Dual-Pivot Quicksort and Beyond

Analysis of Multiway Partitioning and Its Practical Potential

Sebastian Wild

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Vortrag zur wissenschaftlichen Aussprache
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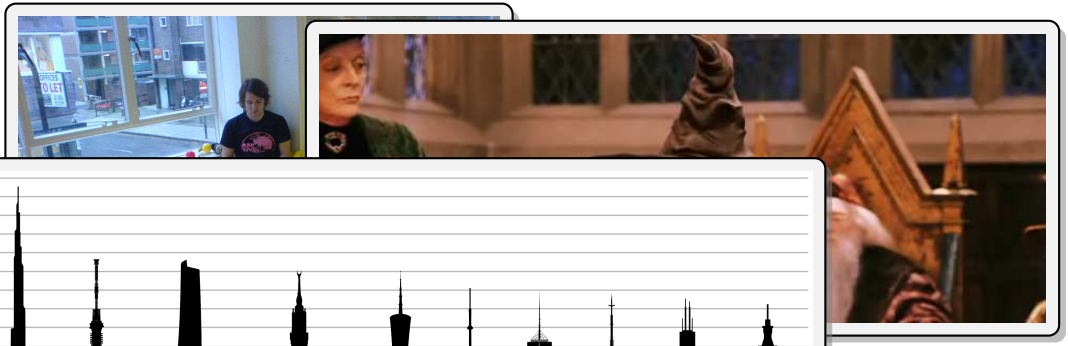

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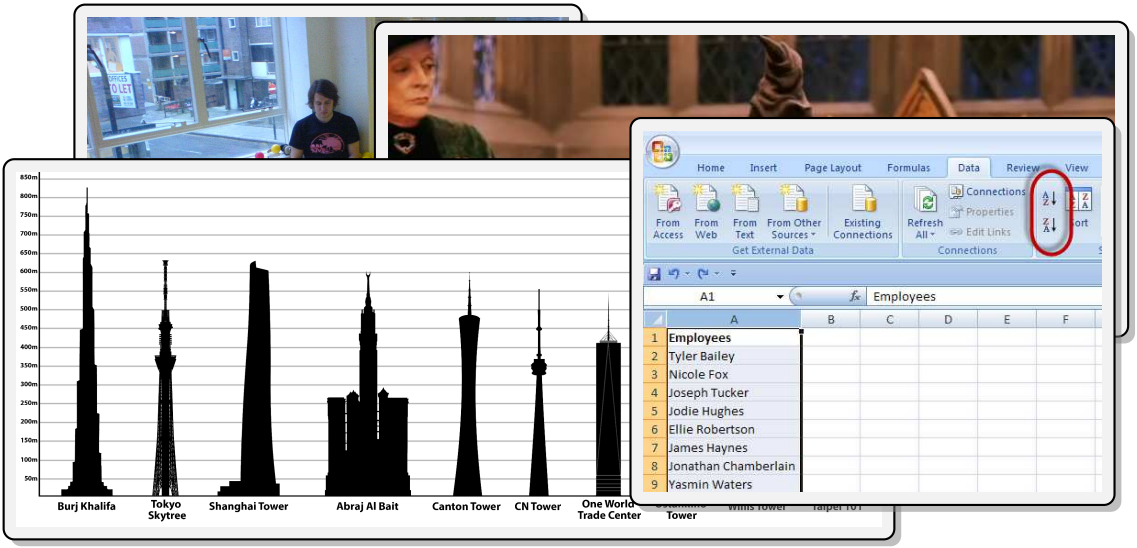

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⇝ Two meanings of **sorting**

1. Separate different **sorts** of things

2. Put items into **order** (alphabetical, numerical, . . .)

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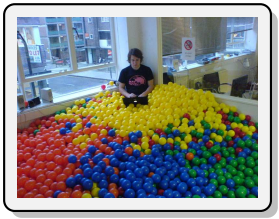

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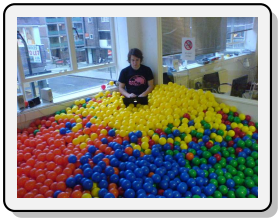

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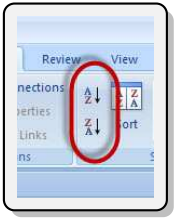

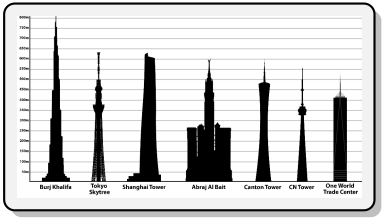

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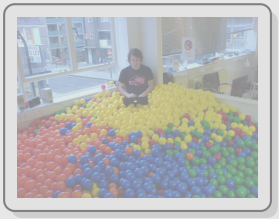

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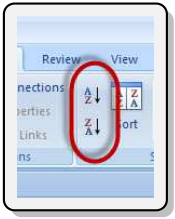

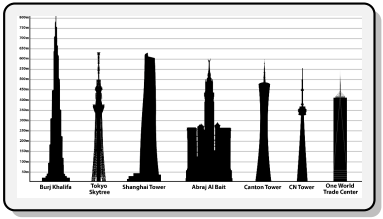

Model of Computation

How do computers sort?

1	2	3	4	5	6	7	8	9
50	2	75	17	10	78	33	72	98

Input given as **array** of n items

- index-based access (5th item $A[5]$ is 10)
- we can read and write cells
- Only info about data: **pairwise comparisons**
  Is $A[5] < A[8]$? Yes, $10 < 72$

$\rightsquigarrow$ only relative ranking of elements important

$\rightsquigarrow$ may assume, e.g., numbers $1, \dots, n$

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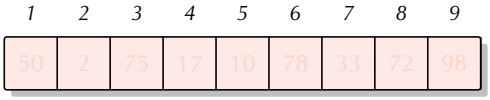

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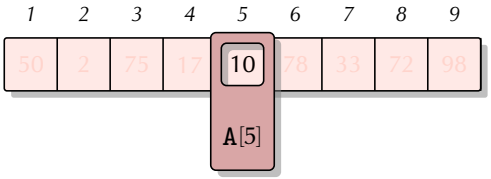

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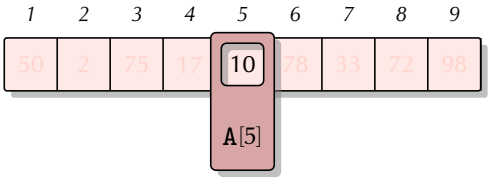

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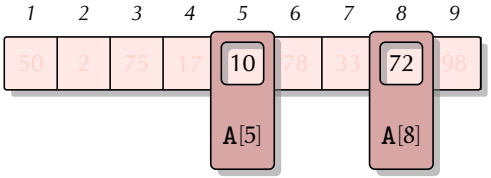

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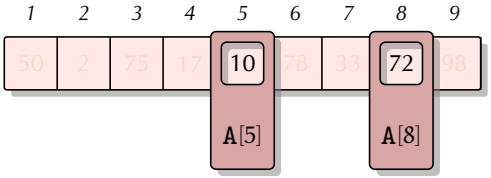

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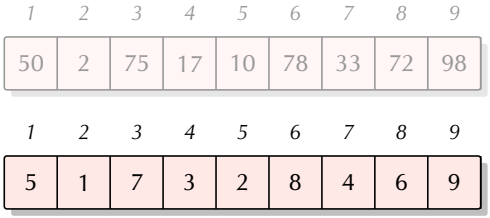

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Find smallest item and put it in place.

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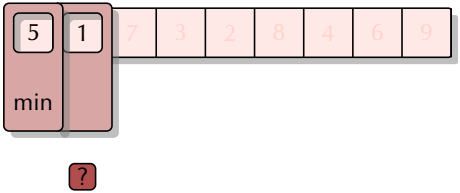

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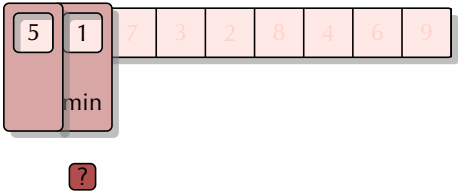

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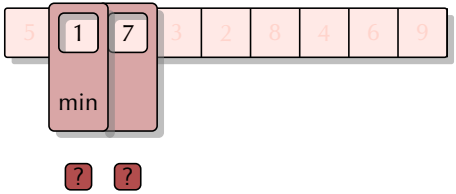

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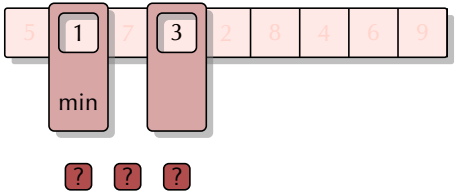

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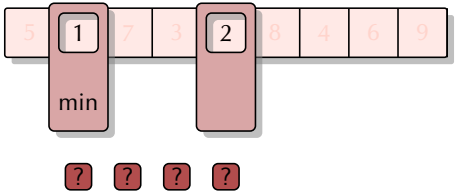

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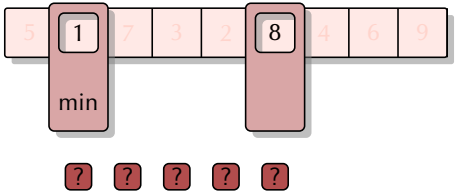

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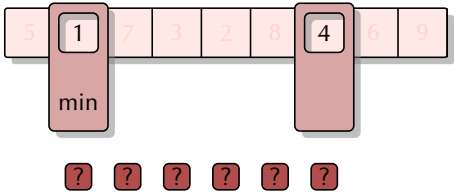

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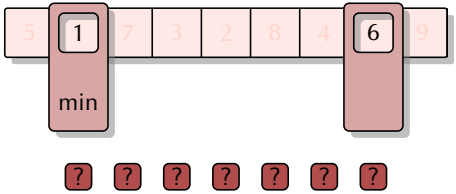

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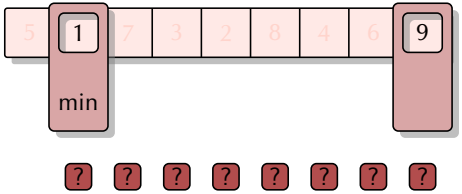

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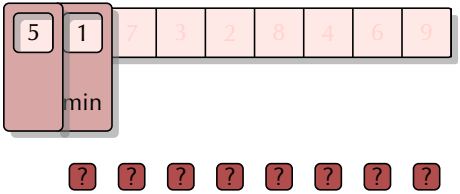

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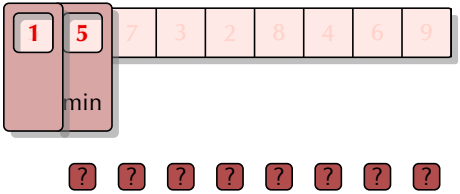

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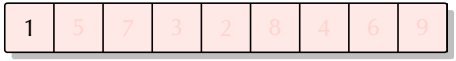



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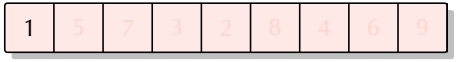



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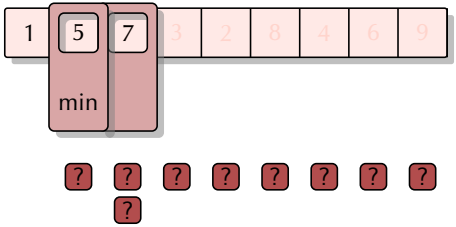

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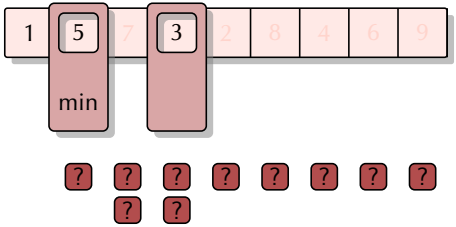

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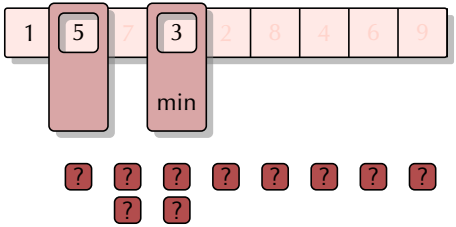

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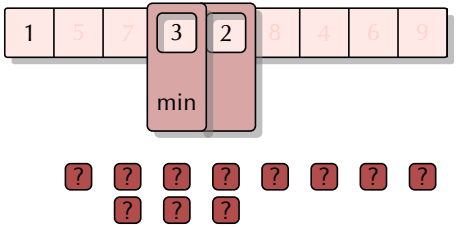

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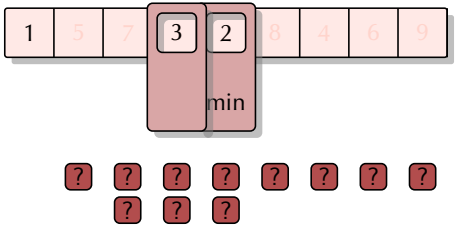

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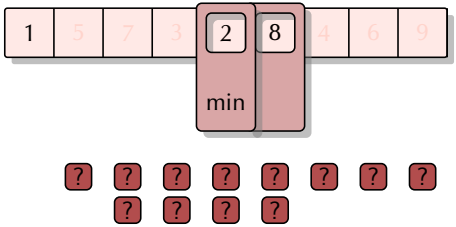

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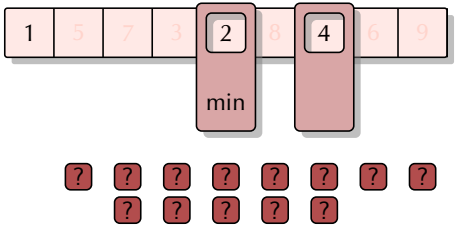

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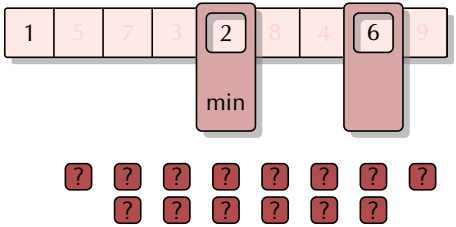

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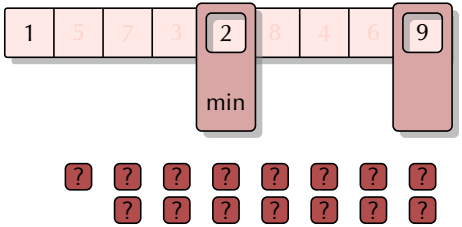

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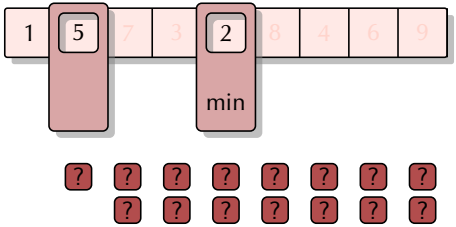

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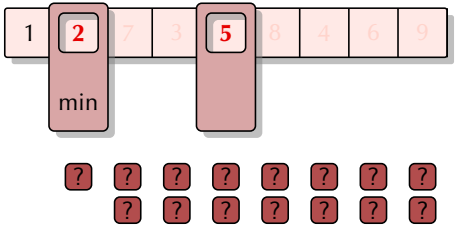

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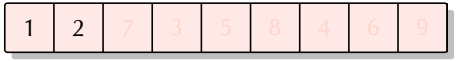



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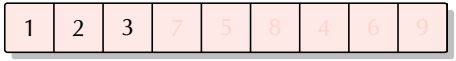



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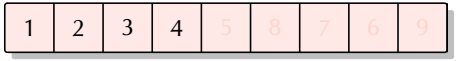

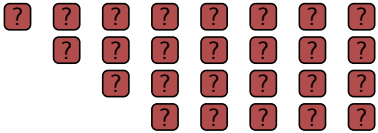

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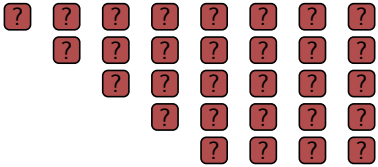

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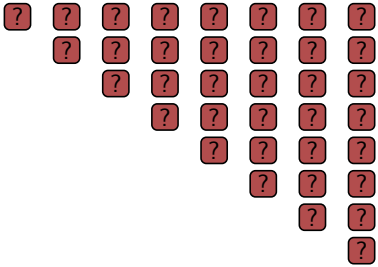

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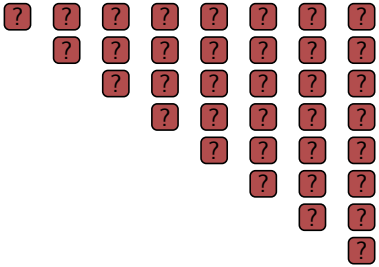

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Find smallest item and put it in place.



