Quickselect under Yaroslavskiy's Dual-Pivoting Algorithm

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Menorca

Selection

Consider the following problem

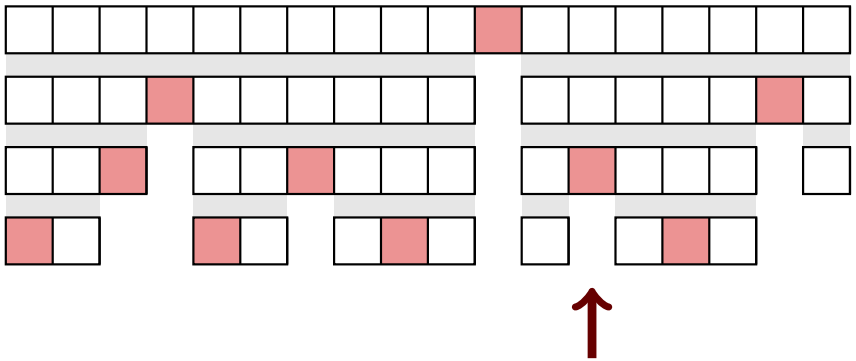
The Selection Problem

Given an **unsorted** array `A` of n data elements,
find the **m-th smallest** element.

- **Applications**
  - estimate distribution of data collection in statistics
  - world rankings in sports
- **Algorithms**
  - **trivial solution:** `sort(A); return A[m];` $\mathcal{O}(n \log n)$
  - linear worst case possible: *Median-of-Medians* (Blum et al.)
    but rather slow in practice (compared to sorting!)
  - Hoare 1961: **Quicksort** idea usable for selection

From Quicksort to Quickselect

Sort whole array by Quicksort

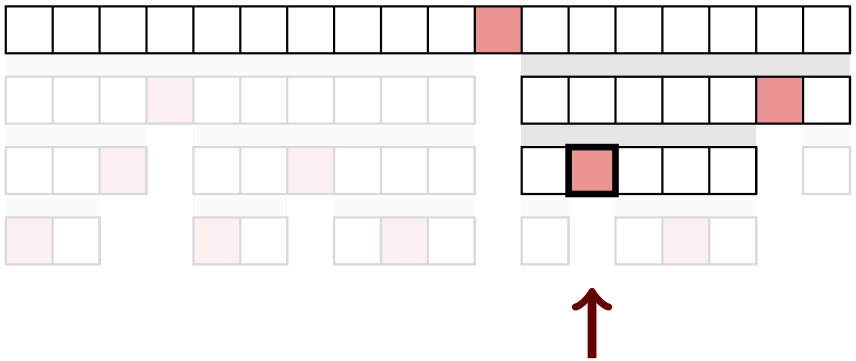


- Markus Nebel just showed:
  Yaroslavskiy's partitioning can **speed up Quicksort**

Does success of dual partitioning in sorting carry over to selection?

From Quicksort to Quickselect

Can prune many branches



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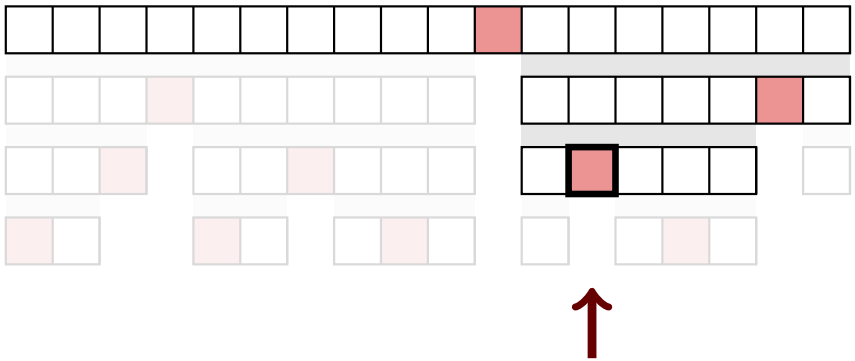
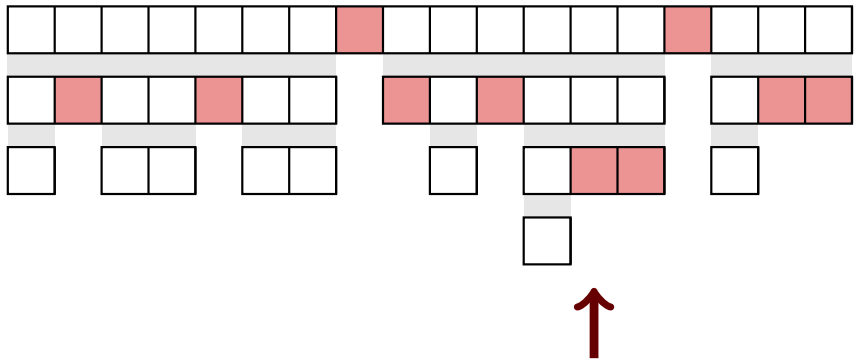


Dual Pivot Quicksort



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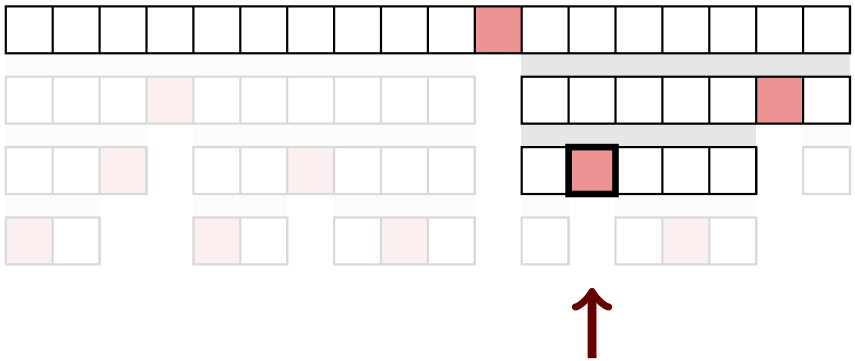
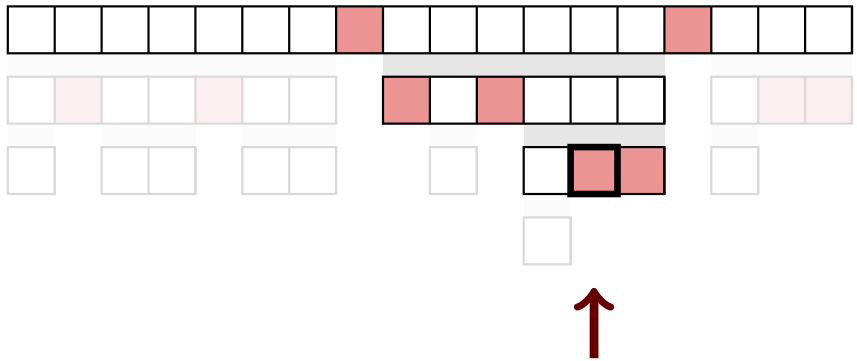


