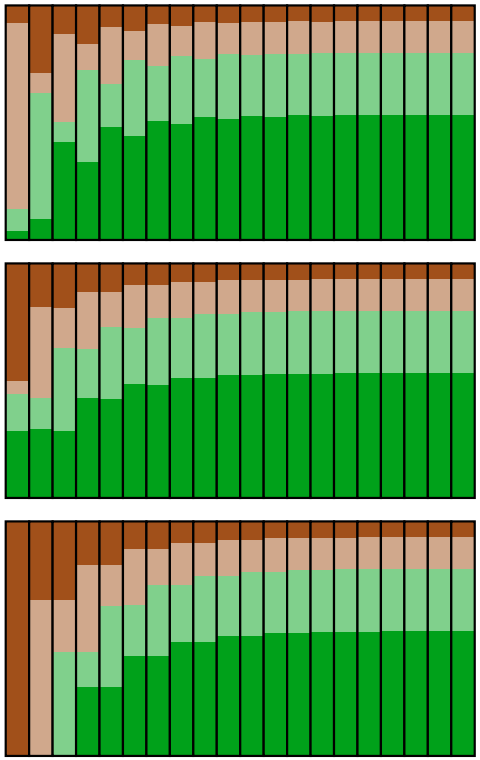

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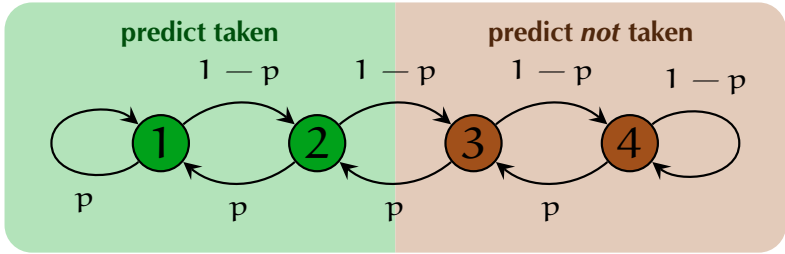

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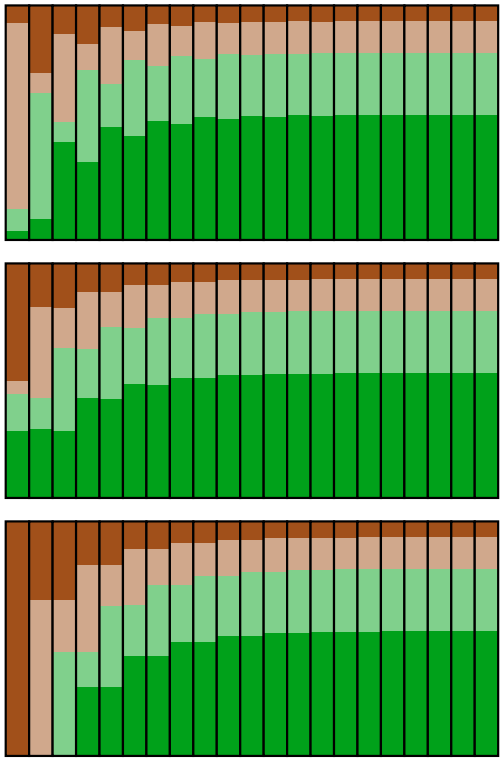

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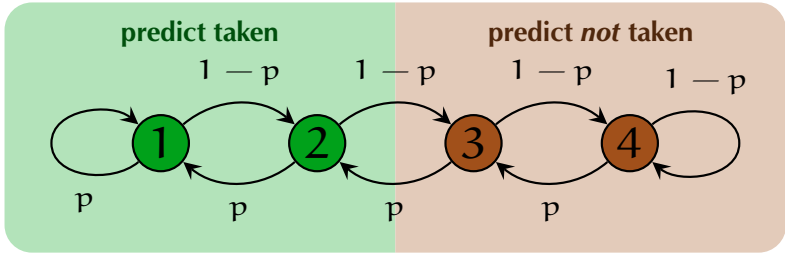