Each round puts one item in place.
$\rightsquigarrow$ Remaining list **one** smaller.

Can we do better?

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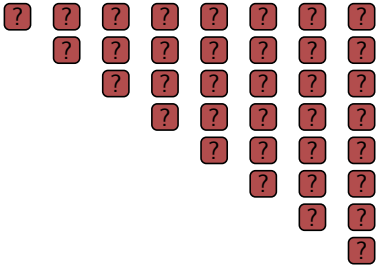

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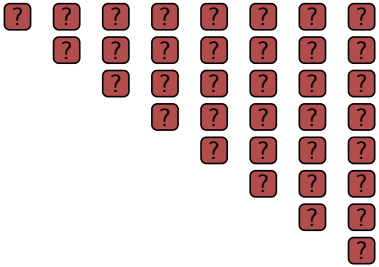

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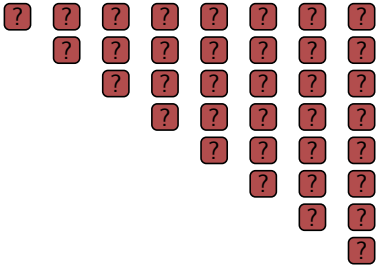

Each round puts one item in place.
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Can we do better?

Yes! Finding the minimum costs the same as computing the **rank** of **any** element.

number of elements
smaller than given element

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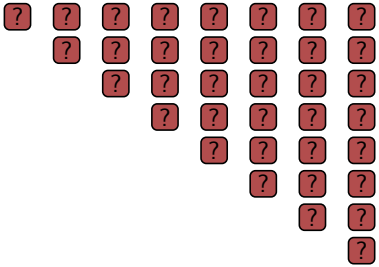

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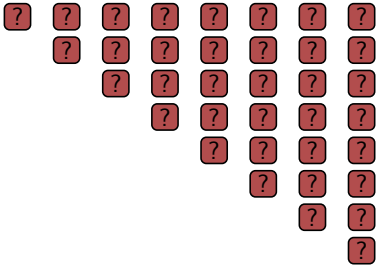

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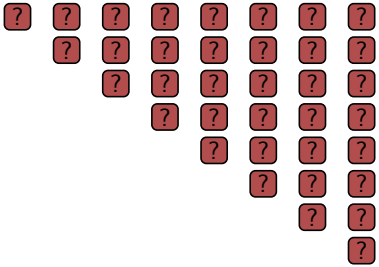

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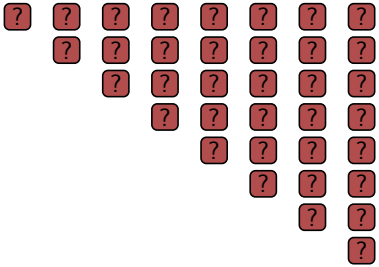

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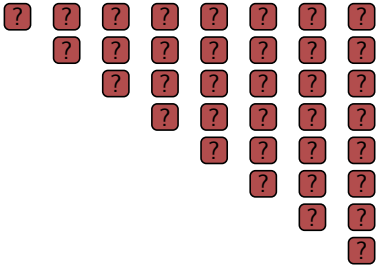

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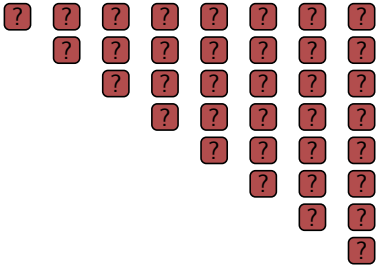

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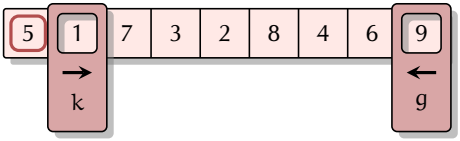

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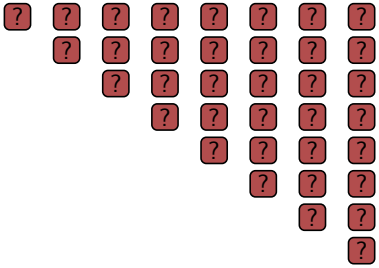

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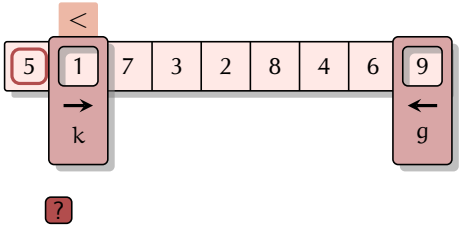

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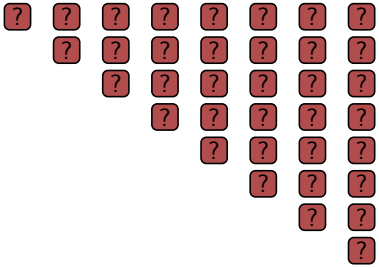

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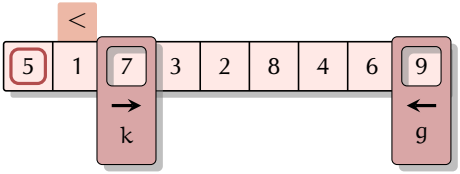



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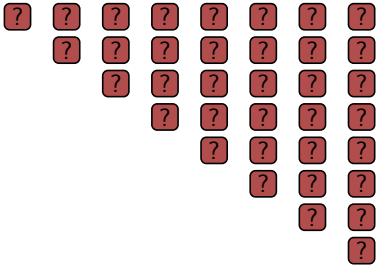

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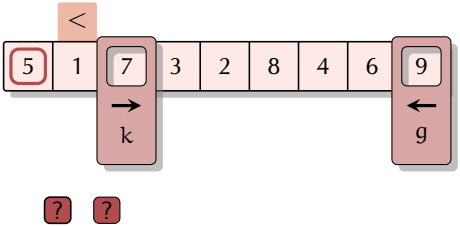

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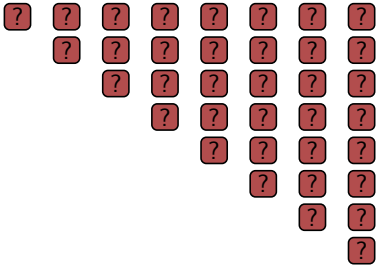

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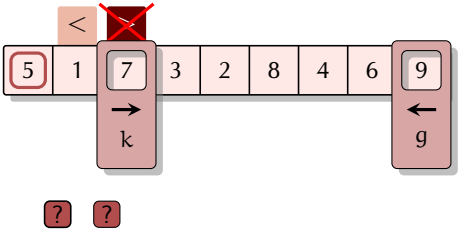

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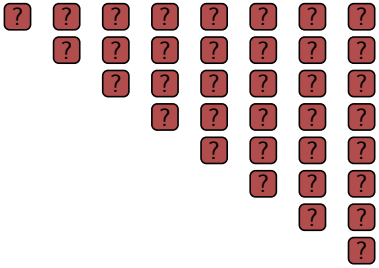

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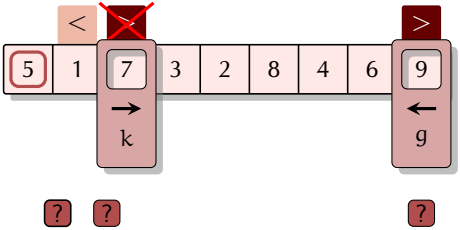

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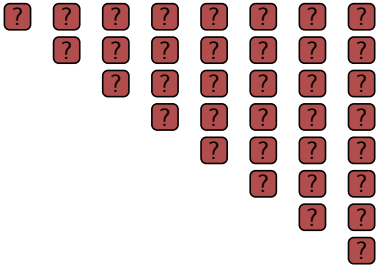

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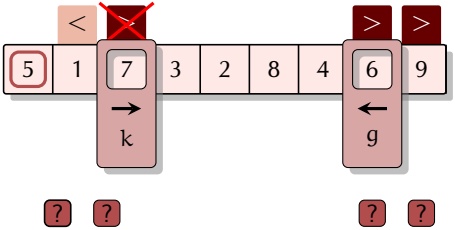

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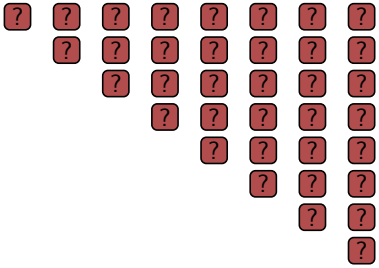

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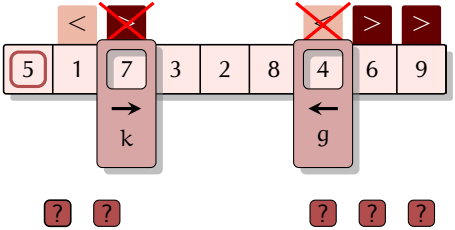

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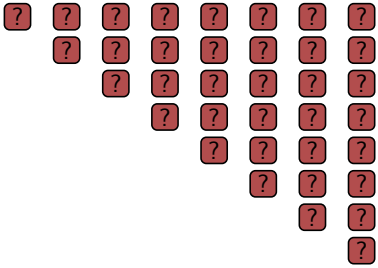

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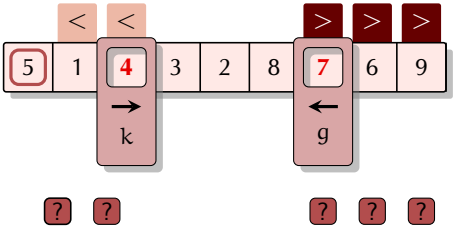

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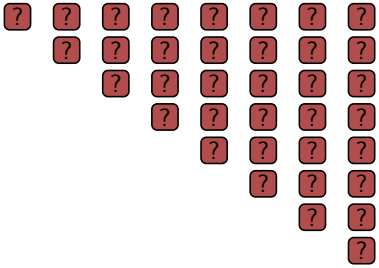

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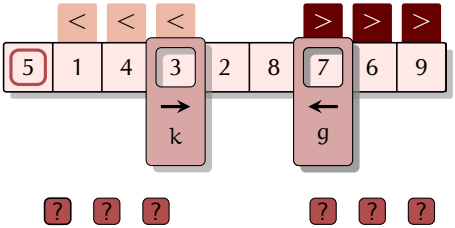

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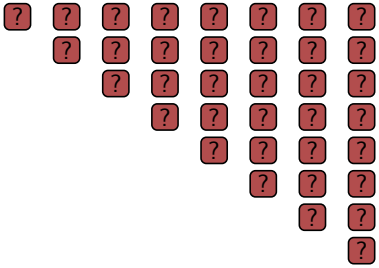

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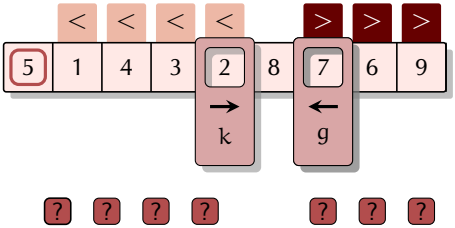

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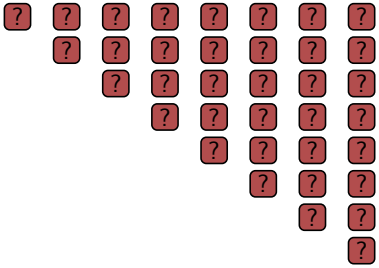

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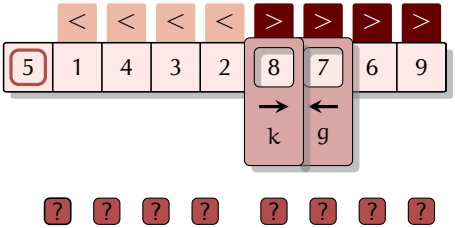

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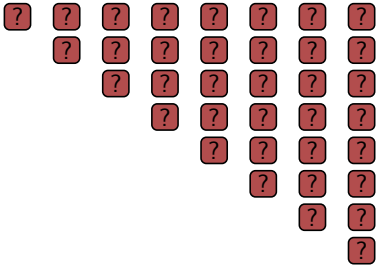

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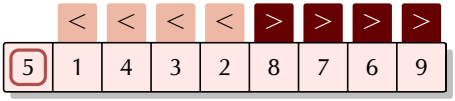



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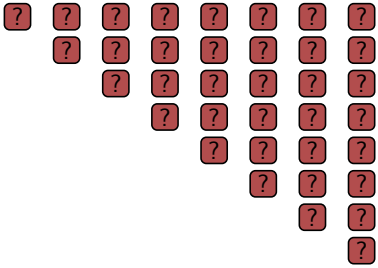

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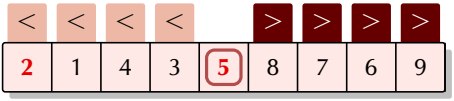



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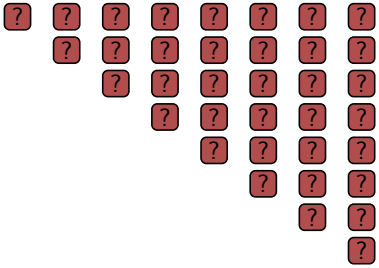

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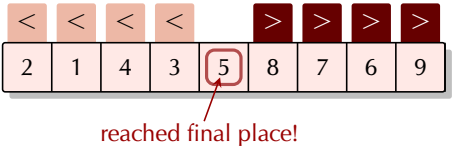



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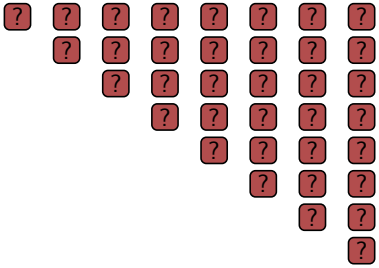

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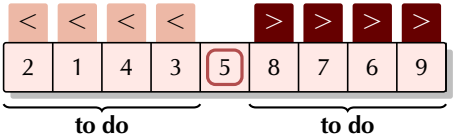



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Selectionsort

Find smallest item and put it in place.