Prune even more branches



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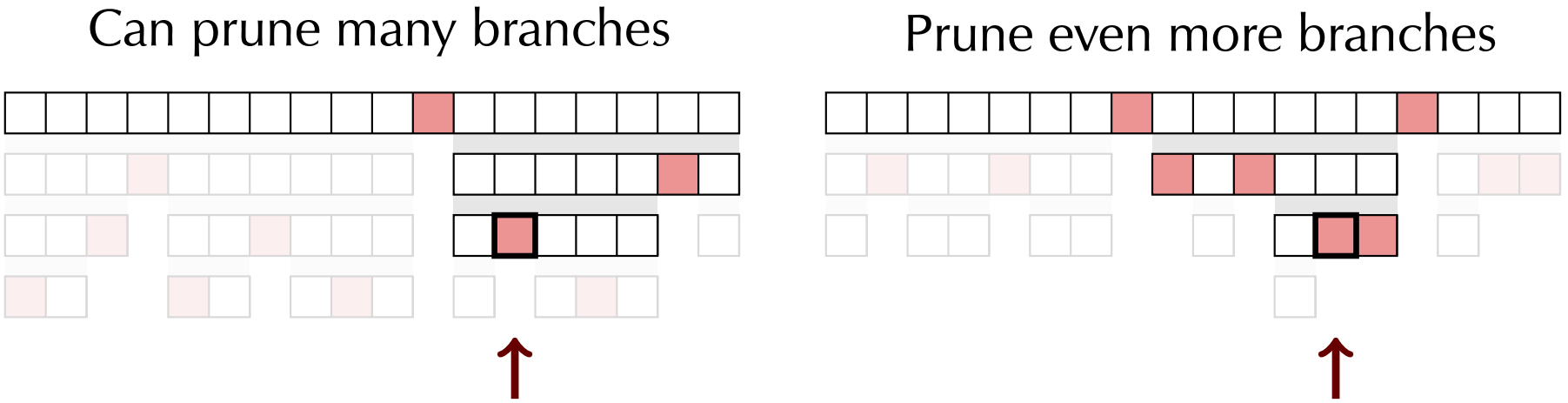

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Does success of dual partitioning in sorting carry over to selection?

Notation

- Analyse **(random)** number of data **comparisons** $C_n^{(m)}$ to select from random permutation

- $P < Q$: (random) **ranks** of the two pivots

$\rightsquigarrow$ **subarray sizes** determined by P and Q:

- selection continues recursively in

  - left subarray, if $m < P$
  - middle subarray, if $P < m < Q$
  - right subarray, if $Q < m$
  - or terminates if $m = P$ or $m = Q$

corresponding **events**

$\mathcal{E}_1 := \{m < P\}$
$\mathcal{E}_2 := \{P < m < Q\}$
$\mathcal{E}_3 := \{Q < m\}$

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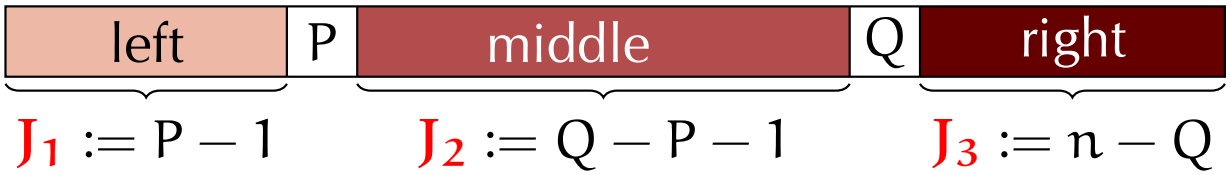

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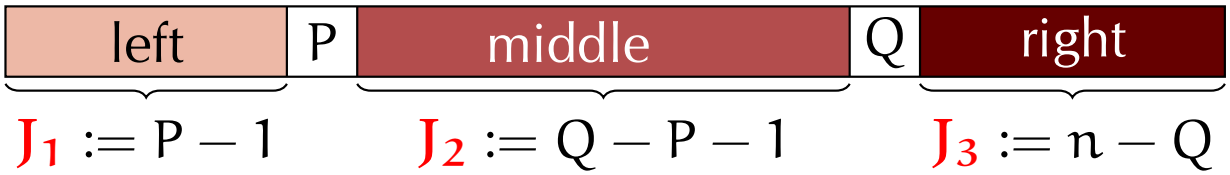

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Quickselect Recurrence

- Yaroslavskiy's partitioning preserves randomness

$\rightsquigarrow$ can set up **recurrence** for the **distribution** of $C_n^{(m)}$:

$$C_n^{(m)} \overset{\mathcal{D}}{=} T_n + \begin{cases} C_{J_1}^{(m)} & \text{if } m \text{ lies in left subarray} \\ C_{J_2}^{(m-P)} & \text{if } m \text{ lies in middle subarray} \\ C_{J_3}^{(m-Q)} & \text{if } m \text{ lies in right subarray} \end{cases}$$

$$= T_n + \mathbb{1}_{\mathcal{E}_1} \cdot C_{J_1}^{(m)} + \mathbb{1}_{\mathcal{E}_2} \cdot C_{J_2}^{(m-P)} + \mathbb{1}_{\mathcal{E}_3} \cdot C_{J_3}^{(m-Q)}$$

- T_n is the (random) number of comparisons during **partitioning**

Getting Rid of m

- **Complication:** **second** parameter m in recurrence

$$C_n^{(m)} \overset{\mathcal{D}}{=} T_n + \mathbb{1}_{\mathcal{E}_1} \cdot C_{J_1}^{(m)} + \mathbb{1}_{\mathcal{E}_2} \cdot C_{J_2}^{(m-P)} + \mathbb{1}_{\mathcal{E}_3} \cdot C_{J_3}^{(m-Q)}$$

$\rightsquigarrow$ consider simpler quantities

1. **Random Rank:** $m = M_n \overset{\mathcal{D}}{=} \mathit{Uniform}\{1, \dots, n\}$ $\leftarrow$ *focus in talk*
   abbreviate $\bar{C}_n := C_n^{(M_n)}$

   $\rightsquigarrow$ $\bar{C}_n \overset{\mathcal{D}}{=} T_n + \mathbb{1}_{\mathcal{E}_1} \bar{C}_{J_1} + \mathbb{1}_{\mathcal{E}_2} \bar{C}_{J_2} + \mathbb{1}_{\mathcal{E}_3} \bar{C}_{J_3}$