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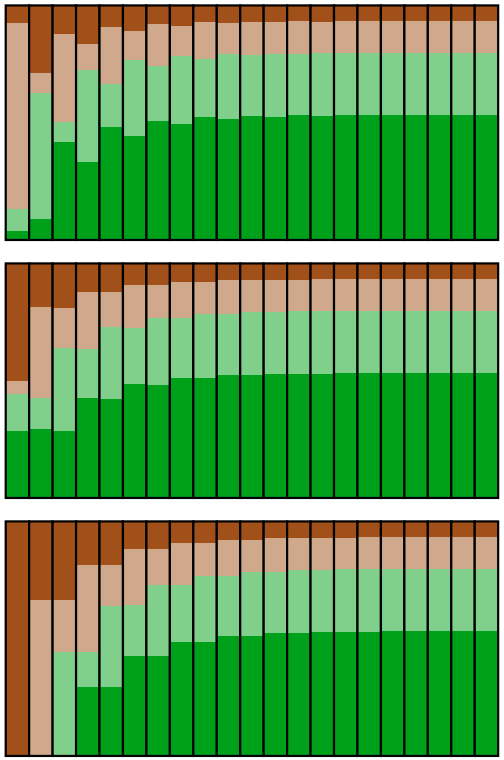

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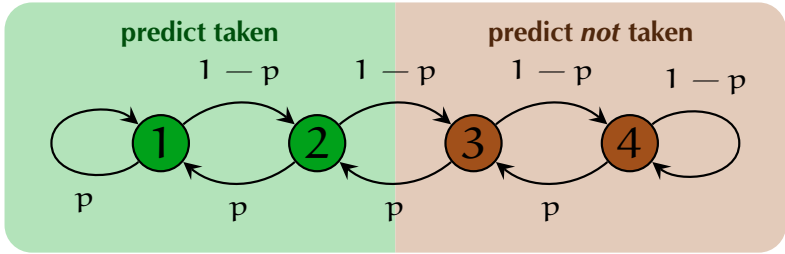

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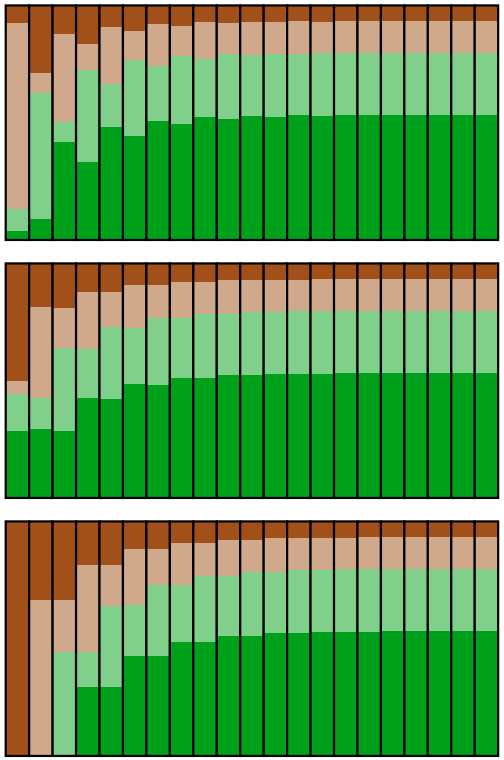

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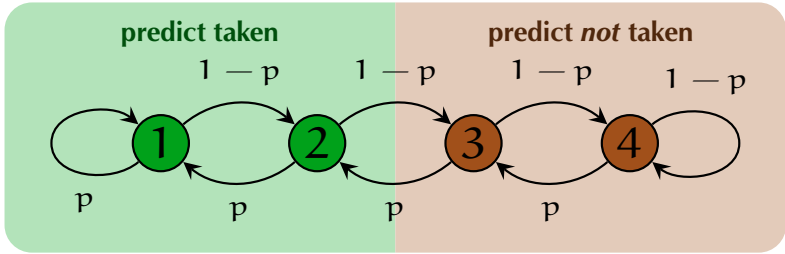

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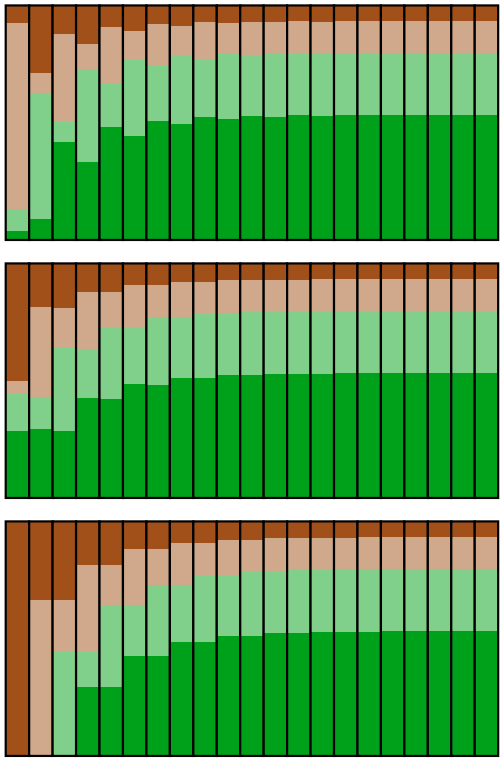