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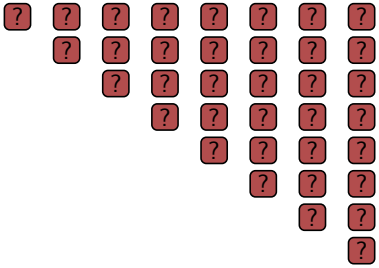

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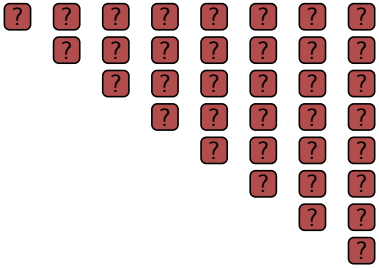

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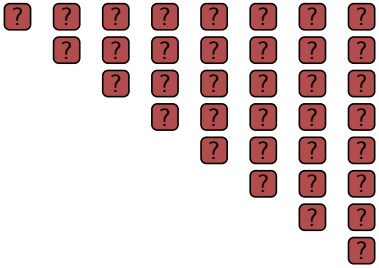

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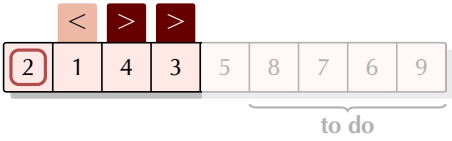

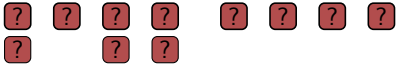

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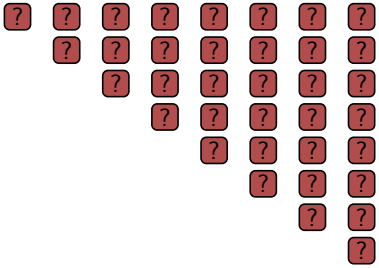

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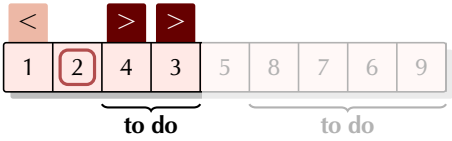

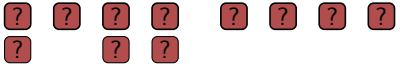

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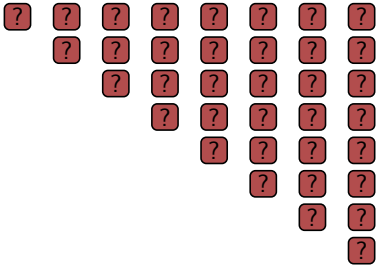

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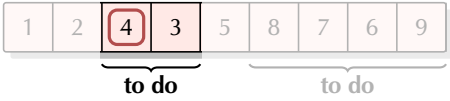

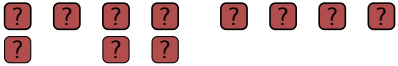

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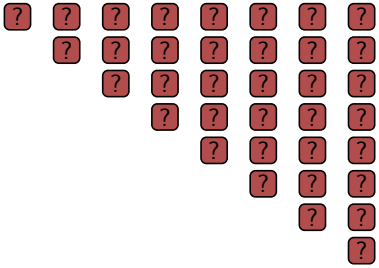

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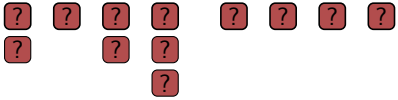

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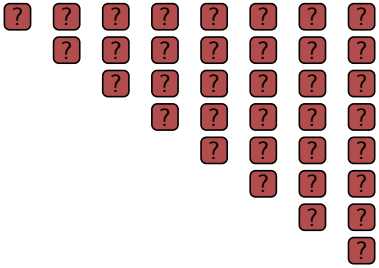

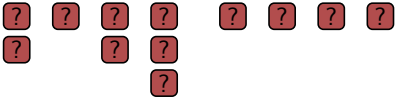

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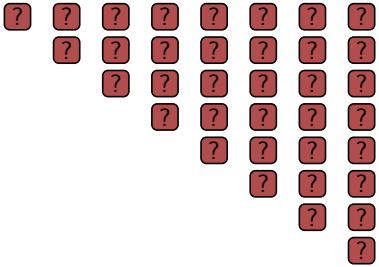

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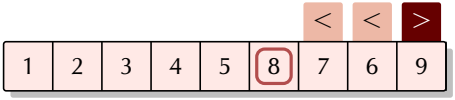

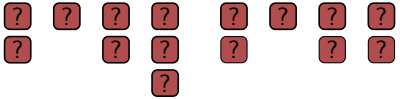

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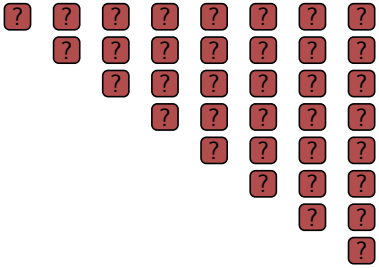

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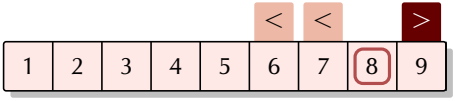

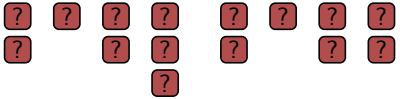

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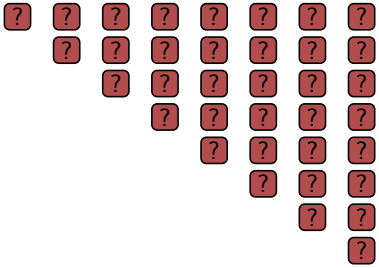

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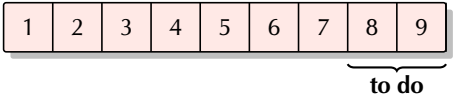

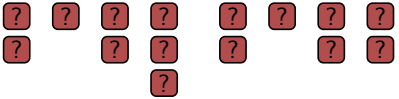

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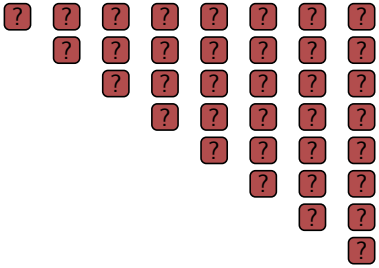

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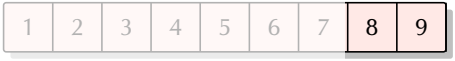

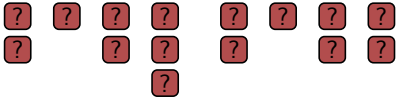

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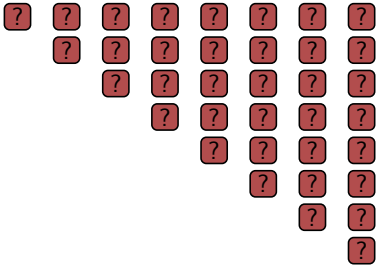

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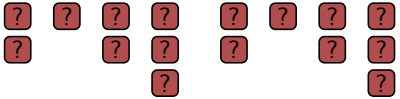

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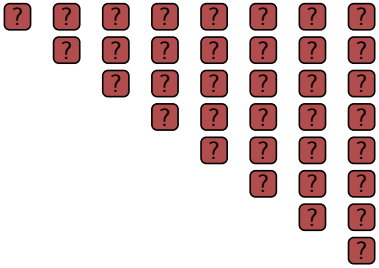

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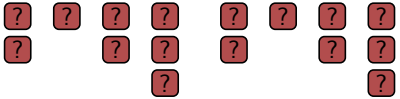

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Much **less** comparisons in Quicksort

. . . unless pivots are max/min!

But **average/expected** behavior is good.

⇝ Quicksort is method of choice in practice.

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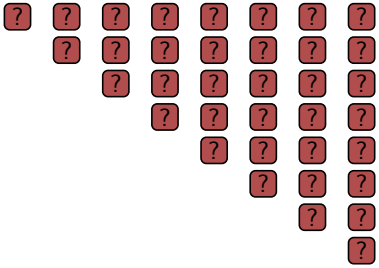

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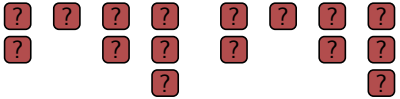

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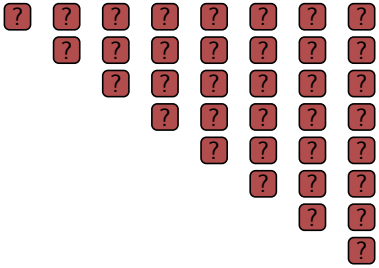

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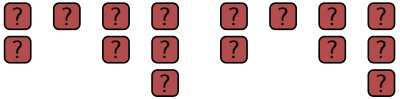

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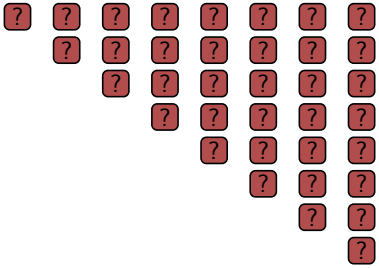

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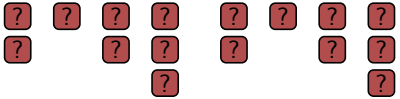

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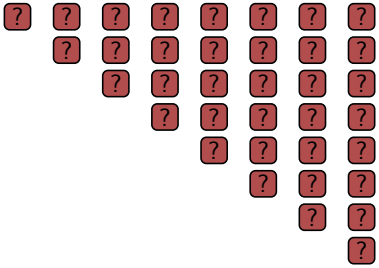

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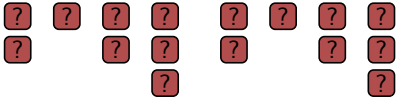

Variants of Quicksort

It is tempting to try to develop ways to improve quicksort:
a faster sorting algorithm is computer science's "better mousetrap,"
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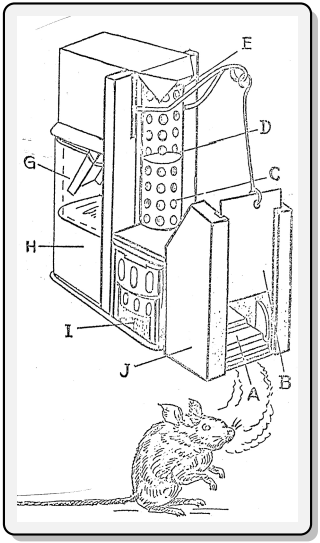

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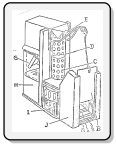

Many variants tried

Two important tuning options:

1. Choose **pivots** wisely ***(pivot sampling)***
   - for example: middle element of three *(median-of-three)*

   $\rightsquigarrow$ pivot never min or max!
2. Split into $s \geqslant 2$ **subproblems** at once ***(multiway partitioning)***
   - for example: 2 pivots $\rightsquigarrow$ 3 classes: small, medium and large

   $\rightsquigarrow$ subproblems are smaller,
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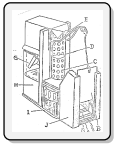

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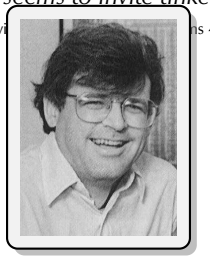

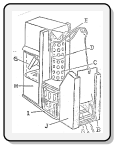

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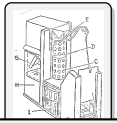

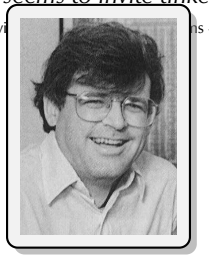

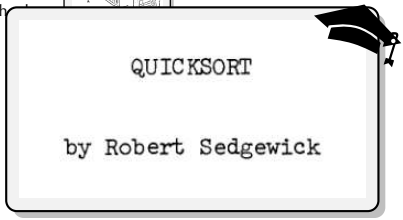

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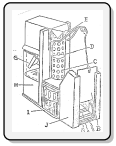



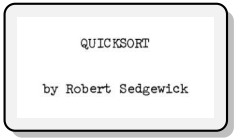

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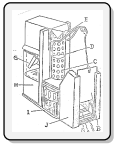



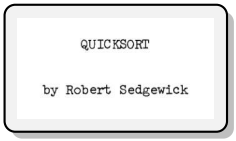

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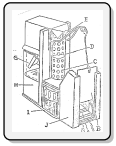



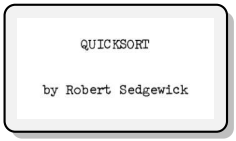

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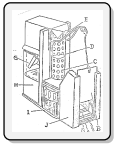



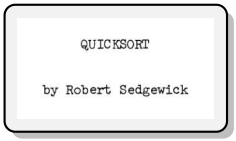

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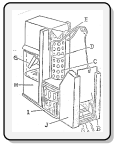



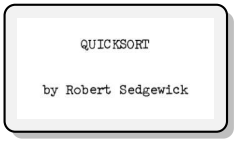

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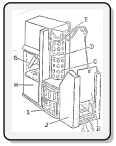



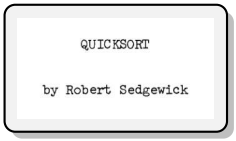

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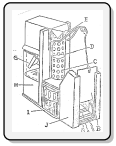



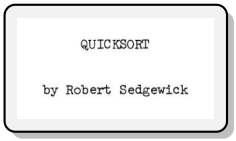

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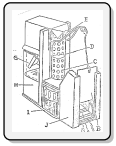



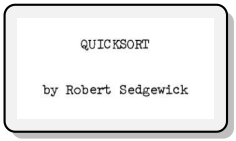

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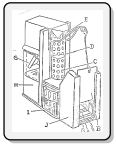



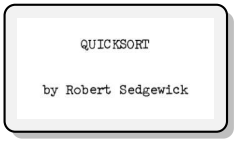

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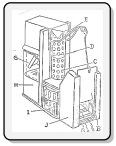



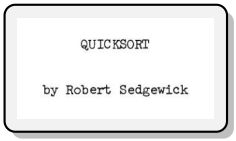

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used e.g. for Android apps

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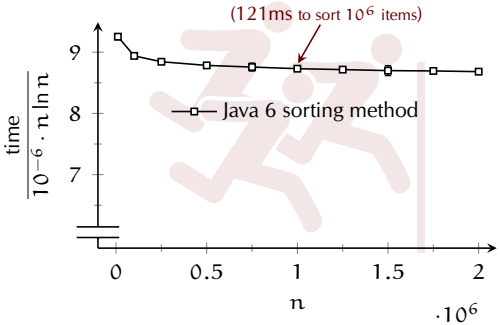

Normalized Java runtimes (in *ms*).