2. **Extreme Rank:** $m = 1$
   abbreviate $\hat{C}_n := C_n^{(1)}$ $\rightsquigarrow$ $\hat{C}_n \overset{\mathcal{D}}{=} T_n + \hat{C}_{P-1}$

- explicit solution for **expectations** possible (generating functions)
- **But:** Requires tedious computations

*More elegant shortcut to **asymptotics** of $\mathbb{E}[\bar{C}_n]$ and $\mathbb{E}[\hat{C}_n]$?*

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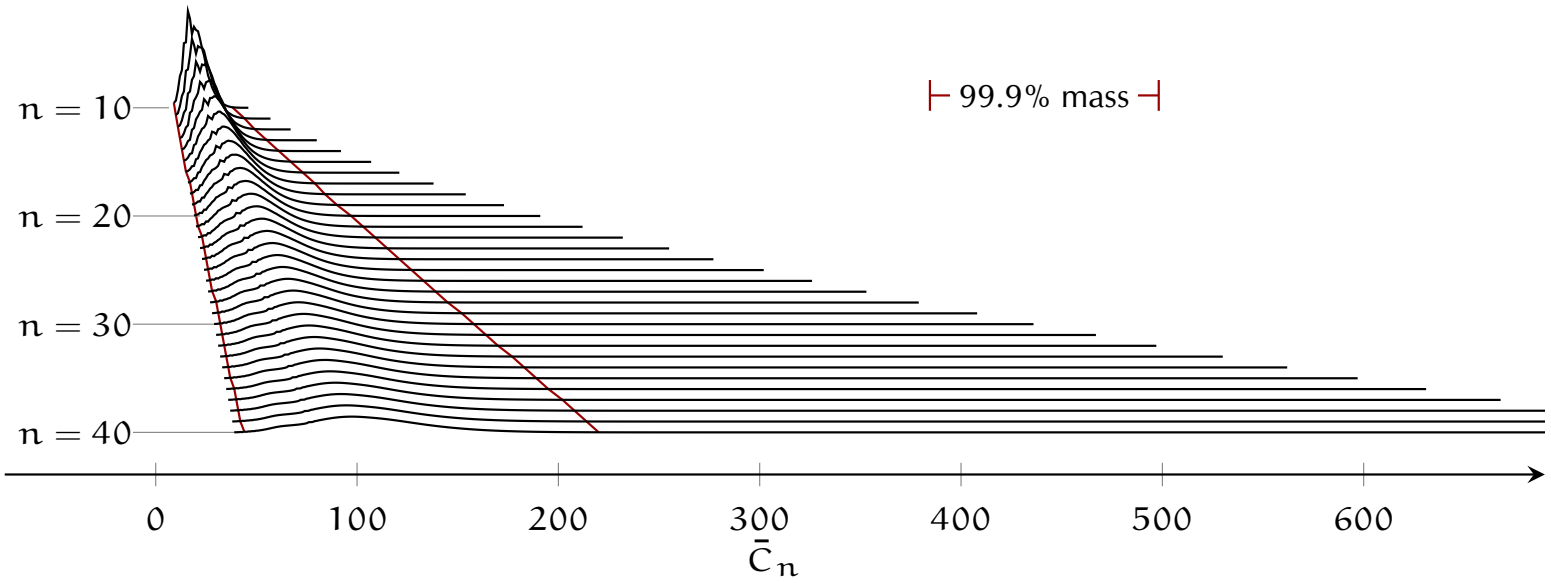
Convergence

- **Idea:** Computation easy if we can **drop** index n
  i. e. when recurrence **"in equilibrium"**

- For that, we need stochastic **convergence** of $\bar{C}_n$



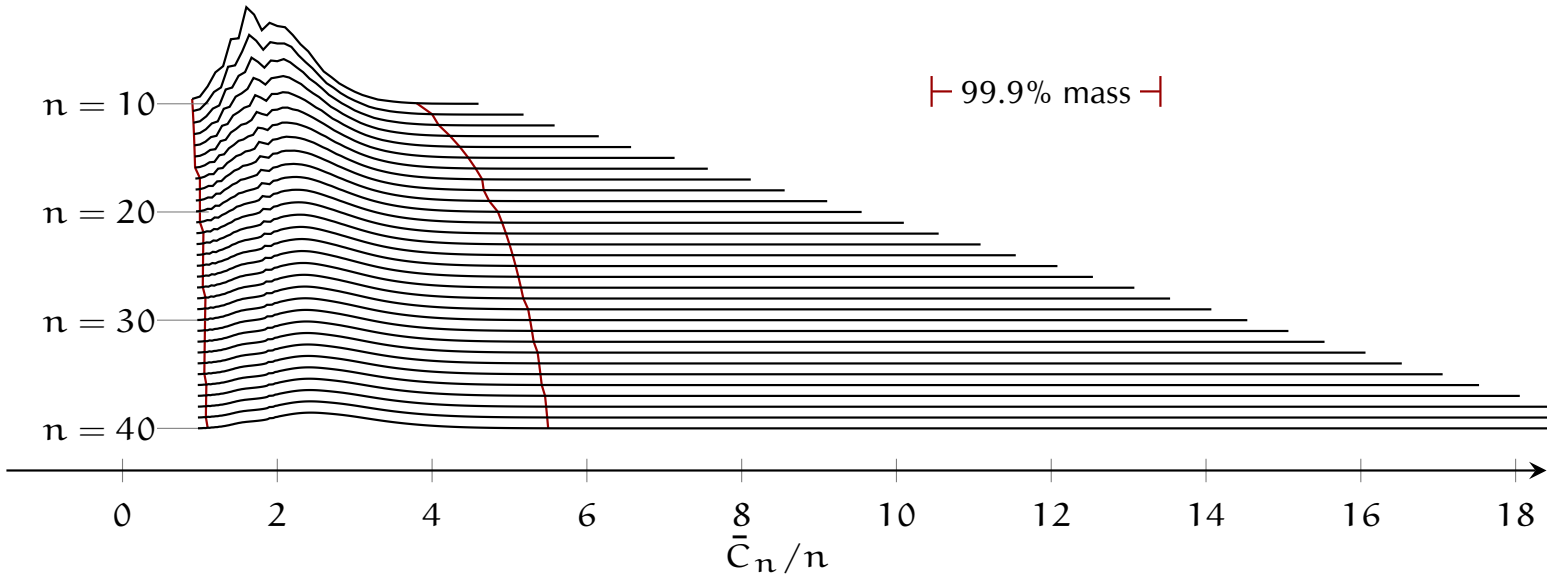
But:

