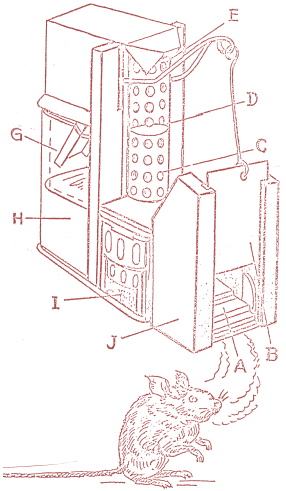
Dual-Pivot Quicksort and Beyond



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KAISERSLAUTERN

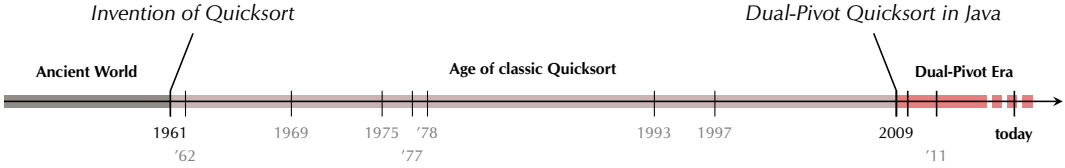
Forschungsseminar
Hasso-Plattner-Institut

6. September 2016

It is tempting to try to develop ways to improve quicksort:
a faster sorting algorithm is computer science's "better mousetrap,"
and quicksort is a venerable method that seems to invite tinkering.

R. Sedgewick and K. Wayne, Algorithms 4th ed.

Sorting History



Sorting History

1961,62 **Hoare:** publication, first analysis

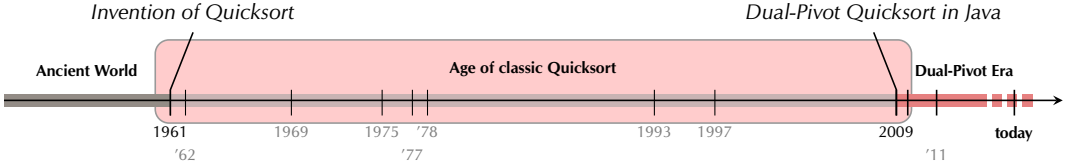
1969 **Singleton:** median-of-three & Insertionsort on small subarrays

1975-78 **Sedgewick:** analysis of many optimizations

1993 **Bentley, McIlroy:** duplicate elements & "ninther"

1997 **Musser:** $\mathcal{O}(n \log n)$ worst case by truncating recursion

$\rightsquigarrow$ Basic algorithm settled since 1961; latest tweaks from 1990's.



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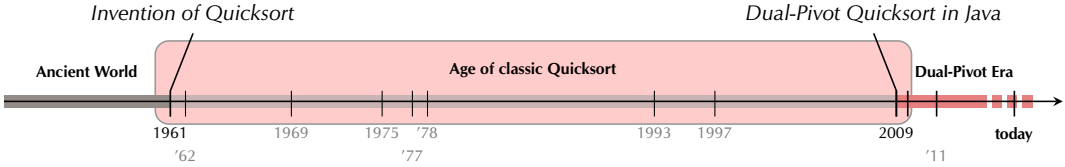

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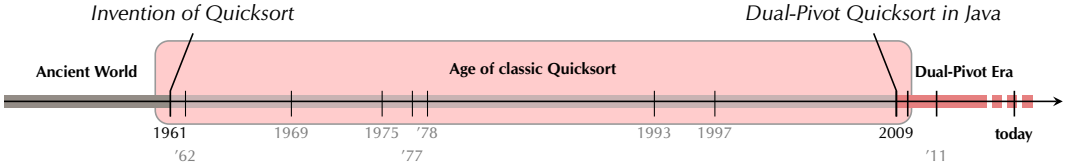
Since then: Almost identical in all programming libraries!



Sorting History

2008 – 2009 **Vladimir Yaroslavskiy** (developer at Sun)
experiments with Quicksort with **two pivots**

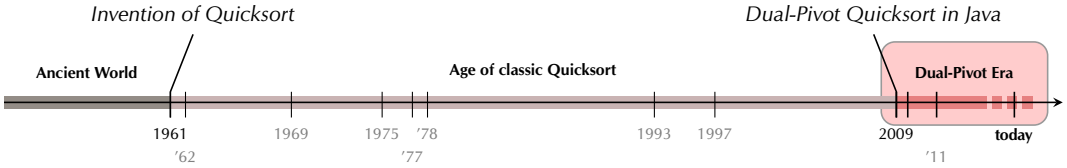
2009 – 2011 optimizations by Joshua Bloch, Jon Bentley and others

11 Sep 2009 announcement on Java core library mailing list

29 Oct 2009 **inclusion** in development version of OpenJDK

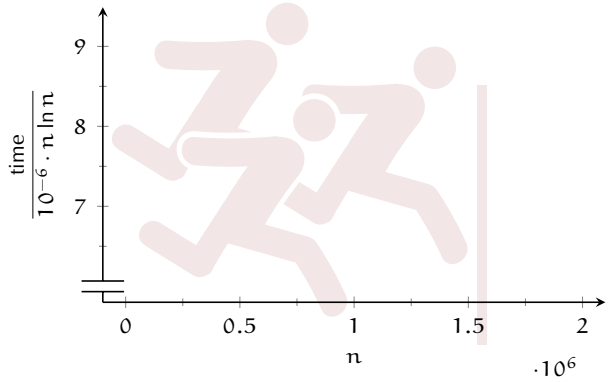
28 July 2011 **public release** of Java 7 with ***Yaroslavskiy-Bentley-Block (YBB) Quicksort***

(same in Java 8)



Running Time Experiments

Why change a well-tested implementation?