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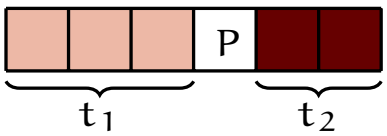

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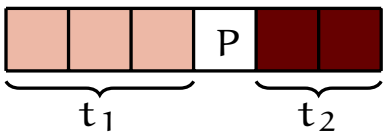

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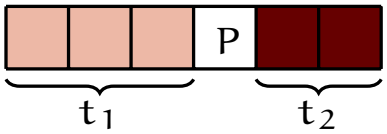

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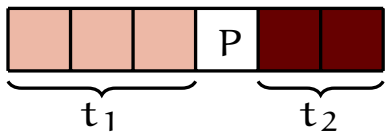

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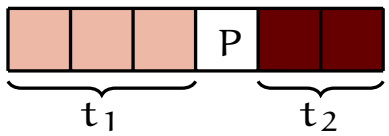

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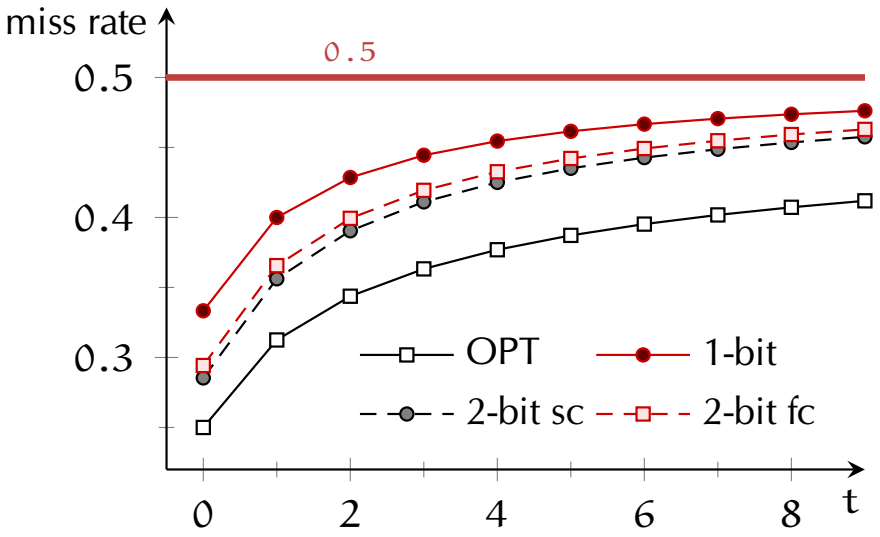

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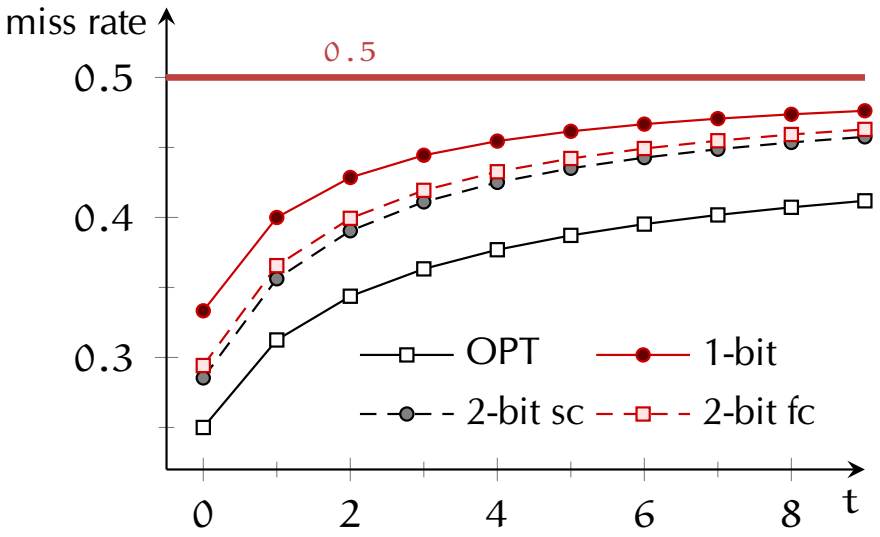