- $\mathbb{E}[\bar{C}_n]$ grows with n
- high mass region moves with n

} $\bar{C}_n$ does **not** converge

Convergence

- **Idea:** Computation easy if we can **drop** index n
  i. e. when recurrence **"in equilibrium"**
- Try **normalized** variables $\bar{C}_n^* := \frac{\bar{C}_n}{n}$ instead:



- $\mathbb{E}[\bar{C}_n^*]$ seems constant
- range of high mass looks bounded

} $\bar{C}_n^*$ might converge

- Works because of property of Quickselect: $\mathbb{E}[\bar{C}_n] = \mathcal{O}\left(\sqrt{\mathrm{Var}(\bar{C}_n)}\right)$

The Contraction Method

- Rewrite recurrence: (Write as sum)

$$\bar{C}_n \overset{\mathcal{D}}{=} T_n + \mathbb{1}_{\varepsilon_1} \bar{C}_{J_1} + \mathbb{1}_{\varepsilon_2} \bar{C}_{J_2} + \mathbb{1}_{\varepsilon_3} \bar{C}_{J_3}$$

- **Contraction Method:**
  Convergence of coefficients + contraction condition implies convergence to fixpoint solution.

⇝ have to show: (all convergences in L_1-norm)

1. $T_n^* \to T^*$
2. $\mathbb{1}_{\varepsilon_i} \frac{J_i}{n} \to A_i^*$
3. $\sum_{i=1}^{3} \mathbb{E}[|A_i^*|] < 1$

The Contraction Method

- Rewrite recurrence: (Divide by n)

$$\bar{C}_n \overset{\mathcal{D}}{=} T_n + \sum_{i=1}^{3} \mathbb{1}_{\mathcal{E}_i} \bar{C}_{J_i}$$

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The Contraction Method

- Rewrite recurrence: (Rearrange)

$$\frac{\bar{C}_n}{n} \overset{\mathcal{D}}{=} \frac{T_n}{n} + \sum_{i=1}^{3} \frac{\mathbb{1}_{\mathcal{E}_i} \bar{C}_{J_i}}{n} \cdot \frac{J_i}{J_i}$$

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The Contraction Method

- Rewrite recurrence: (use normalized variables)

$$\frac{\bar{C}_n}{n} \overset{\mathcal{D}}{=} \frac{T_n}{n} + \sum_{i=1}^{3} \mathbb{1}_{\mathcal{E}_i} \frac{J_i}{n} \cdot \frac{\bar{C}_{J_i}}{J_i}$$

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The Contraction Method

- Recurrence for $\bar{C}^*_n$

$$\bar{C}^*_n \overset{\mathcal{D}}{=} T^*_n + \sum_{i=1}^{3} \mathbb{1}_{\mathcal{E}_i} \frac{J_i}{n} \cdot \bar{C}^*_{J_i} \qquad \text{with } T^*_n := \frac{T_n}{n}$$

- **Contraction Method:**
  Convergence of coefficients + contraction condition implies convergence to fixpoint solution.

$\rightsquigarrow$ have to show: (all convergences in L_1-norm)

1. $T^*_n \to T^*$
2. $\mathbb{1}_{\mathcal{E}_i} \frac{J_i}{n} \to A^*_i$
3. $\sum_{i=1}^{3} \mathbb{E}\left[|A^*_i|\right] < 1$

The Contraction Method

- Recurrence for $\bar{C}_n^*$

$$\bar{C}_n^* \overset{\mathcal{D}}{=} T_n^* + \sum_{i=1}^{3} \underbrace{\mathbb{1}_{\mathcal{E}_i} \frac{J_i}{n}}_{\downarrow} \cdot \bar{C}_{J_i}^*$$

$$\downarrow \qquad\qquad \downarrow$$

$$T^* \qquad\qquad A_i^*$$

- **Contraction Method:**
  Convergence of coefficients *+ contraction condition implies convergence to fixpoint solution.*

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$$T^* \qquad A_i^*$$

$$\sum_{i=1}^{3} \mathbb{E}\left[|A_i^*|\right] < 1$$

- **Contraction Method:**
  *Convergence of coefficients + **contraction condition** implies convergence to fixpoint solution.*

$\rightsquigarrow$ have to show: (all convergences in L_1-norm)

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The Contraction Method

- Recurrence for $\bar{C}^*_n$

$$\begin{array}{ccccc} \bar{C}^*_n & \stackrel{\mathcal{D}}{=} & T^*_n + \sum_{i=1}^{3} \underbrace{\mathbb{1}_{\mathcal{E}_i} \frac{J_i}{n}} \cdot \bar{C}^*_{J_i} \\ \downarrow & & \downarrow \quad \downarrow \quad \downarrow \\ \bar{C}^* & \stackrel{\mathcal{D}}{=} & T^* + \sum_{i=1}^{3} A^*_i \cdot \bar{C}^* \end{array} \qquad \sum_{i=1}^{3} \mathbb{E}\left[|A^*_i|\right] < 1$$

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- **"Equilibrium" equation for $\bar{C}^*$**

$$\bar{C}^* \quad \stackrel{\mathcal{D}}{=} \quad T^* + \sum_{i=1}^{3} A_i^* \quad \cdot \bar{C}^*$$

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The Contraction Method

- **"Equilibrium" equation for $\bar{C}^*$**

  (Take **expectation** on both sides)

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The Contraction Method

- **"Equilibrium" equation for $\bar{C}^*$**

  (and **solve** for $\mathbb{E}\,\bar{C}^*$)

$$\mathbb{E}\,\bar{C}^* \;=\; \mathbb{E}\,T^* + \sum_{i=1}^{3} \mathbb{E}\,A_i^* \cdot \mathbb{E}\,\bar{C}^*$$

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- **"Equilibrium" equation for $\bar{C}^*$**

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$$\mathbb{E}\,\bar{C}^* = \frac{\mathbb{E}\,T^*}{1 - \sum_{i=1}^{3} \mathbb{E}\,A_i^*}$$

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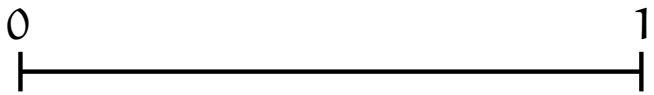
2. $\mathbb{1}_{\mathcal{E}_i}\frac{J_i}{n} \to A_i^*$ ... *upcoming*
3. $\sum_{i=1}^{3}\mathbb{E}\big[|A_i^*|\big] < 1$

Probabilistic Model

Need "convenient" probabilistic model:
same probability space *for finite* n *and limit case*

- **Natural model:** input elements $U_1, \dots, U_n$ **i. i. d.** ***Uniform***$(0, 1)$



first two = pivots values

- **Our equivalent model:**
  1. Draw spacings S_1, S_2, S_3 on $(0, 1)$ with $S \overset{D}{=} \mathit{Dirichlet}(1, 1, 1)$
  2. Draw subarray sizes J_1, J_2, J_3 with $J \overset{D}{=} \mathit{Multinomial}(n - 2; S_1, S_2, S_3)$
  3. Continue recursively