Average and standard deviation of
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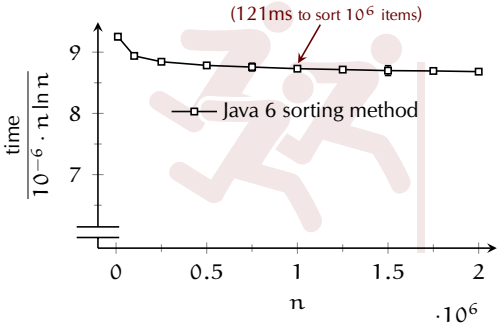

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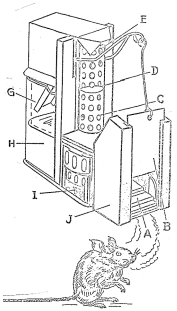

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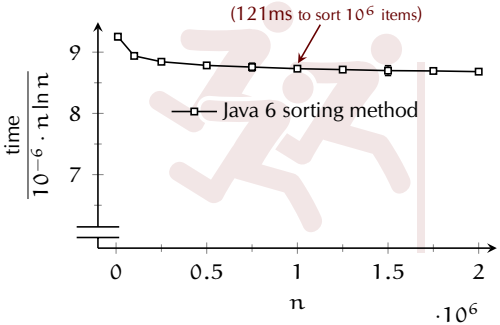

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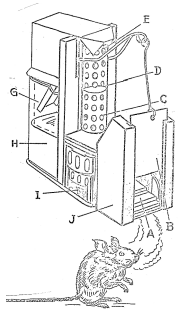

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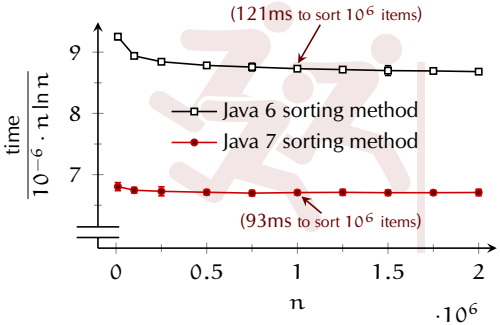

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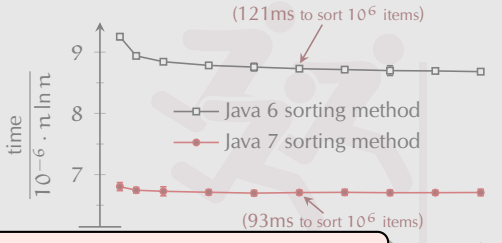

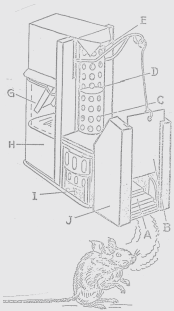

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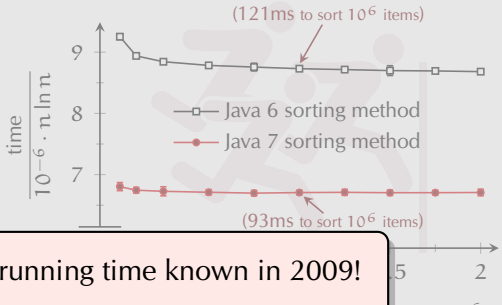

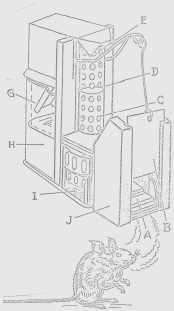

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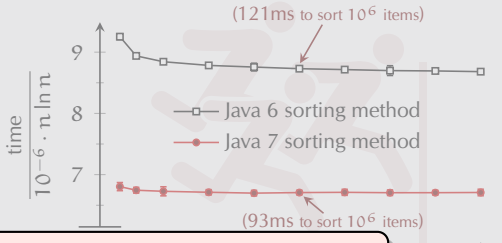

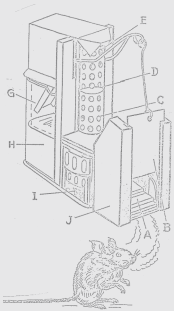

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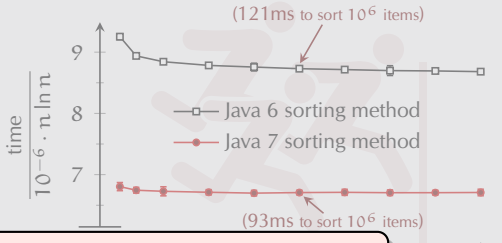
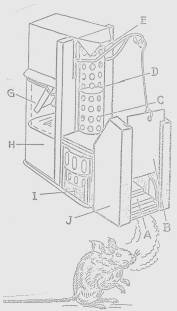
used e.g. for Android apps

What happened in 2009? *Java 7 sorting method was released!*

as in example before
+ median-of-three
+ Insertionsort

- till 2009, Java used ***classic Quicksort***
- Java 7 uses **new** ***dual-pivot Quicksort***





No theoretical explanation for running time known in 2009!

Only lucky experiments?

Why did no-one come up with this earlier?

That was the starting point for my work!

Overview of my work

My Thesis

Multiway partitioning has genuine potential to speed up Quicksort.

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Multiway partitioning has genuine potential to speed up Quicksort.

Mathematical Analysis

- prove eternal truths
- **not** experiments

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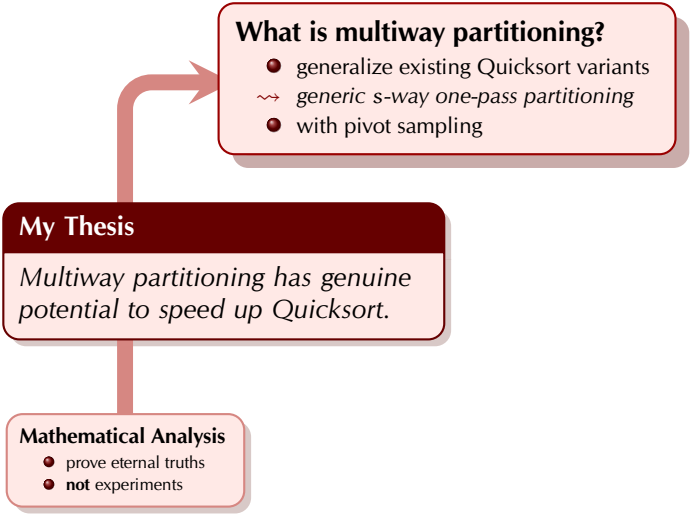

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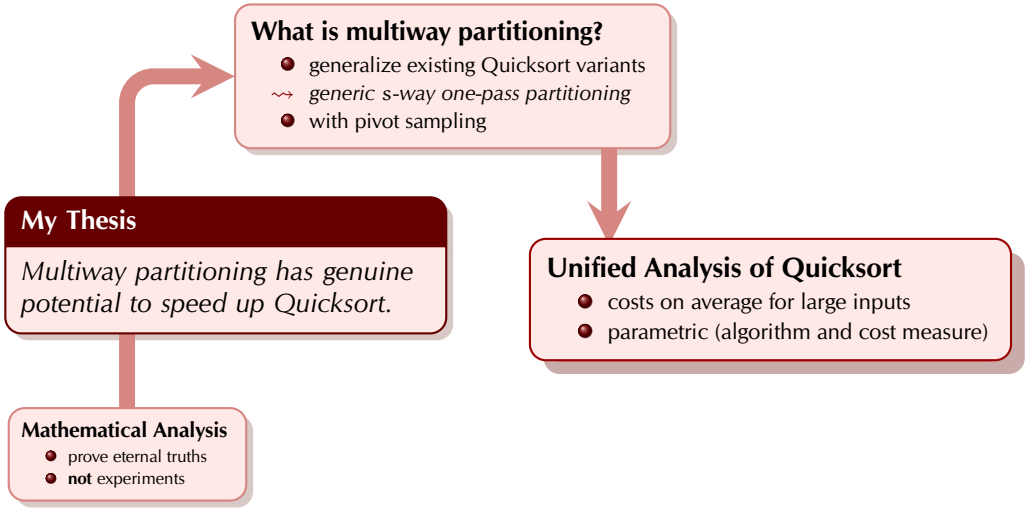

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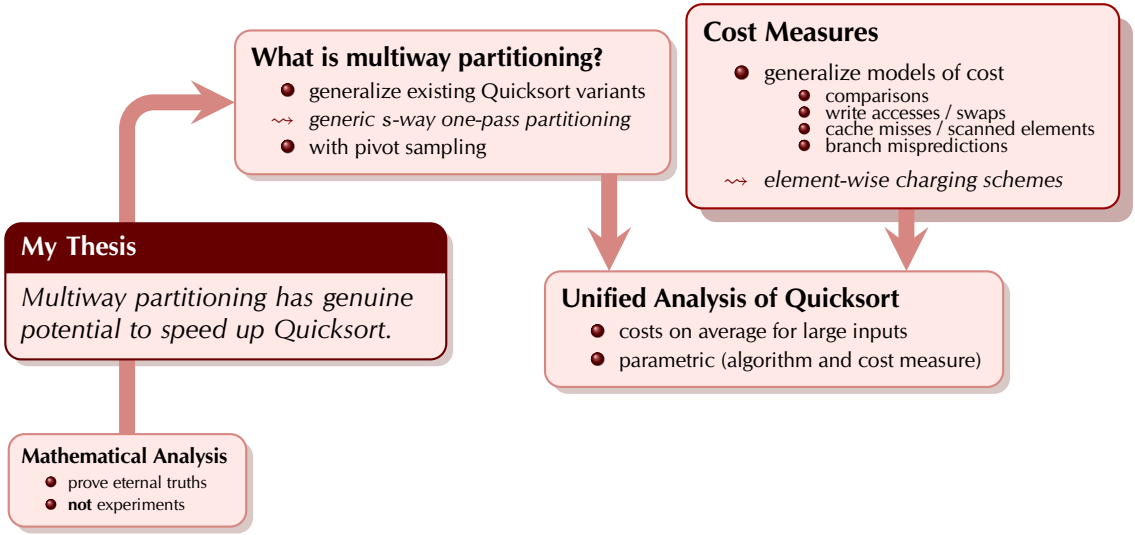

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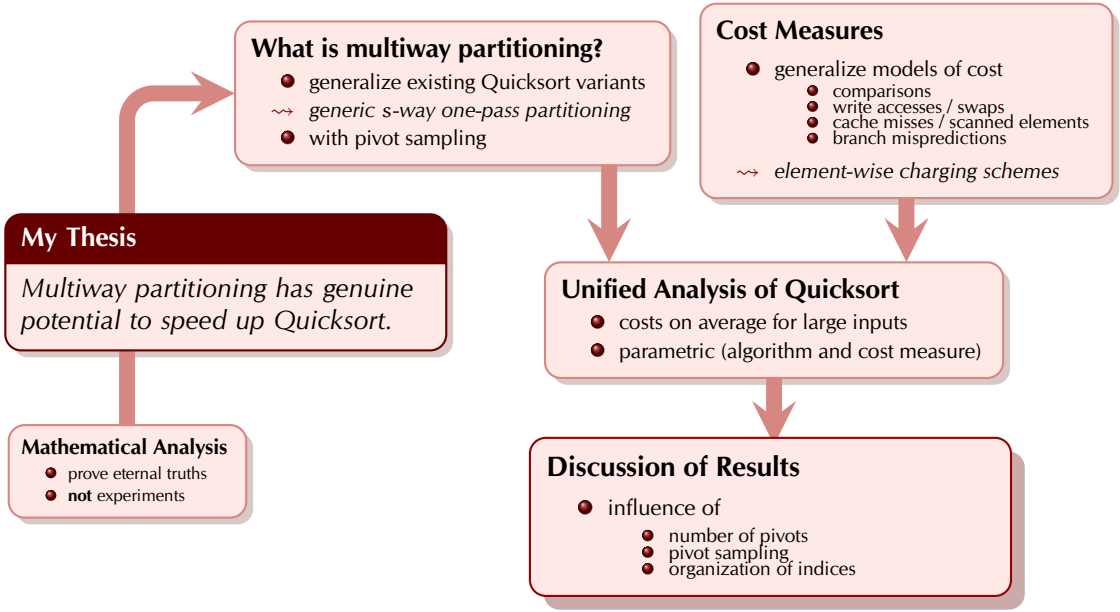

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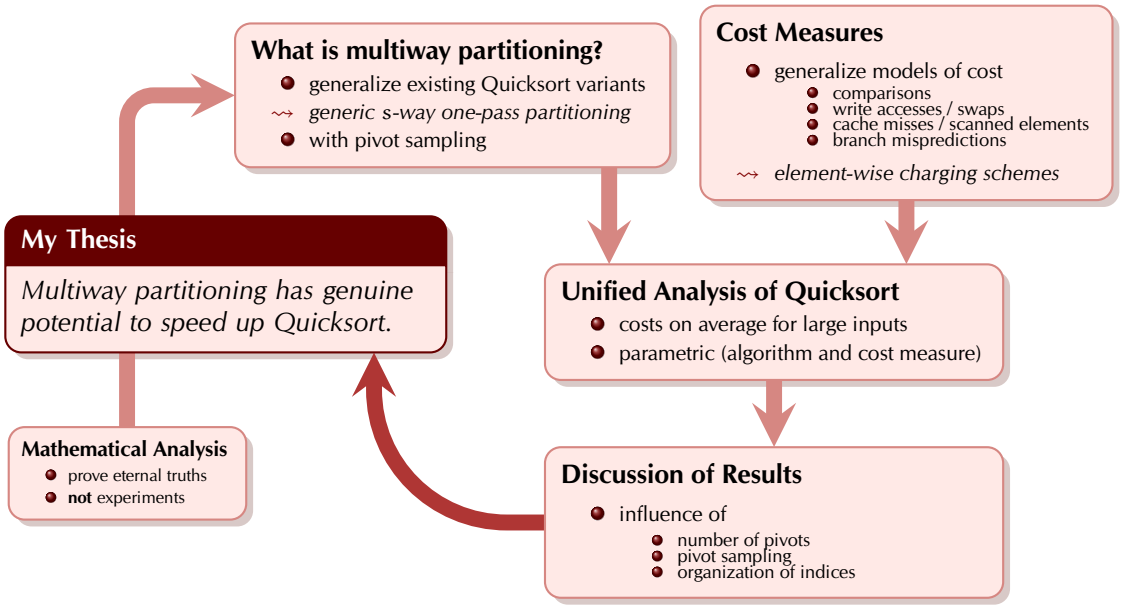

Entropy

What is entropy?

1. **Thermodynamics:** unusable part of thermal energy
2. **Mathematics:** information content of a random variable

Definition *(Entropy)*

Let X be a random variable with $\mathbb{P}[X = i] = p_i$ for $i \in \{1, \dots, u\}$.
Its *entropy* is defined as $\mathcal{H}(X) = \sum_{i=1}^{u} p_i \log_2(1/p_i)$.

…all clear?
Let's see an application.

- Urn with **colored** balls.
- Alice draws ball Ⓧ uniformly at random from urn.
- Bob wants to **guess its color** C by **Yes/No-Questions**.

⇝ **Theorem** *(Shannon's Source Coding Theorem)*

On average, Bob **can't do with less** than $\mathcal{H}(C)$ questions.