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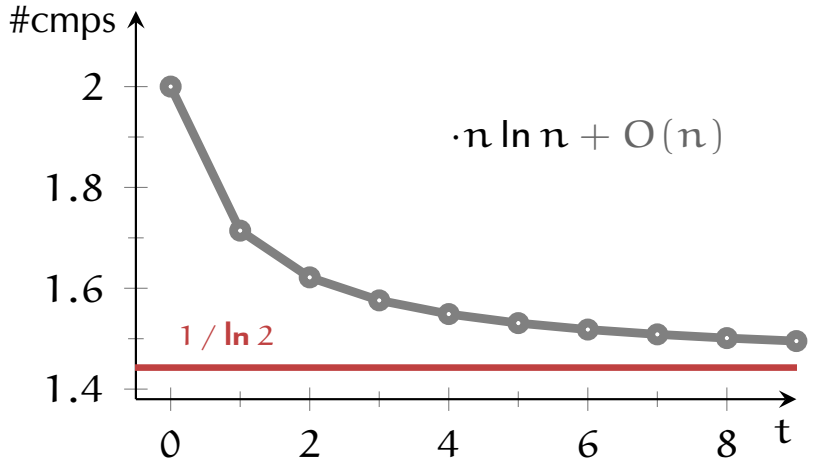

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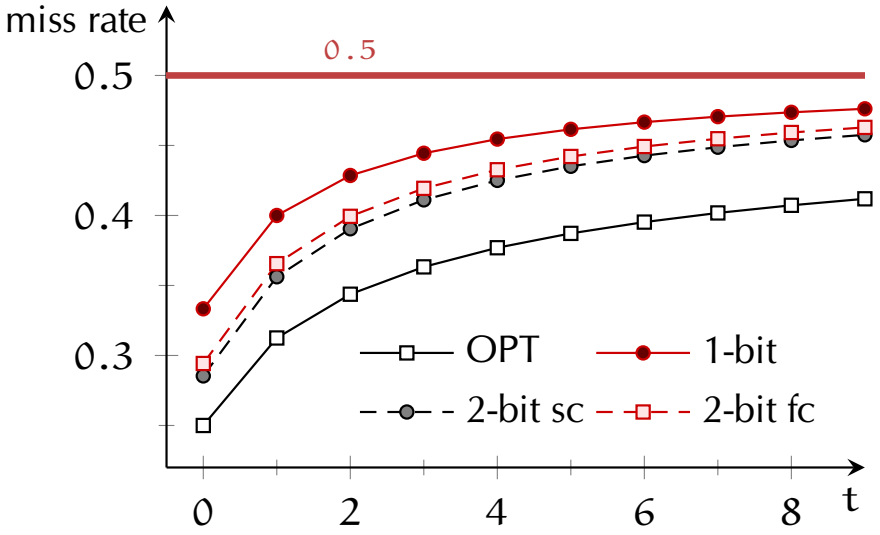

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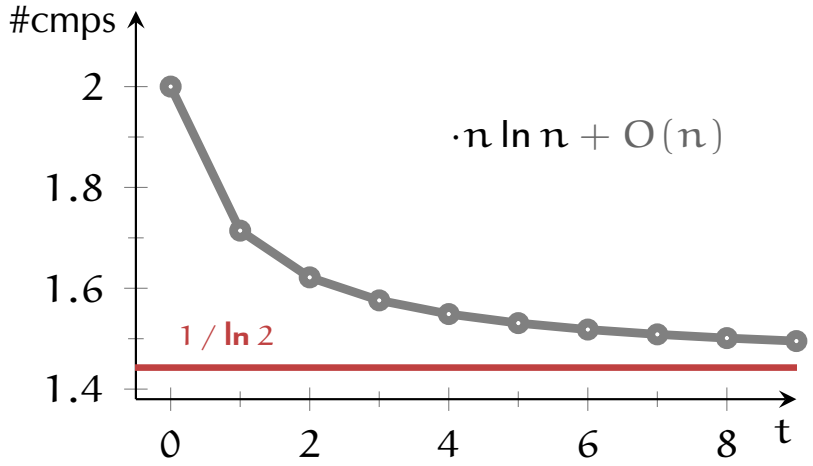