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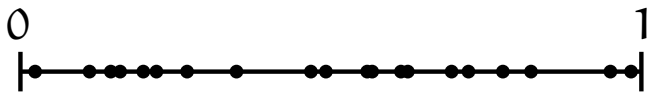

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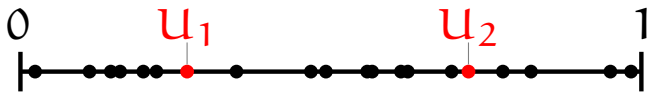

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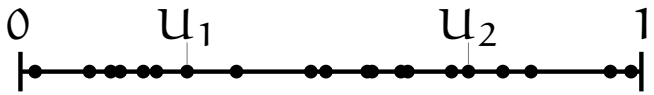

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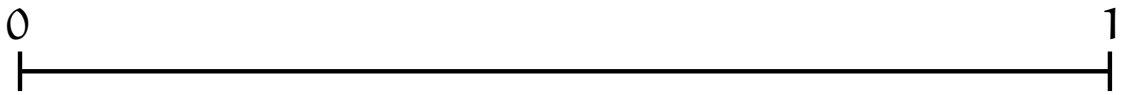

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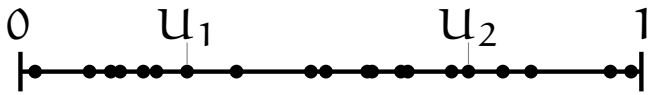

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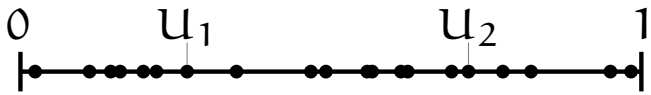

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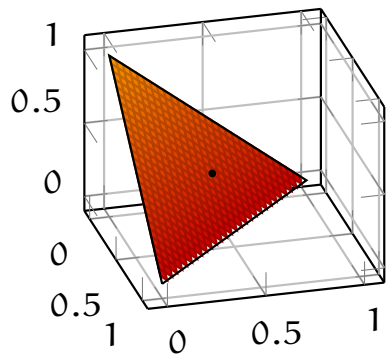

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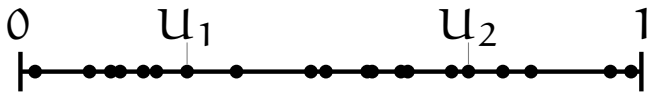

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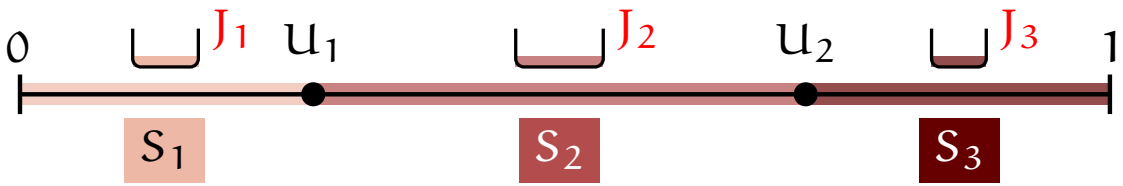

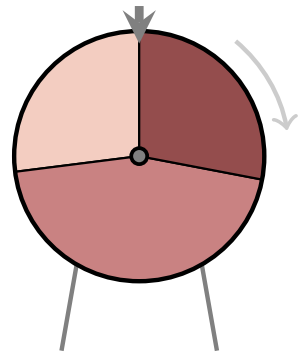

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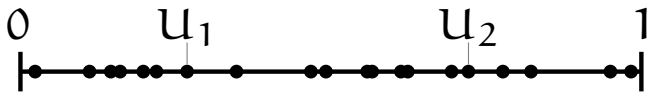

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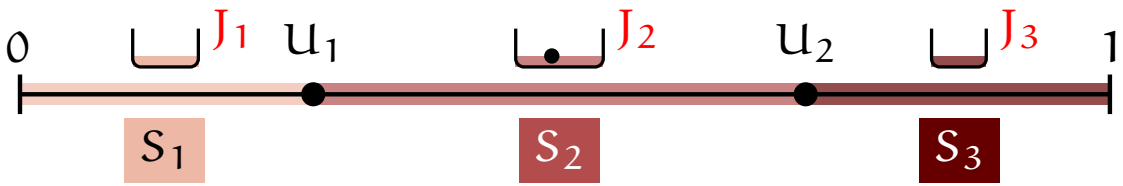

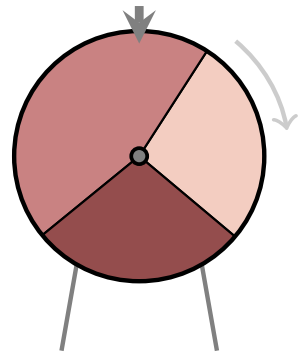

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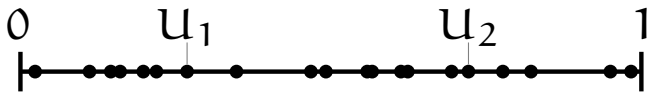

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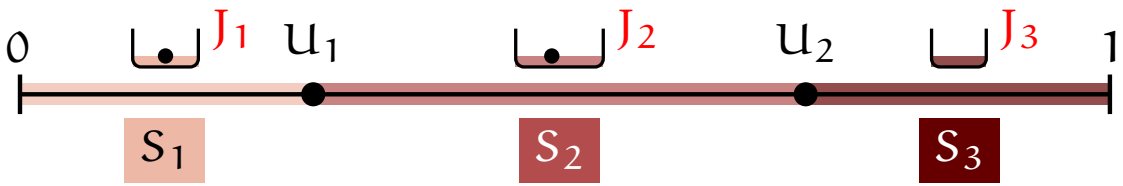

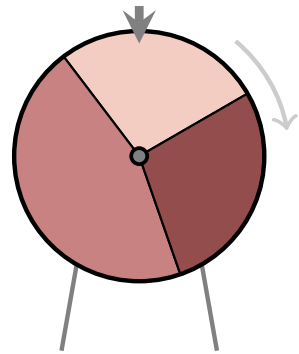

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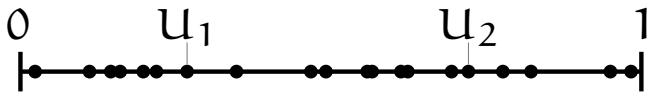