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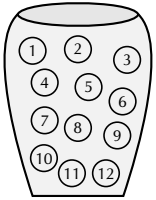

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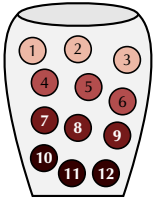

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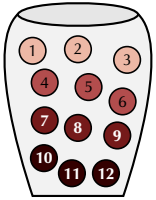

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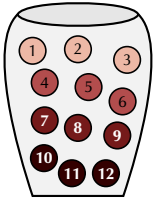

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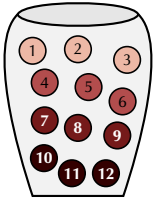

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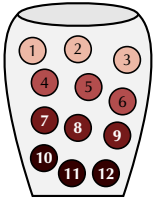

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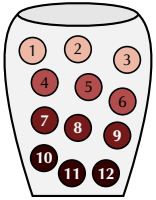

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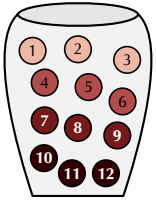

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Information Theory & Sorting

Where is the connection to sorting?



$n(n-1)\cdots 2\cdot 1$

- Now, Alice draws one of the $n!$ possible **orderings** of n items.
- Bob guesses the order by asking comparisons.

Assumptions:
(1) All orderings equally likely.
(2) Only comparisons allowed.

asymptotic error term for $n \to \infty$

Theorem *(Information-Theoretic Lower Bound for Sorting)*

$\rightsquigarrow$ We cannot sort with less than $\log_2(n!) = n\log_2(n) \pm O(n)$ comparisons.

$\mathcal{H}(C)$

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We can reverse this idea to analyze Quicksort.

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Comparisons in Quicksort

How many comparisons does Quicksort use on average?

Entropy-based analysis:

1. Compute the **average information yield**: $x \frac{\text{bit}}{\text{comparison}}$. $(0 < x \leqslant 1)$
   What is x? *. . . stay tuned.*
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not mathematically rigorous ...
but correct result!

rigorous proof $\rightarrow$ dissertation

Partitioning Entropy

What is x?

Classic Quicksort:

- Always compare with the pivot P

- Outcome C is either small or large.

⇝ Information yield is $\mathcal{H}(C) = -\mathbb{P}[<] \log_2(\mathbb{P}[<]) - \mathbb{P}[>] \log_2(\mathbb{P}[>])$

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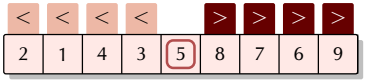

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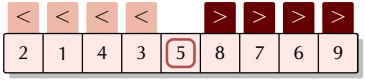

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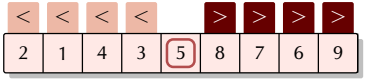

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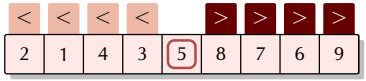

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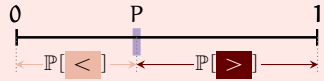

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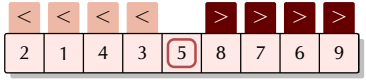

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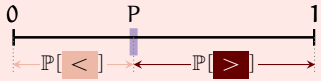

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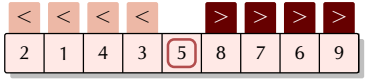

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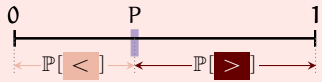

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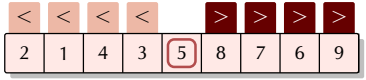

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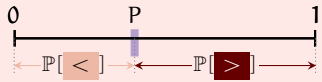

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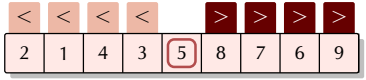

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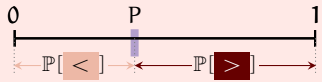

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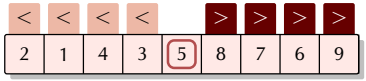

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<	<	<	<		>	>	>	>
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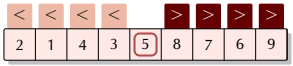

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s-way Quicksort with pivot sampling:

- $s-1$ pivots: $P_1, \ldots, P_{s-1}$
  - s classes $c_1, \ldots, c_s$
- **Pivot Sampling**: parameter $\vec{t} = (t_1, \ldots, t_s) \in \mathbb{N}_0^s$
  - **Example:** $s = 3$, $\vec{t} = (1, 2, 3)$

⇝ $\vec{D} \stackrel{\mathcal{D}}{=} \text{Dirichlet}(t_1+1, \ldots, t_s+1)$

$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{t_r+1}$

$$\rightsquigarrow \mathbb{E}[\mathcal{H}(C)] = \sum_{r=1}^{s} \frac{t_r+1}{k+1}\left(H_{k+1} - H_{t_r+1}\right)\cdot\log_2(e)$$

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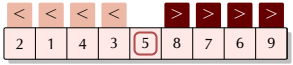

- Outcome C is either small or large.

⇝ Information yield is $\mathcal{H}(C) = -\mathbb{P}[<]\log_2(\mathbb{P}[<]) - \mathbb{P}[>]\log_2(\mathbb{P}[>])$

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- $s-1$ pivots: $P_1, \dots, P_{s-1}$
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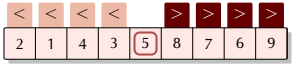

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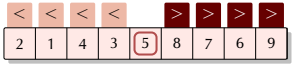

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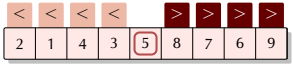

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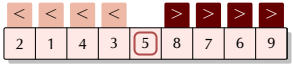

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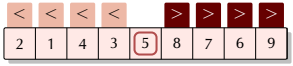

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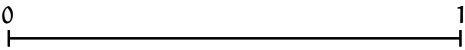

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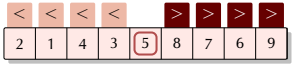

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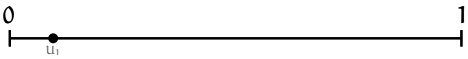

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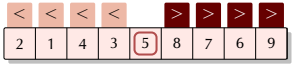

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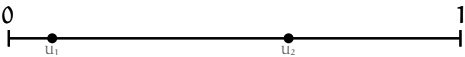

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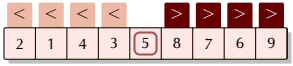

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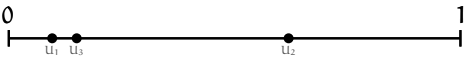

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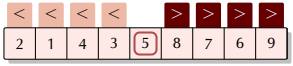

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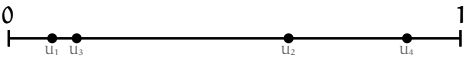

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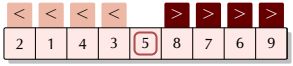

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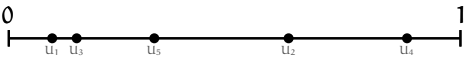

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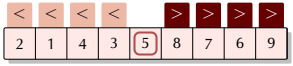

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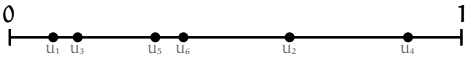

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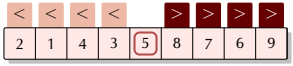

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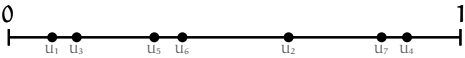

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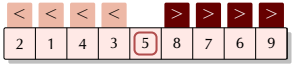

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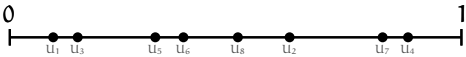

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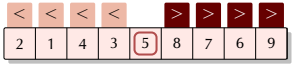

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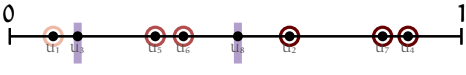

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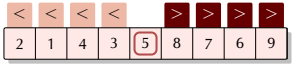

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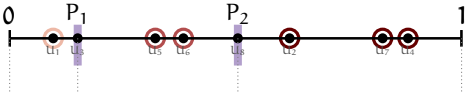

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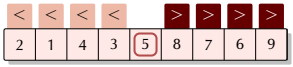

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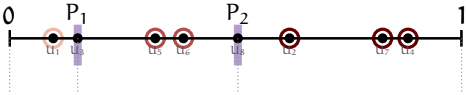

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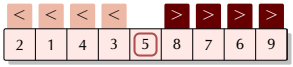

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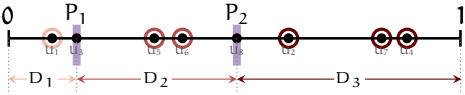

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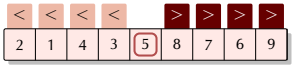

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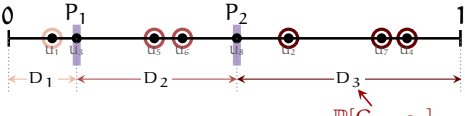

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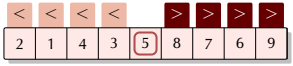

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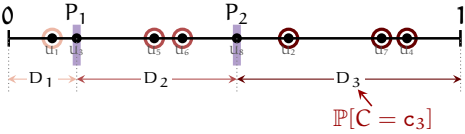

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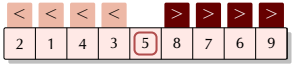

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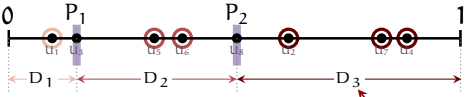

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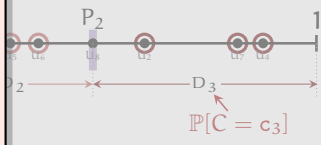

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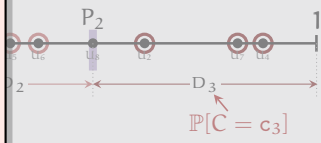

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$H_{t_r+1} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{t_r+1}$

Binary vs. Multiway Partitioning

Now, does multiway partitioning perform better? *Let's try an example.*

4-way Quicksort

- Need $a = 2$ comparisons to classify 1 item.
- $\frac{24}{13} n \ln(n)$ comparisons ≈ 1.846 (without sampling)

Classic Quicksort

- $2 n \ln(n)$ comparisons (without sampling)

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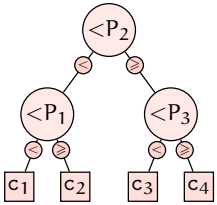

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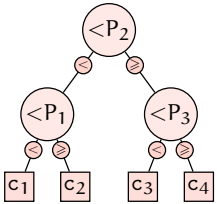

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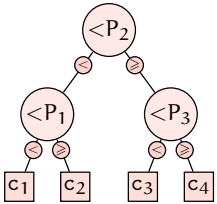

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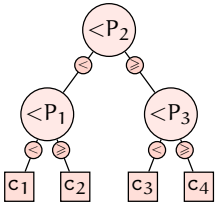

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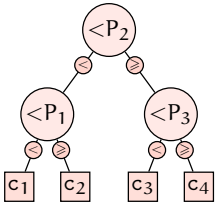

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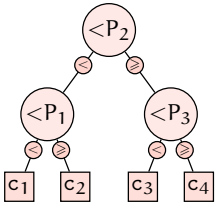

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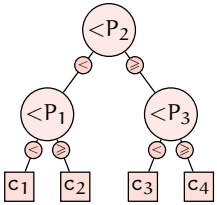

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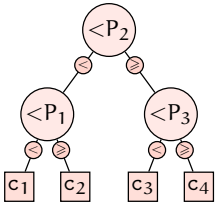

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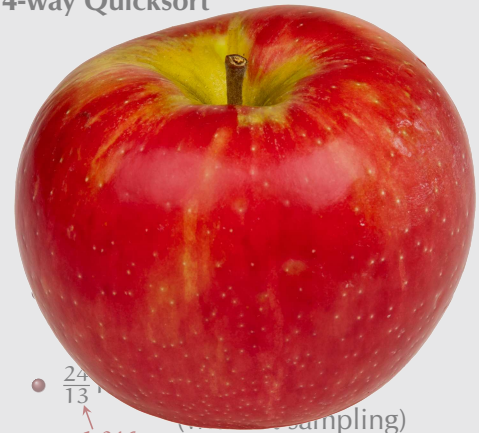

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VS.

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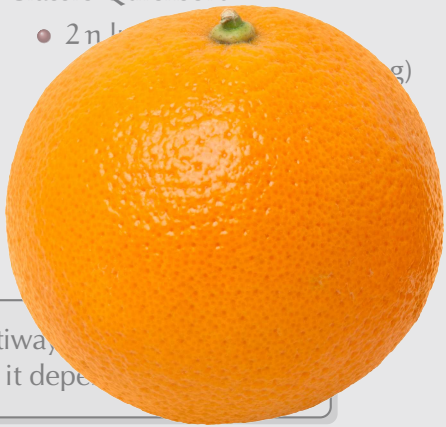

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Binary vs. Multiway Partitioning

Now, d

How to **separate** influence of multiway partitioning from that of sampling?

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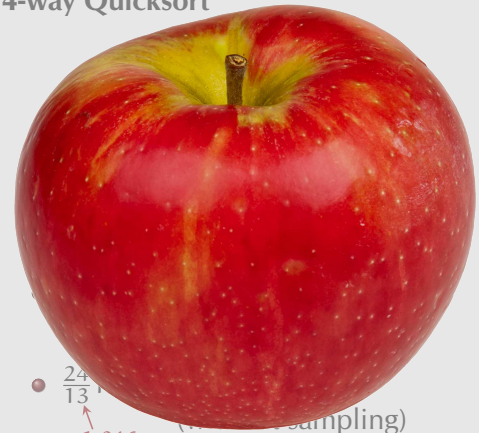

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Key Idea: Simulate by binary partitioning.

One round of 4-way partitioning ...

(without sampling)

... corresponds to three binary partitionings.