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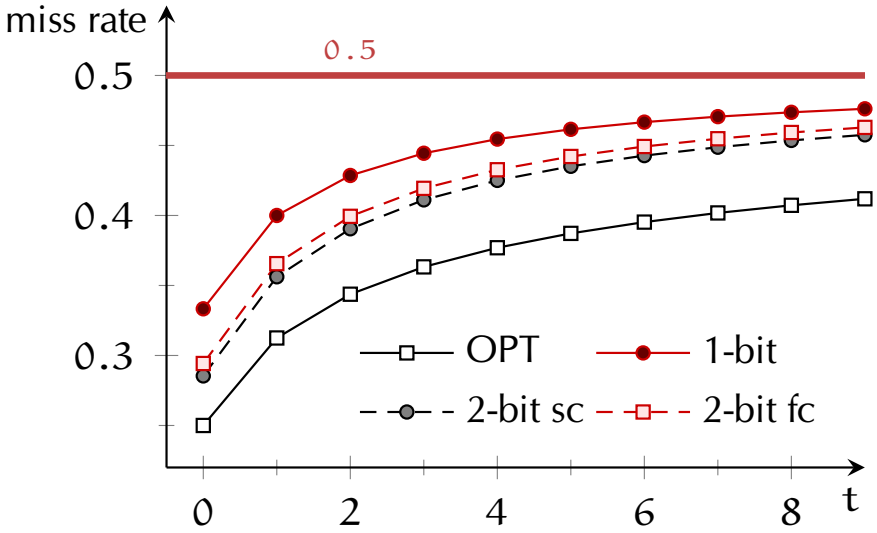

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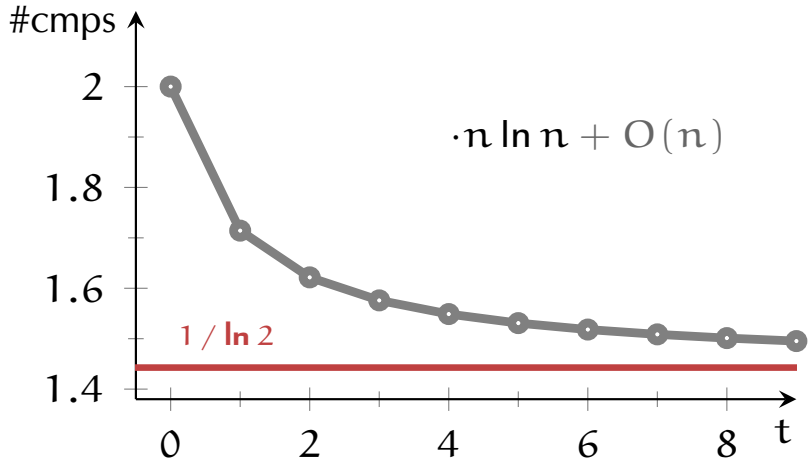