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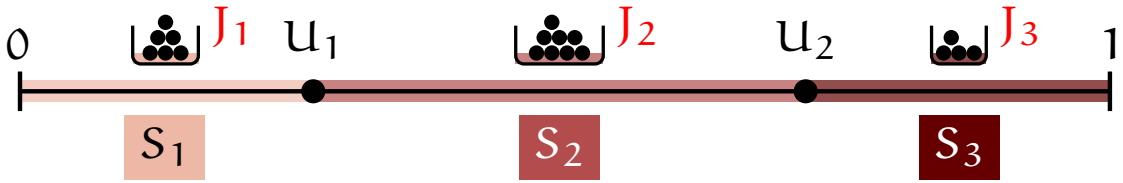

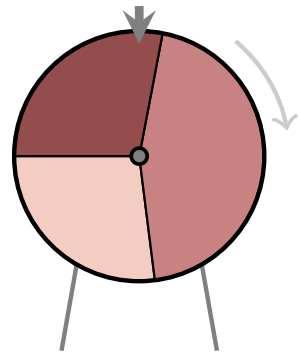

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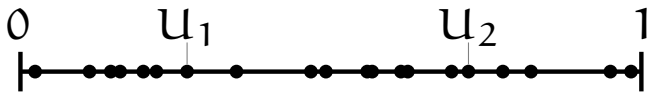

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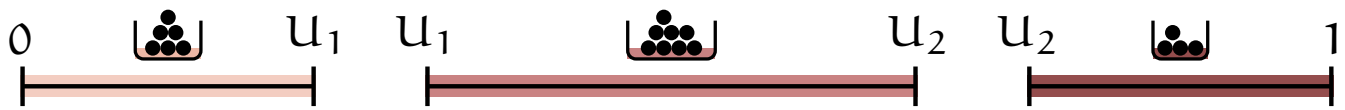

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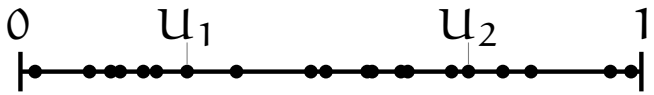

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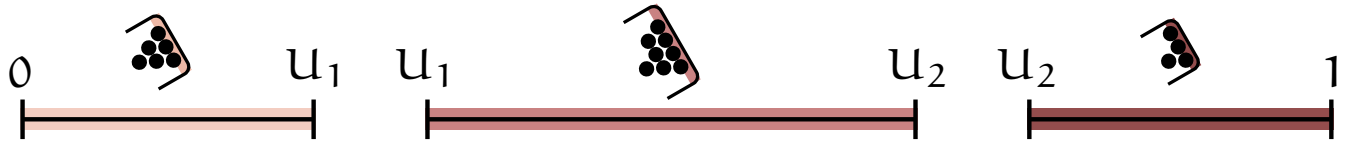

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Distribution of T_n — Repetition

1. $T_n^* \to T^*$

Invariant:

T_n = number of comparisons in **first partitioning step**

$\sim$ n one for every element

$+ J_2$ second for **all** medium elements

$+ \ell @ \mathcal{K}$ second for large elements at c'_k

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- Markus Nebel just showed:

$$\left.\begin{aligned} \ell @ \mathcal{K} &\overset{\mathcal{D}}{=} \mathit{Hypergeometric}(n-2;\ J_3,\ J_1 + J_2) \\ s @ \mathcal{G} &\overset{\mathcal{D}}{=} \mathit{Hypergeometric}(n-2;\ J_1,\ J_3) \end{aligned}\right\} \text{conditional on } \mathbf{J}$$

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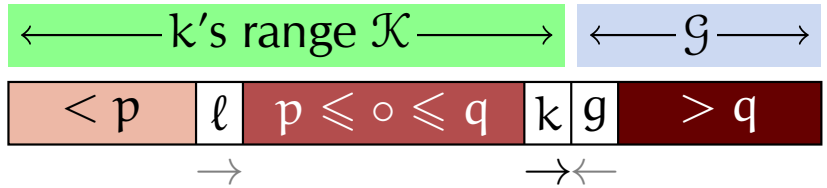

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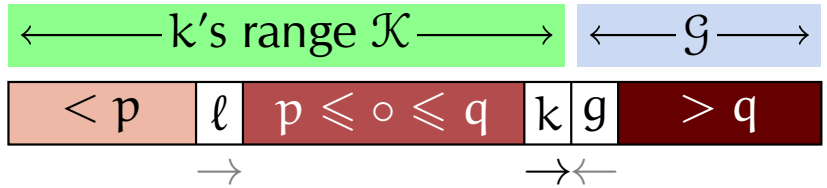

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- $\frac{J_2}{n} \to S_2$ by **law of large numbers**, since $\mathbb{E}[J_i \mid \mathbf{S}] = n \cdot S_i$
- Multinomial and Hypergeometric **concentrated** around mean

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Convergence of Coefficients & Contraction Condition

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- standard computation:
  $\mathbb{1}_{\mathcal{E}_1} \frac{J_1}{n} \to \mathbb{1}_{\mathcal{F}_1} \cdot S_1 \quad \text{with } \mathcal{F}_1 = \{V < S_1\}$
- $i = 2, 3$ similar

3. $\sum_{i=1}^{3} \mathbb{E}\left[|A_i^*|\right] < 1$

- $\mathbb{E}\left[\mathbb{1}_{\mathcal{F}_1} S_1\right] = \frac{1}{6}$

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Computing expectations and plugging in . . .

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Recall:

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- similarly: $\mathbb{E}\, \hat{C}_n \sim 2.375n$

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- $\sum_{i=1}^{3} \mathbb{E}\, A_i^* = \frac{1}{2}$