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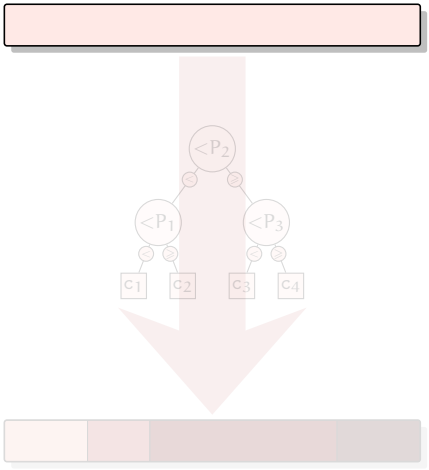

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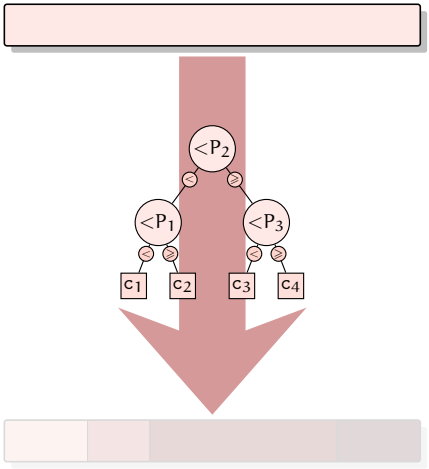

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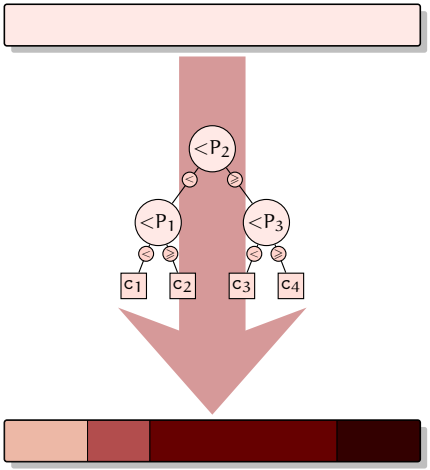

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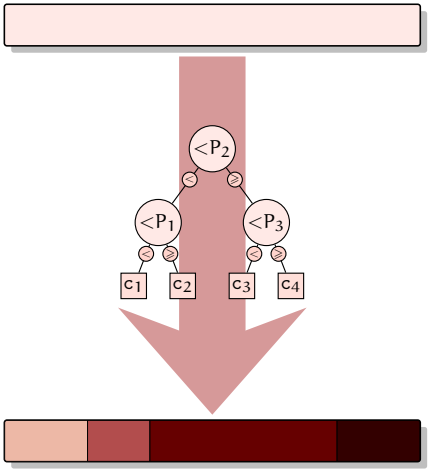

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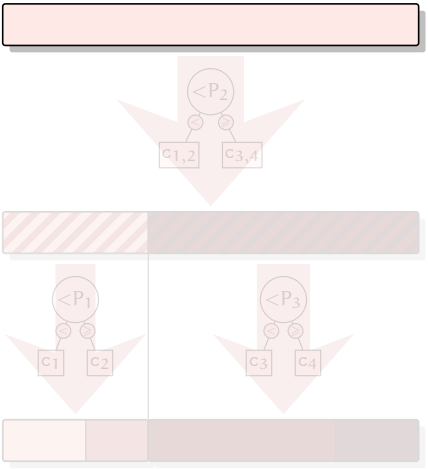

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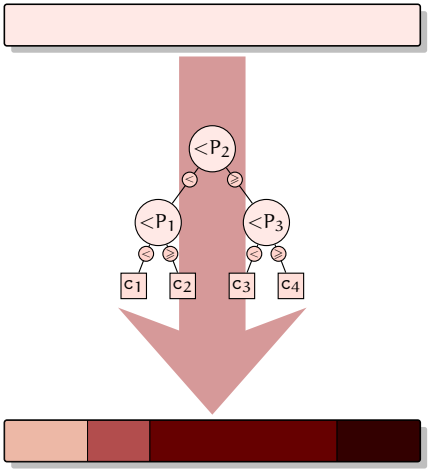

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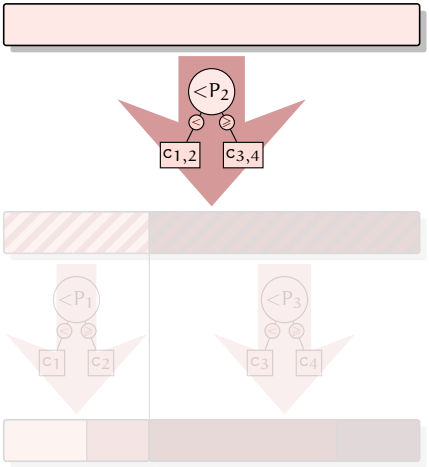

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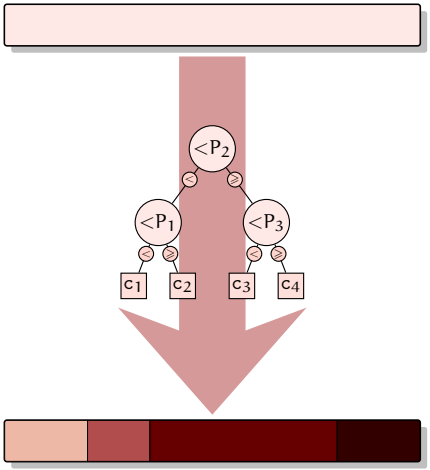

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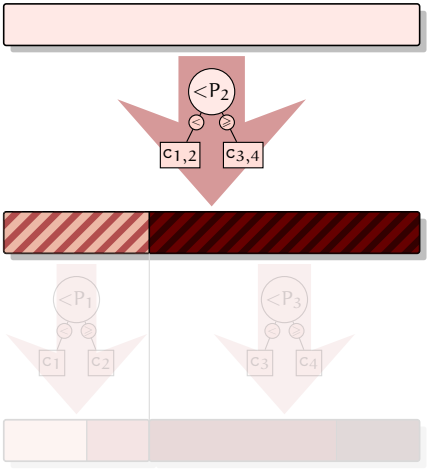

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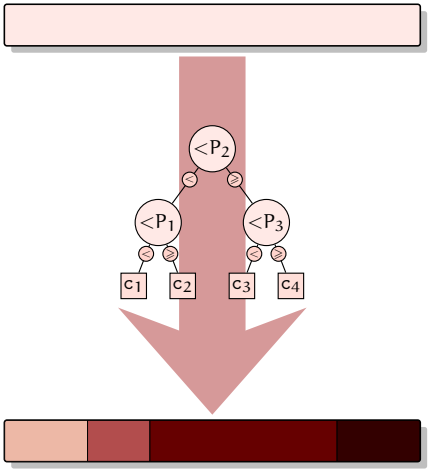

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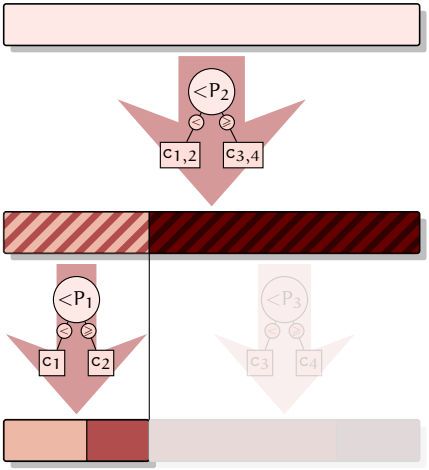

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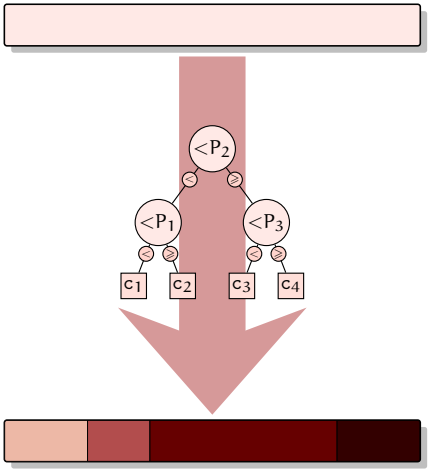

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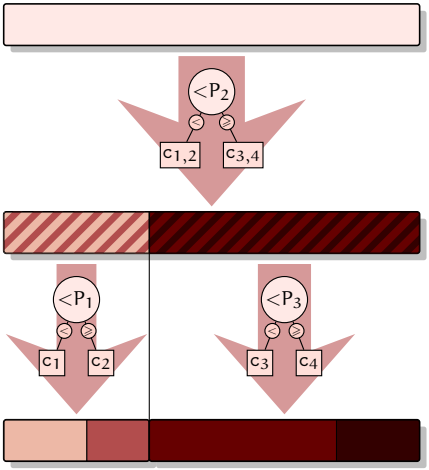

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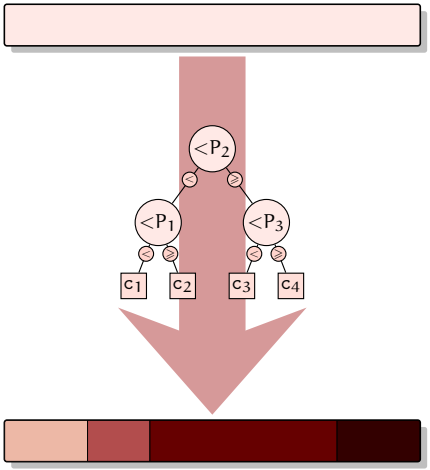

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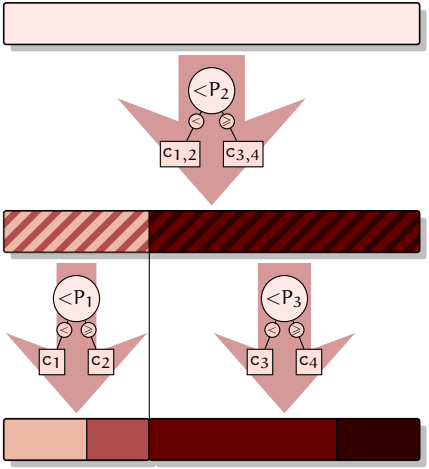

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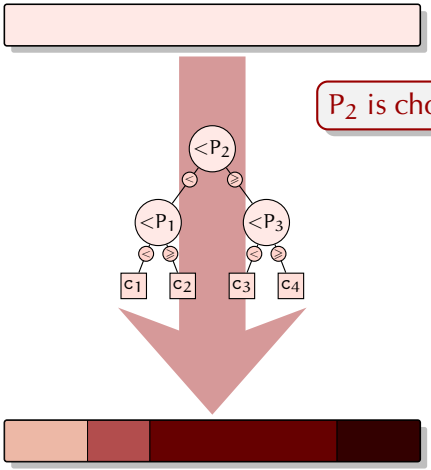

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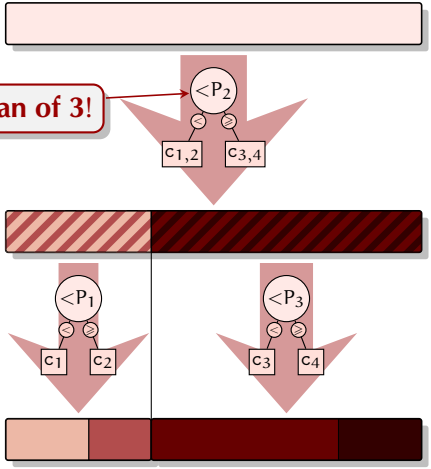
P_2 is chosen as **median of 3**!



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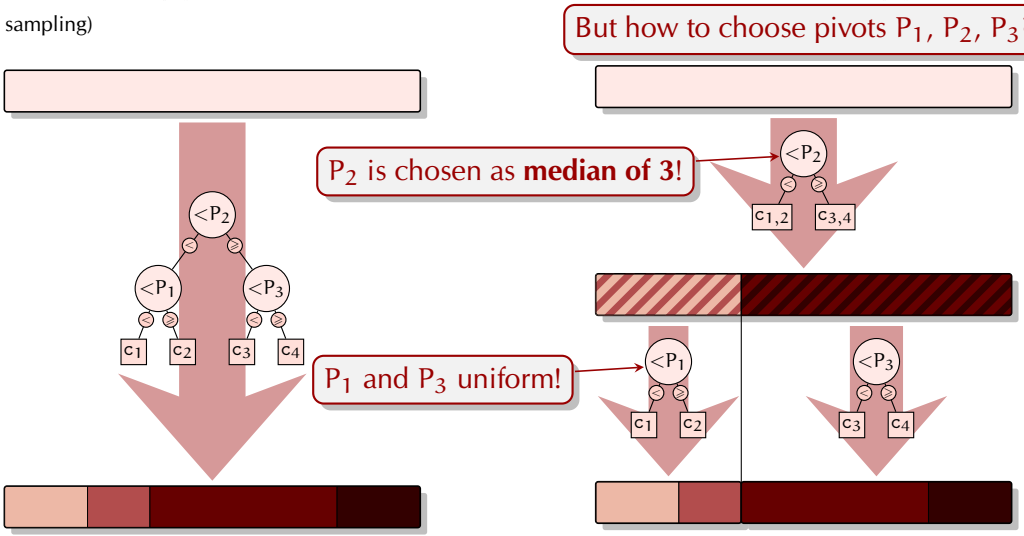

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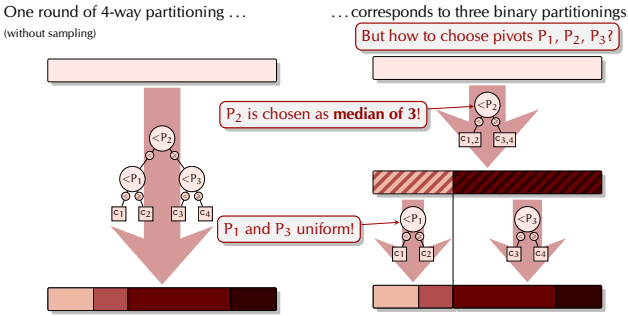

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random-parameter pivot sampling

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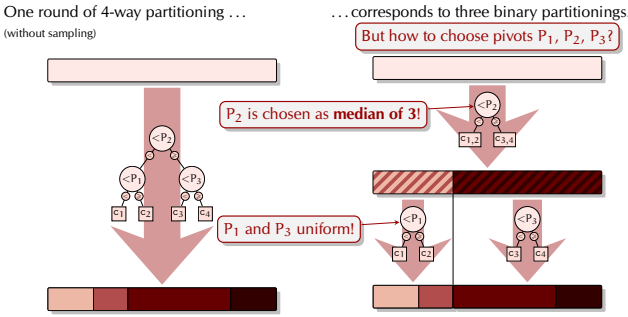

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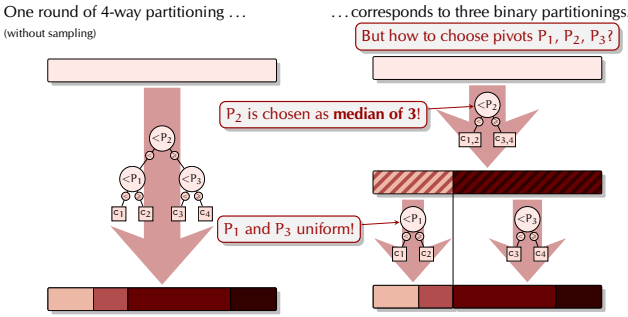

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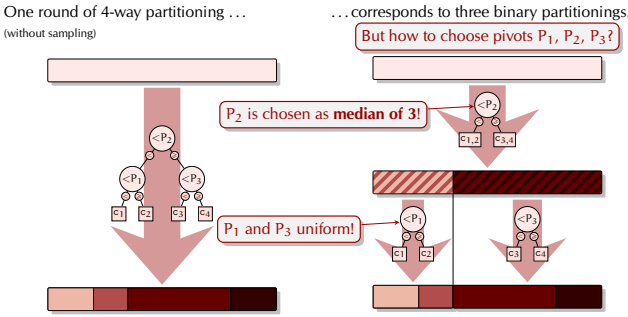

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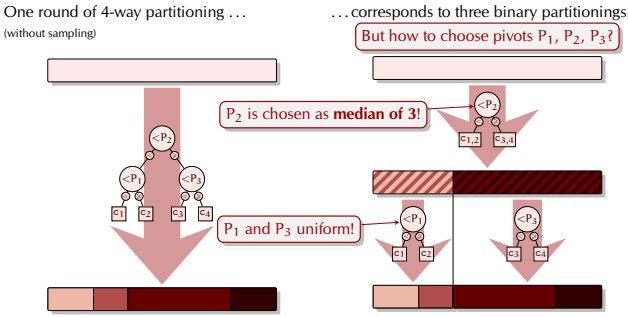

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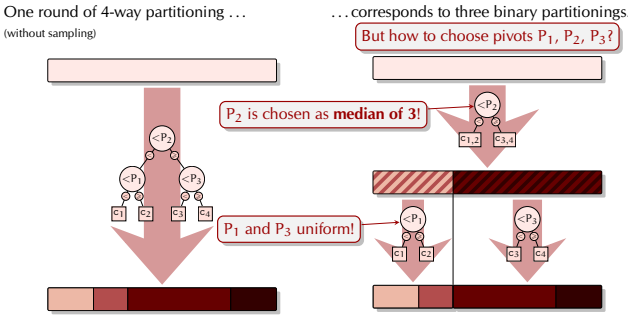

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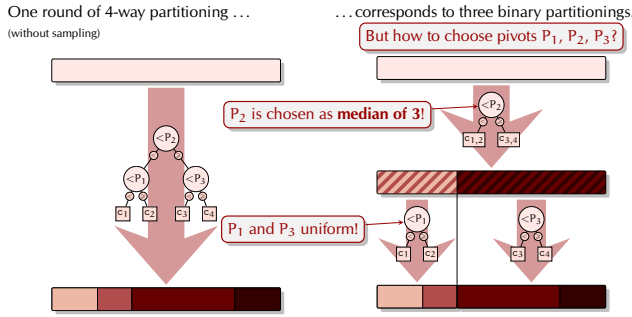

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For every s-way sampling parameter $\vec{t}$ there is a single-pivot random-parameter scheme with the same quality of pivots.