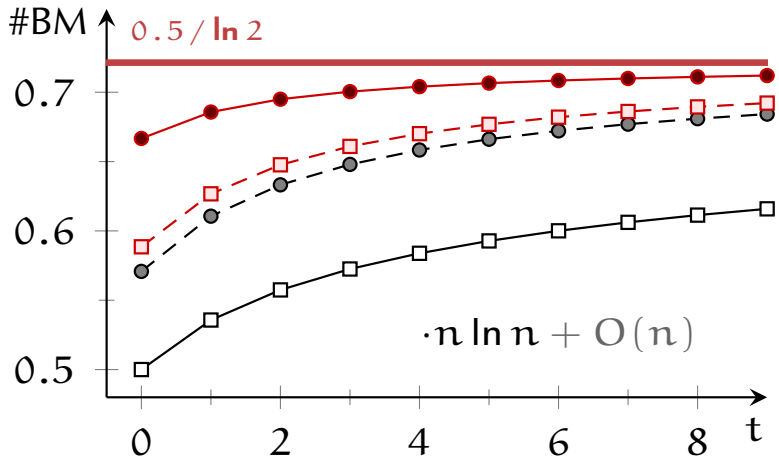

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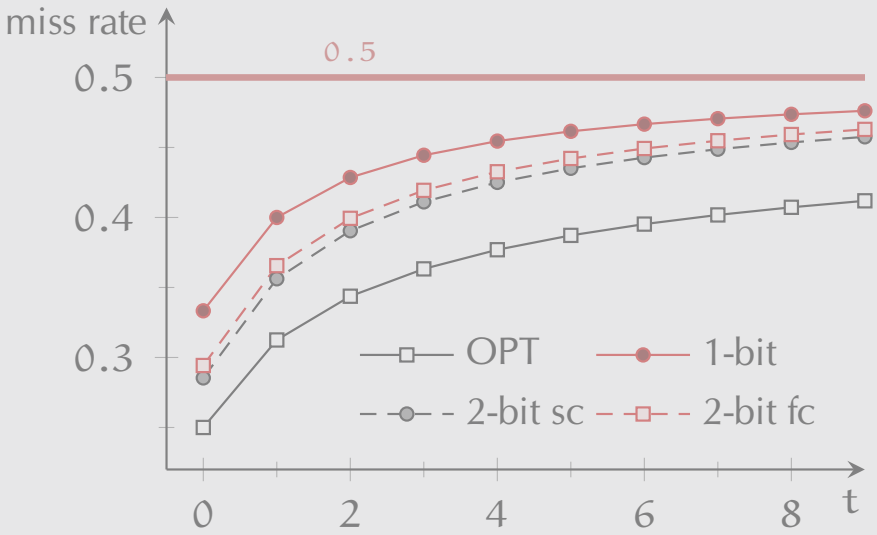
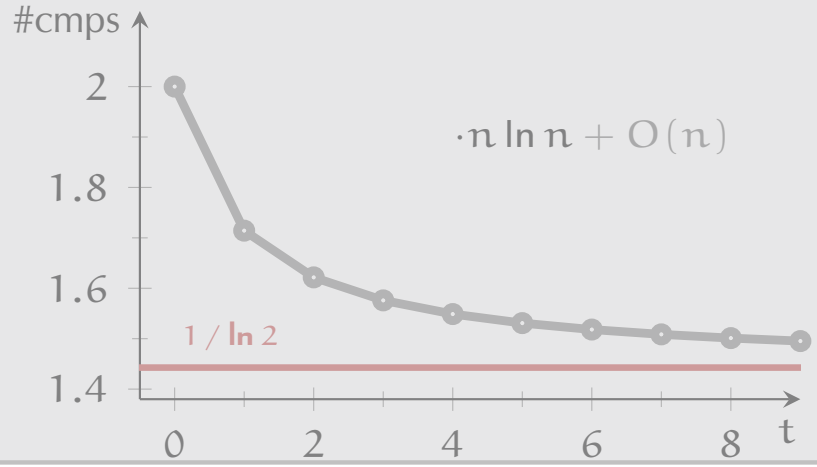


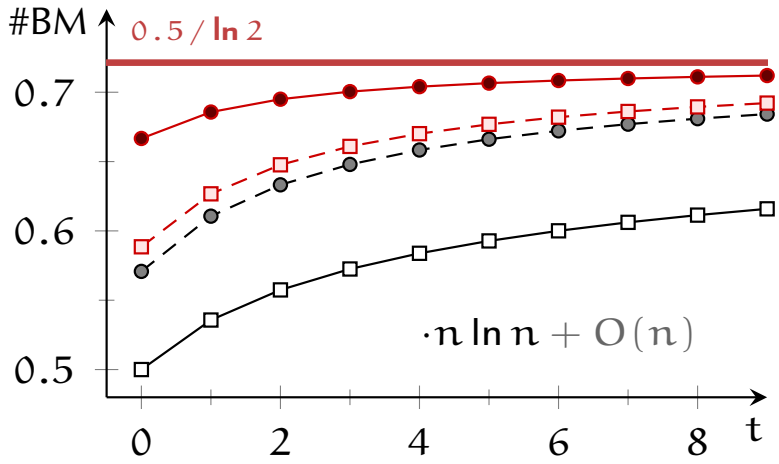
- miss rates quickly get bad (close to guessing!)
- **but:** less **comparisons** in total!



↝ Consider **number of branch misses**:

$$\#\mathrm{BM} = \#\mathit{comparisons} \cdot \text{miss rate}$$

↝ Overall BM still **grows** with t.



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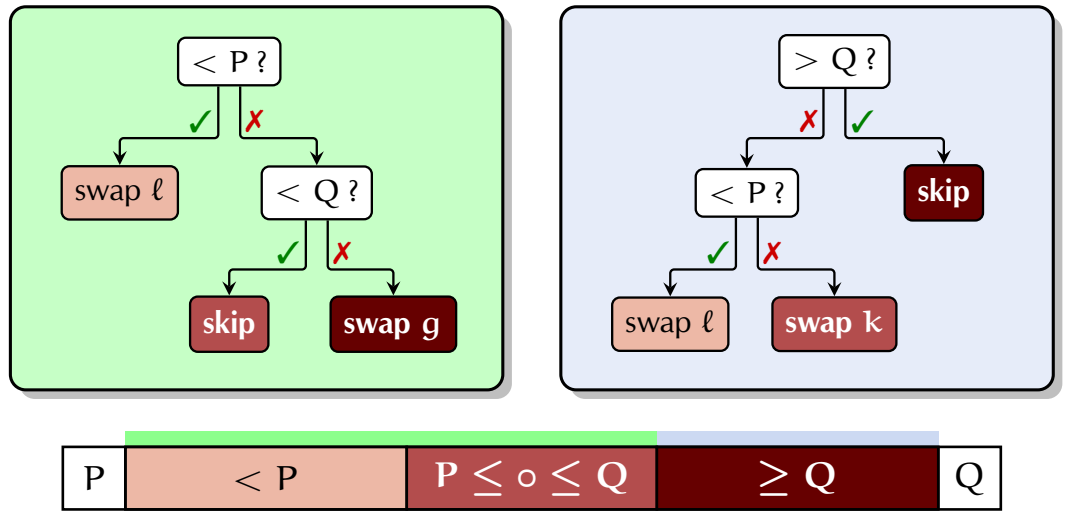