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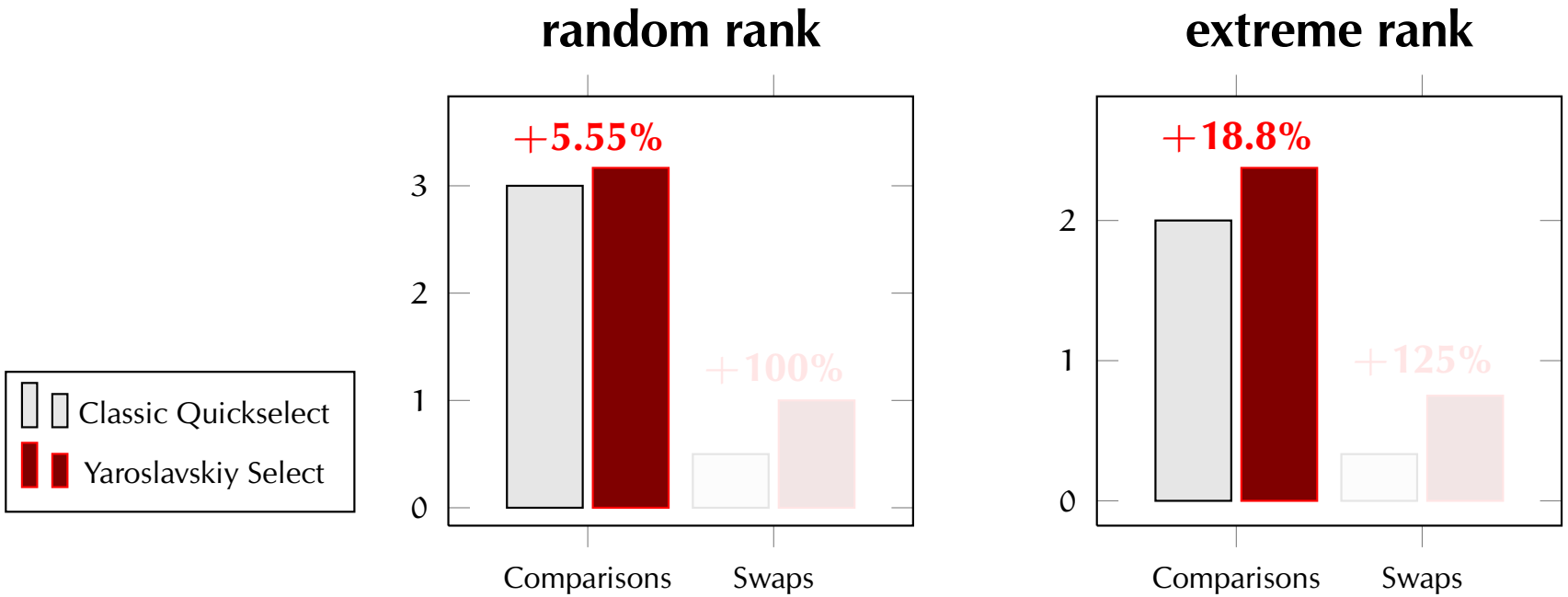
- L_1 convergence $\rightsquigarrow$ $\mathbb{E}\, \bar{C}_n \sim 3.1\overline{6}n$
- similarly: $\mathbb{E}\, \hat{C}_n \sim 2.375n$



Results — Expectations

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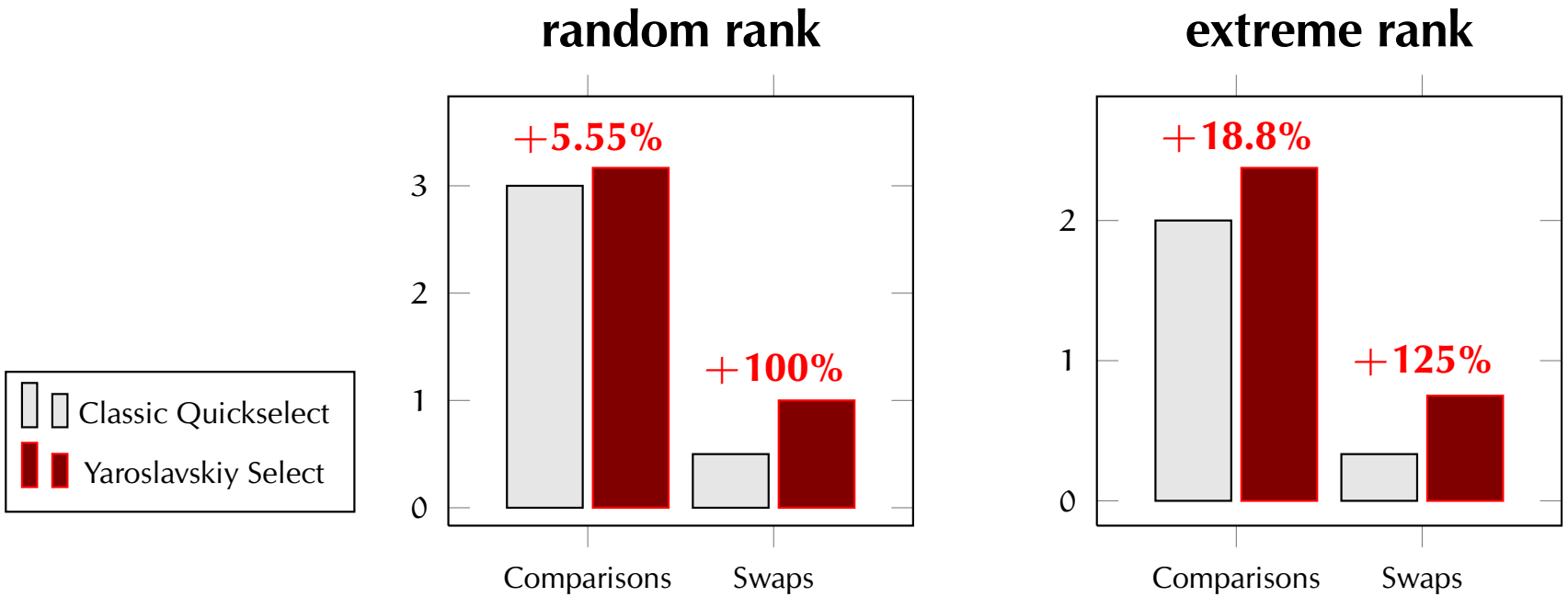

Pivot Sampling

- successful optimization of Quicksort/-select: **median of three**
- **Here:** generalized pivot selection scheme
  1. Choose random sample of k elements
  2. **Sort** the sample
  3. Select pivots s. t. in sorted sample

     t_1 smaller,

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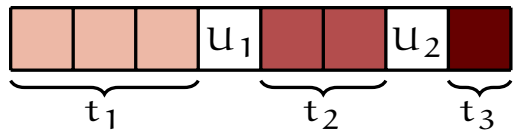
Example with
$k = 8$ and $\mathbf{t} = (3, 2, 1)$:



Our stochastic model nicely **generalizes** to pivot sampling:

- **only** the **distribution of spacings** S changes!

$$S \overset{\mathcal{D}}{=} \mathit{Dirichlet}(t_1 + 1, t_2 + 1, t_3 + 1) \qquad \textit{density: } f(s) = \frac{s_1^{t_1} s_2^{t_2} s_3^{t_3}}{B}$$

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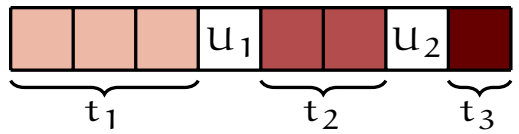

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- **General result:** $\bar{C}_n \sim \dfrac{1 + \frac{t_2+1}{k+1} + \frac{(2(t_1+1)+(t_2+1))(t_3+1)}{(k+1)^{\overline{2}}}}{1 - \sum_{i=1}^{3} \frac{(t_i+1)^{\overline{2}}}{(k+1)^{\overline{2}}}} \cdot n$

  inconvenient for interpretation . . .