(formal meaning in dissertation)

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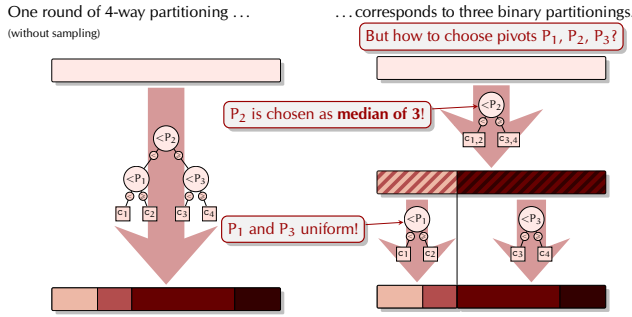

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Key Idea: Sir

One round of 4-wa
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Theorem *(Entropy-Equivalent Sampling)*

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One round of 4-wa
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What does that mean?

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Comparisons $\widehat{=}$ **CPU** effort, but we also need to **load** items from **memory**.

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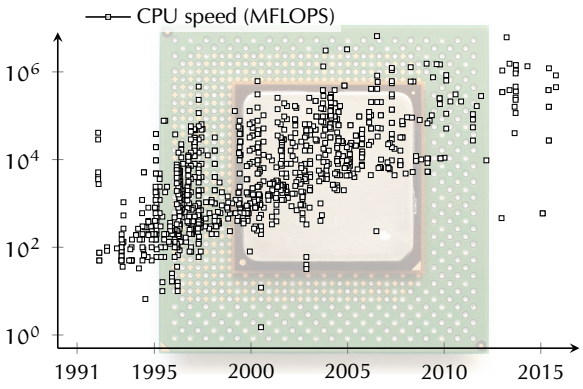

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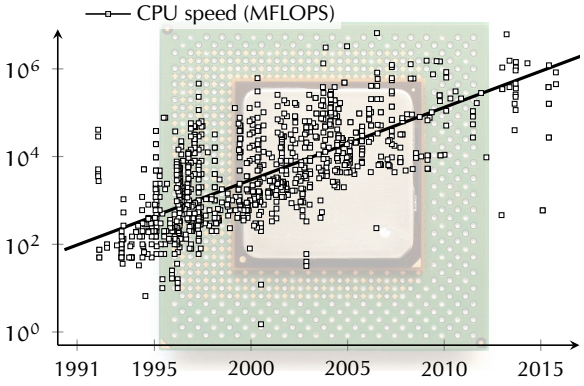

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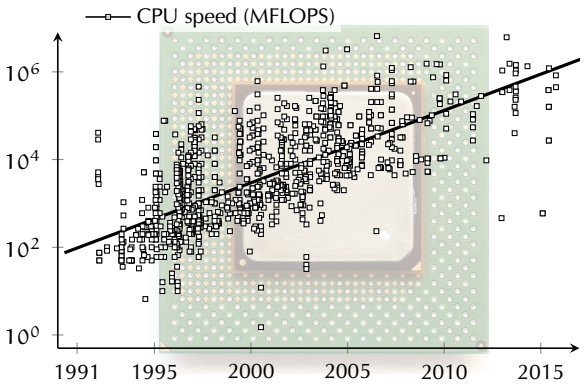

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Processor speed: 46%

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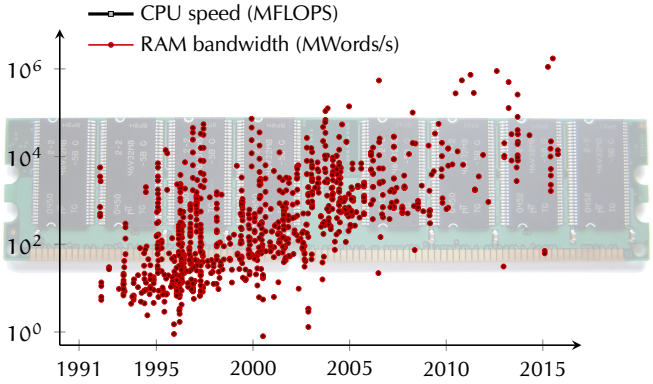

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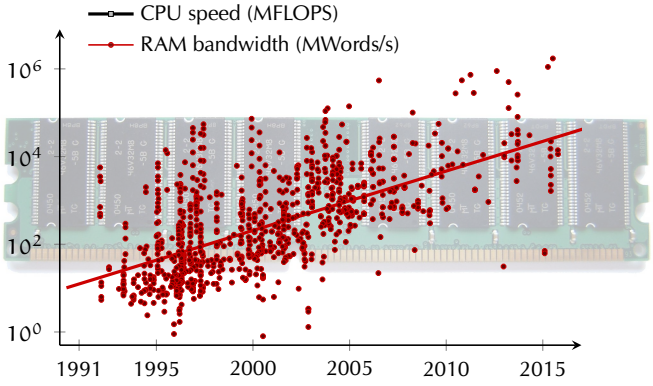

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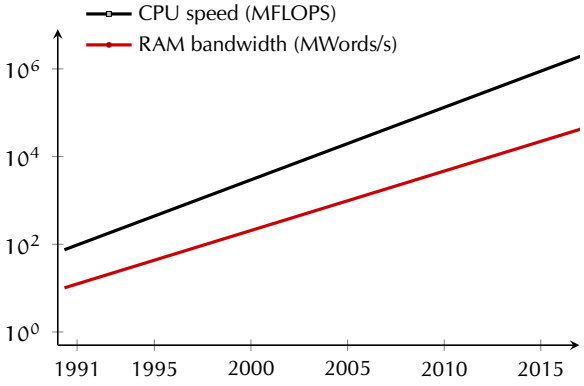

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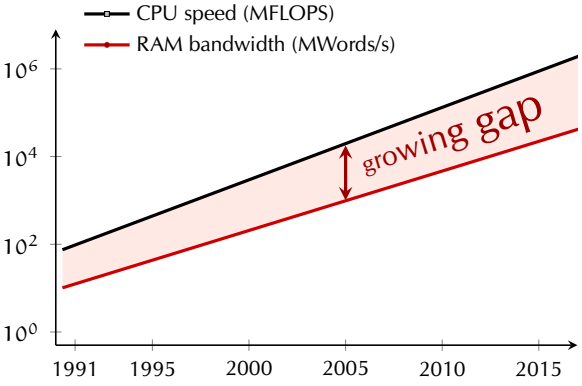

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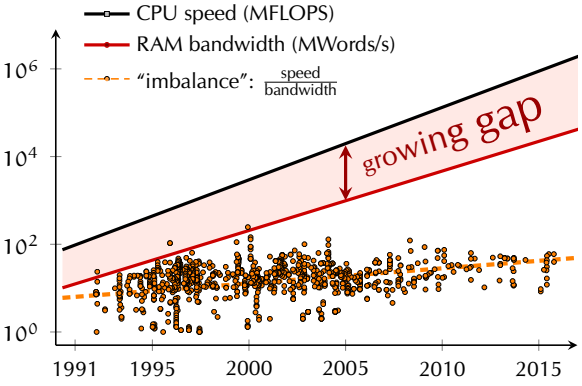

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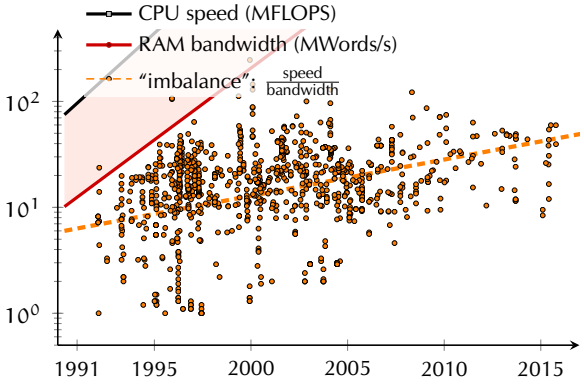

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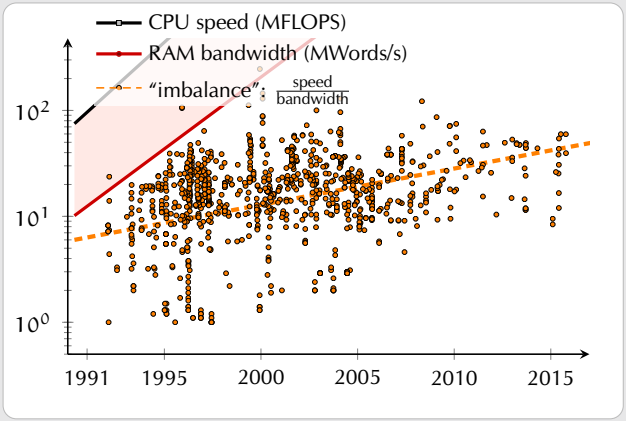

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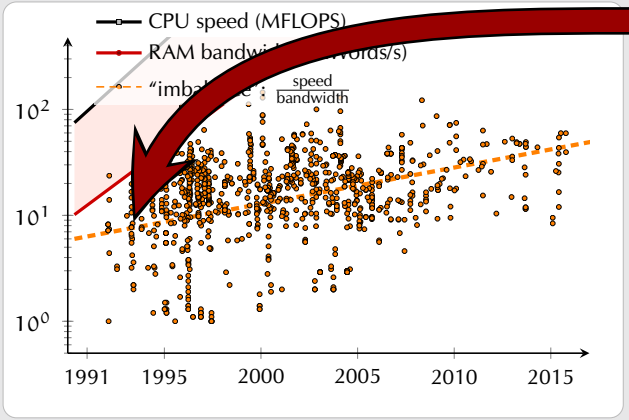

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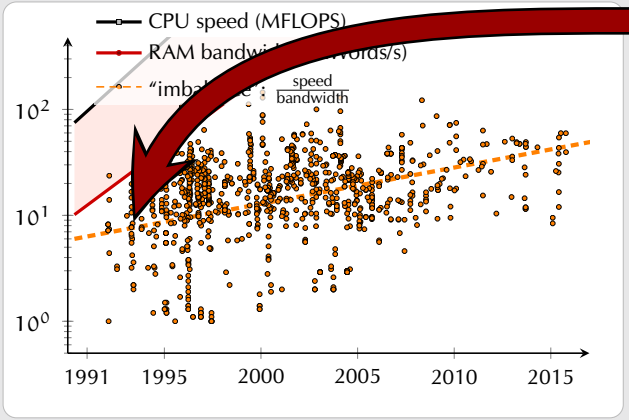

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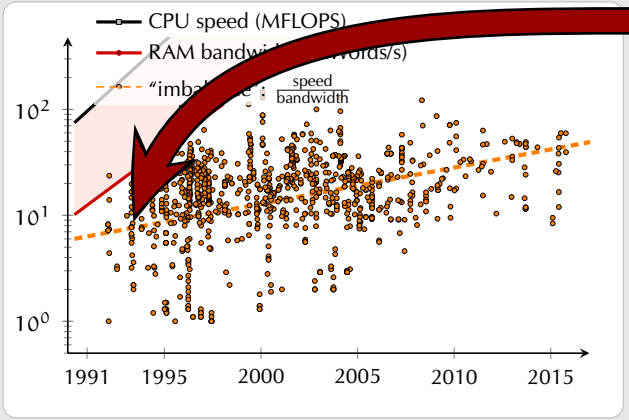

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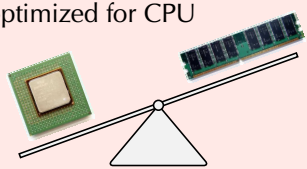

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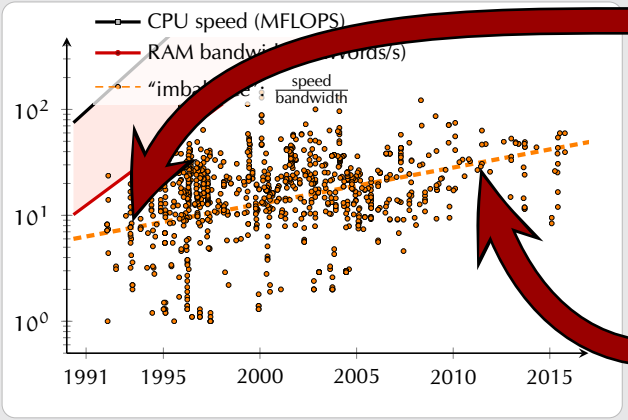

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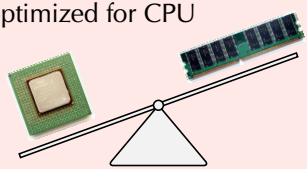

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Dual-Pivot Quicksort, **2009 – today**

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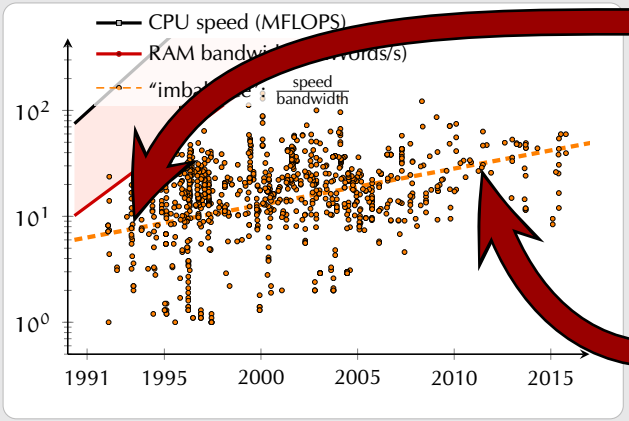

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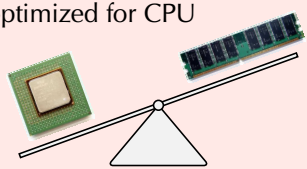

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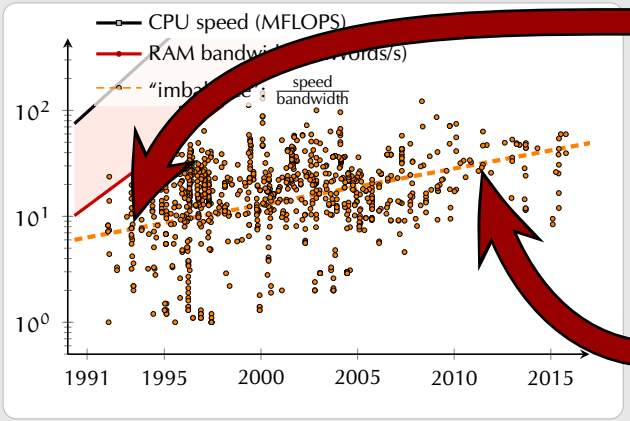