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  - executed $(\mathrm{D}_1 + \mathrm{D}_2)n + \mathrm{O}(1)$ times. (in expectation, conditional on $\mathbf{D}$)
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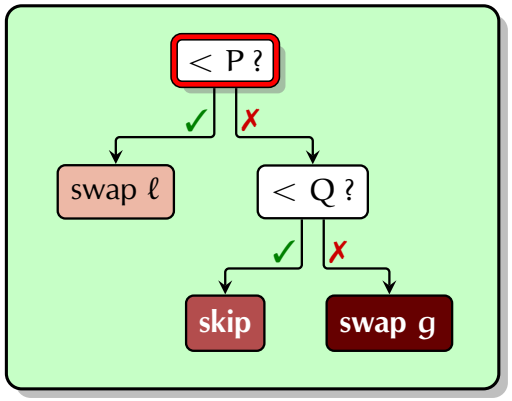

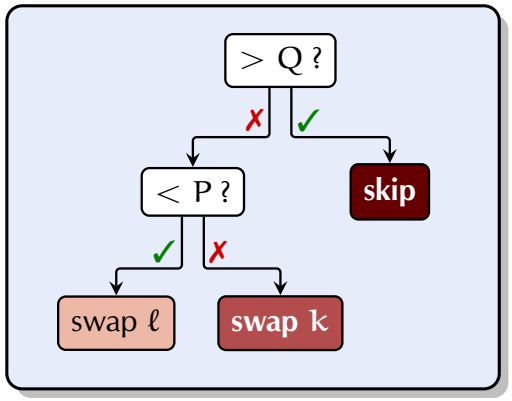

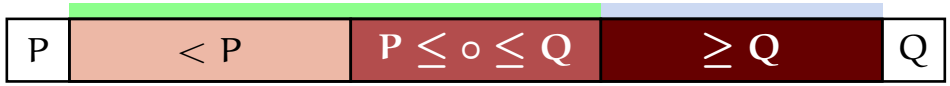

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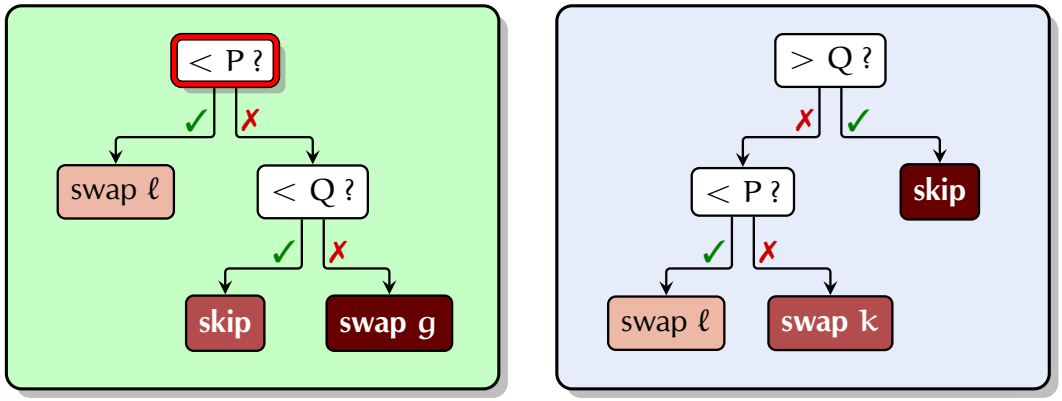