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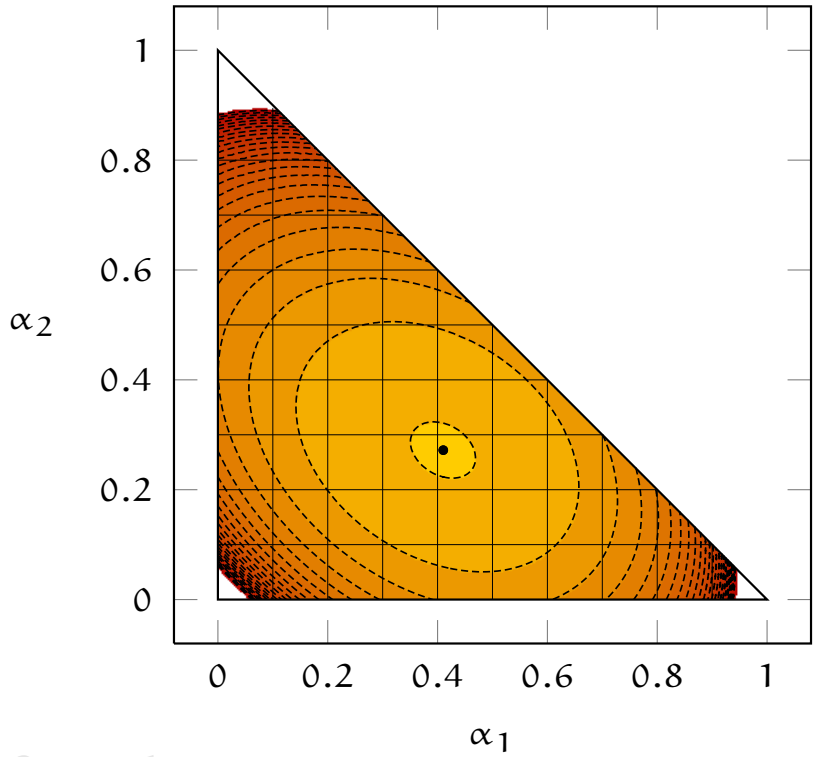

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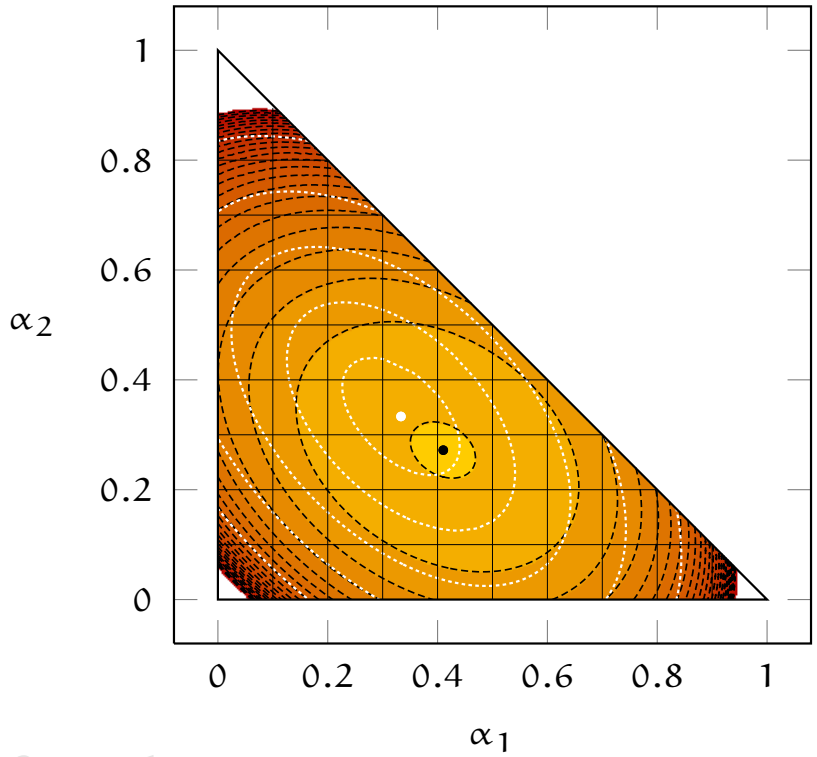

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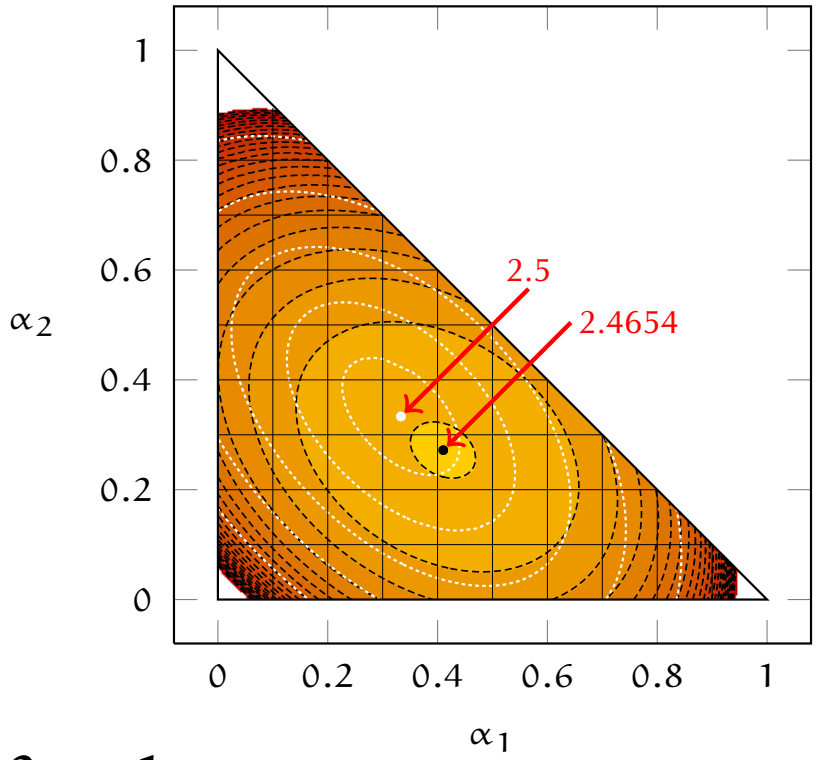

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Conclusion

Practitioners' Lesson:

- Yaroslavskiy's dual-pivoting **not** helpful for selection

Analyzers' Lesson:

- **expected** costs of Quickselect derivable by **contraction method**
- pivot sampling included naturally

References & Further Reading

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