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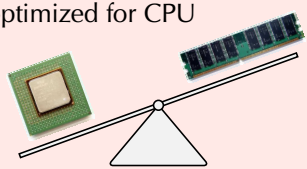

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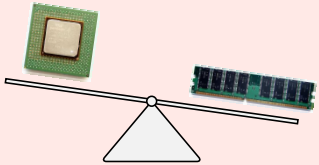

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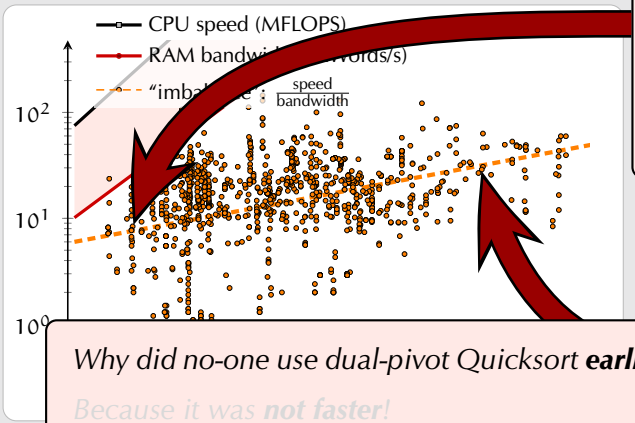

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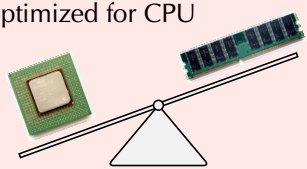

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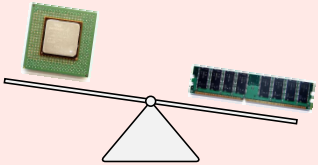

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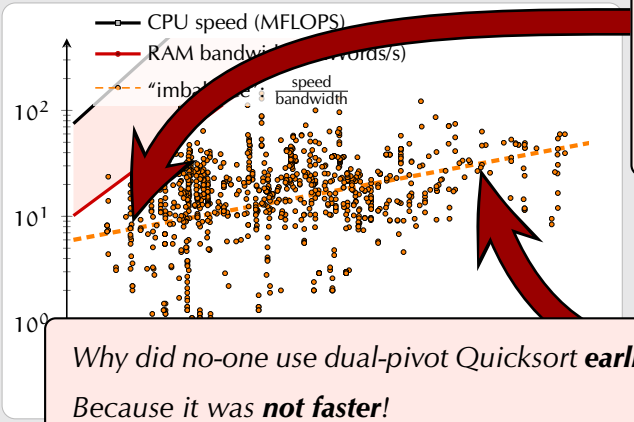

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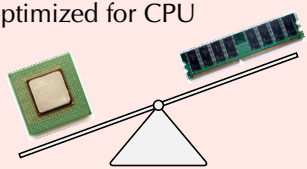

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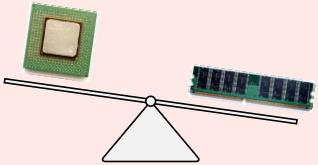

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How to analyze bandwidth consumption?

Need good cost measure!

Proposal: count ***scanned elements***

- Access to array only through **iterators**
  (no random access!)

- Iterators can
  - read left or right (one-directional!)
  - advance to next position
  - read and write current cell
- Cost: number of **advances**

≈ **cache misses**, but hardware independent

Costs of Partitioning

Classic Quicksort

⇝ n scanned elements
(array scanned once)

4-way Quicksort

⇝ $1.5n$ scanned elements
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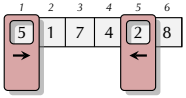

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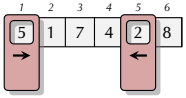

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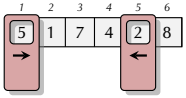

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Proposal: count ***scanned elements***

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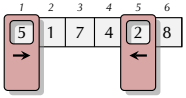

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Costs of Partitioning

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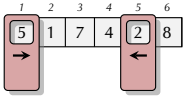

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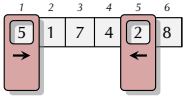

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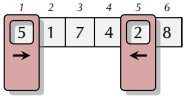

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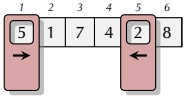

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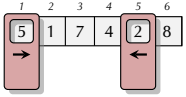

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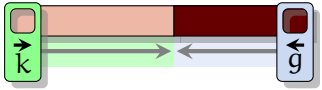

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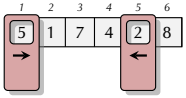

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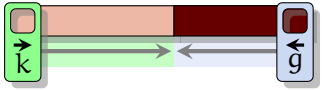

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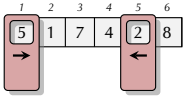

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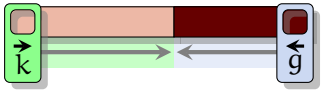

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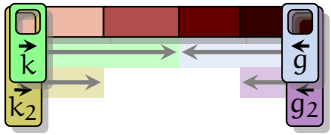
details on how this works in one pass → in dissertation



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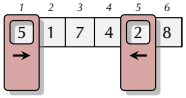

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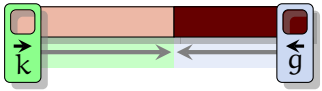

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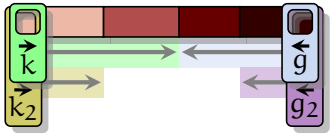

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(on average without sampling)

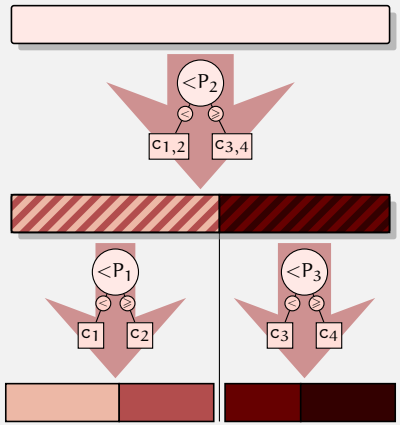
Scanned Elements

Ho

Ne

Pr

4-way simulated by 2-way

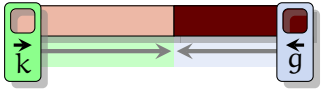


⇝ $2n$ scanned elements

≈ … dent

Costs of Partitioning

Classic Quicksort



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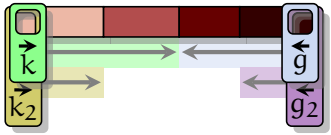

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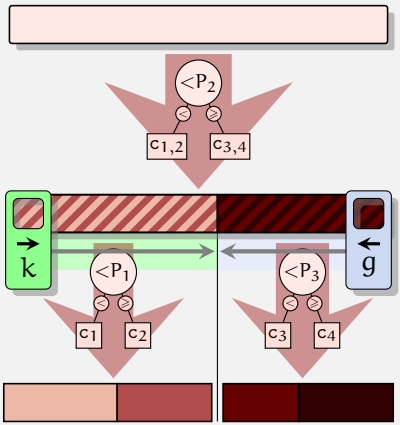

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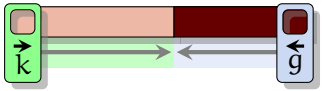

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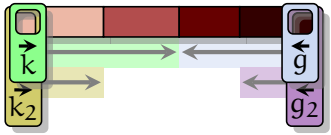

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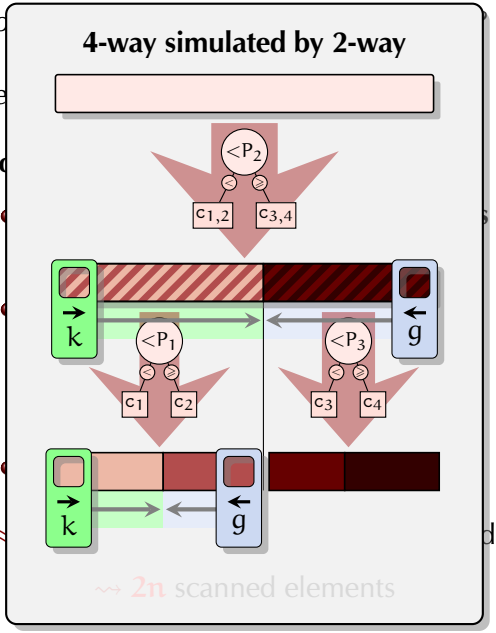

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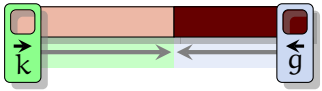

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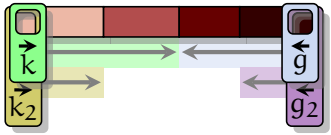

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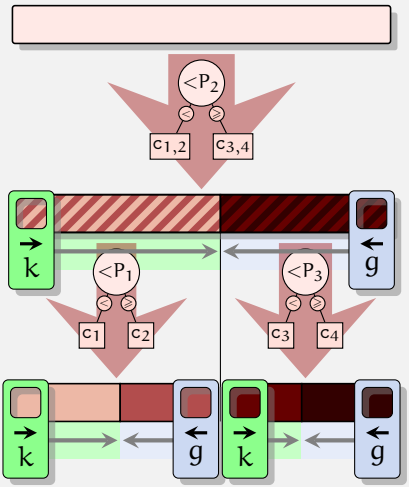

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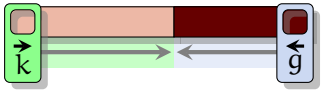

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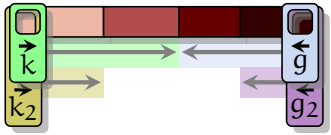

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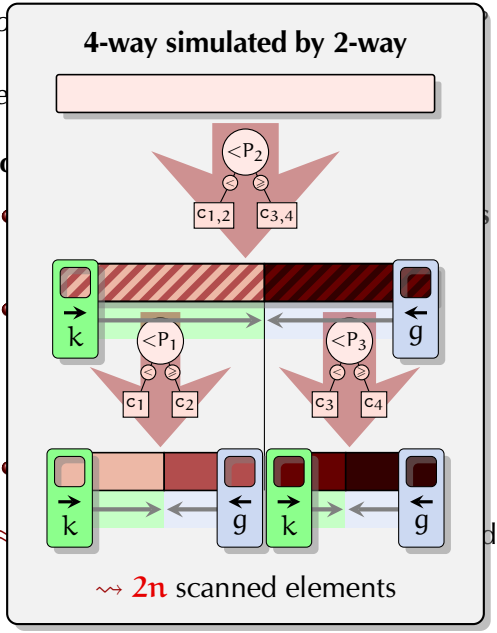

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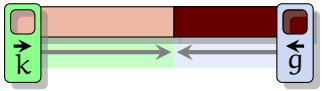

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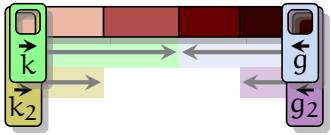

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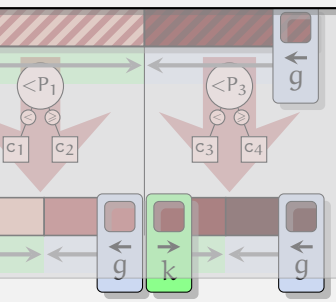

⇝ 2n scanned elements

Multiway uses much **less** **scanned elements**, irrespective of pivot sampling.

Costs of Partitioning

ksort



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4-way Quicksort

details on how this works in one pass → in dissertation



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Conclusion

You have seen:

My Thesis

1. *Multiway partitioning has genuine potential to speed up Quicksort.*

…namely: it reduces memory transfers.

2. Evidence suggesting high impact on running time.

3. Idea of mathematical analysis

$\mathcal{H}(C) = -\mathbb{P}[<] \log_2(\mathbb{P}[<]) - \mathbb{P}[>] \log_2(\mathbb{P}[>])$

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   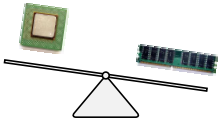

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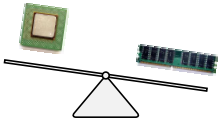

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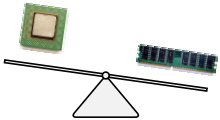

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- Using two comparison trees

- Branch mispredictions

- Treatment of equal keys

- Intriguing interplay of parameters

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   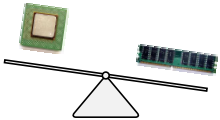

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  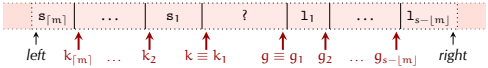

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   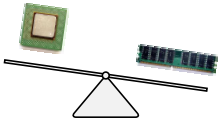

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  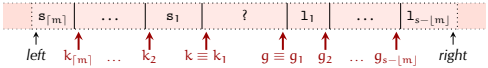

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  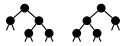

- Branch mispredictions

- Treatment of equal keys

- Intriguing interplay of parameters

Conclusion

You have seen:

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   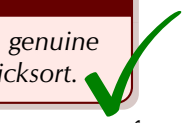

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   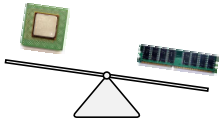

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   $\mathcal{H}(C) = -\mathbb{P}[<] \log_2(\mathbb{P}[<]) - \mathbb{P}[>] \log_2(\mathbb{P}[>])$

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  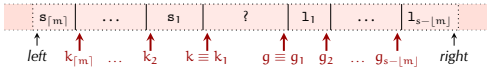

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  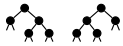

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  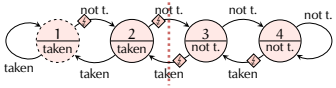

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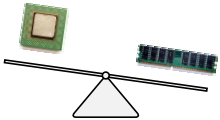

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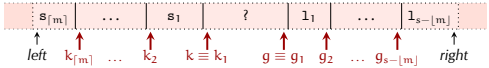

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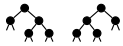

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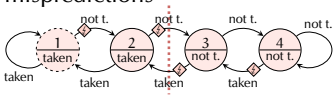

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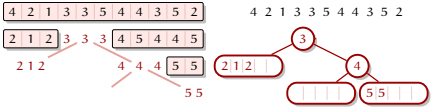

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   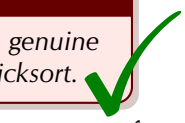

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   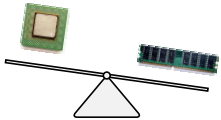

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  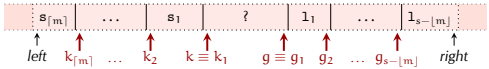

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  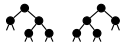

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  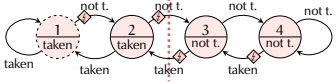

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  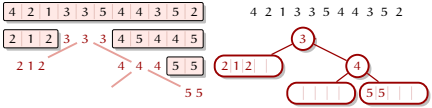

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  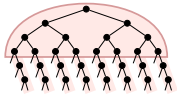



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