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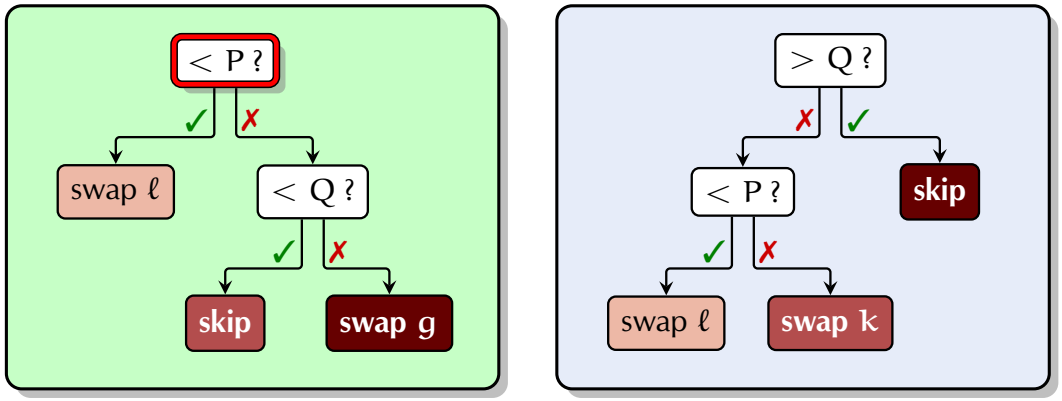

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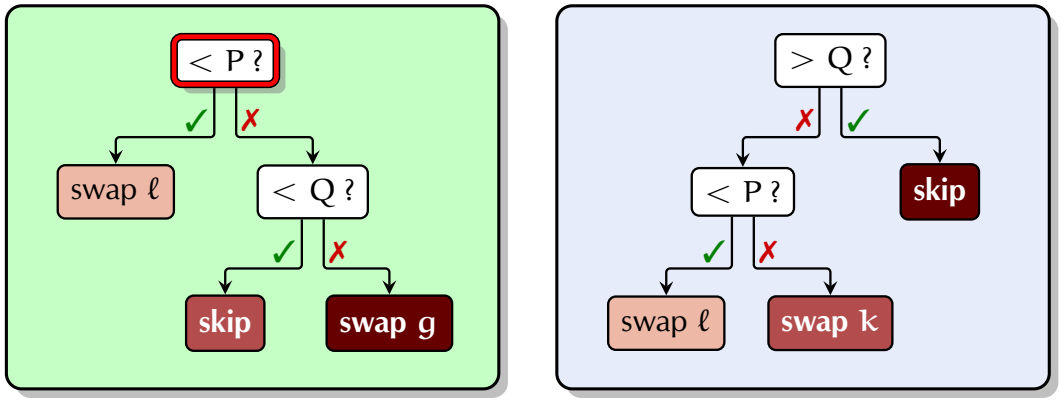

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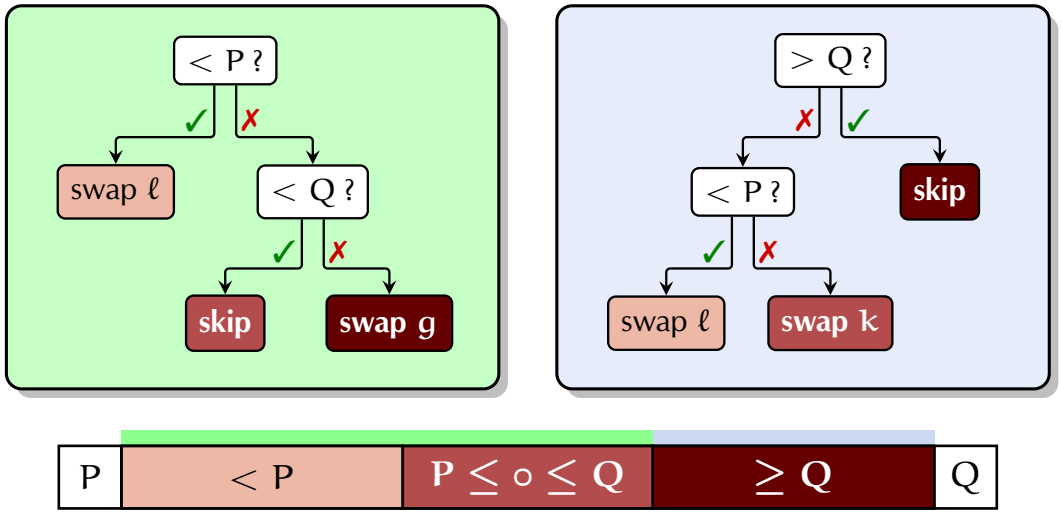

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  - lower bounds on branch miss rates

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