Normalized Java runtimes (in *ms*).

Average of 1000 random permutations per size.

- YBB Quicksort significantly faster!
- . . . just new engineering tricks?

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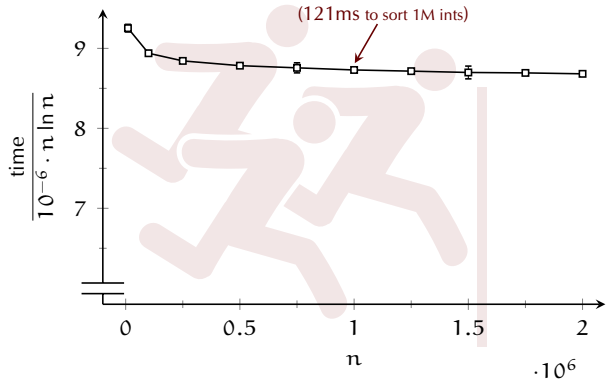

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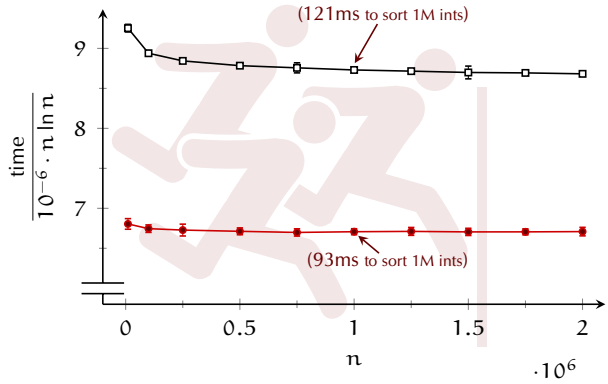

Java 6 Library
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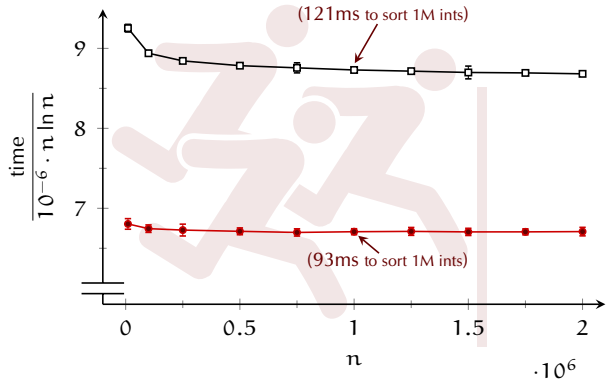

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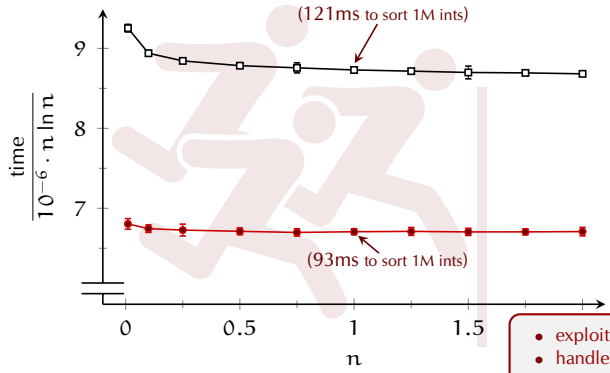

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- handle equal items efficiently
- Insertionsort for small subproblems (actually pair insertion)
- choose good pivots
- use sentinel elements
- low-level optimizations

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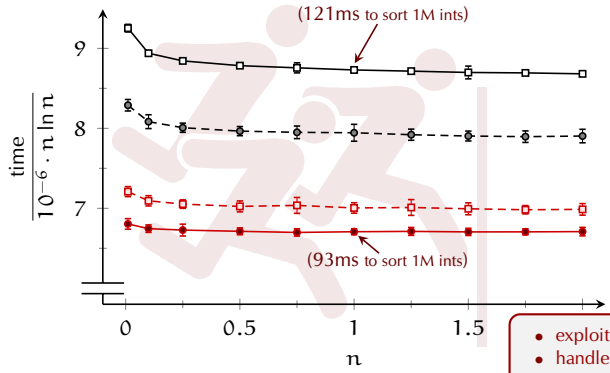

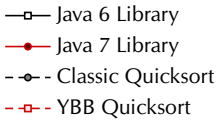

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- **No:** remains true for **basic** variants of algorithms: -●- vs. -□-!

Dual-Pivot Quicksort: Conceptual View

Single-Pivot Quicksort

1. Choose pivots P.
2. For each element x, determine its **class**
   - small for $x < P$
   - large for $P < x$

   by comparing x to P.
3. Arrange elements according to classes:

4. Sort subarrays recursively.

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How to do ❸ efficiently and in place? ⇝ *YBB Partitioning*

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Basic Algorithms

ClassicQuicksort(A, *left*, *right*)

```
1  if right ⩽ left then return
2  P = A[right]
3  k = left − 1;  g = right
4  while true
5      do k = k + 1 while A[k] < P end while
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15         end if
16     end if
17     k = k + 1
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19 ℓ = ℓ − 1;   g = g + 1
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- **Classic Quicksort**
  - crossing-pointer scheme
  - symmetric
- **YBB Quicksort**
  - ...looks messy

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9  end while
10 Swap A[k] and A[right]
11 ClassicQuicksort(A,  left  , k − 1)
12 ClassicQuicksort(A, k + 1,  right )
```

- **Classic Quicksort**
  - crossing-pointer scheme 
  - symmetric 
- **YBB Quicksort**
  - . . . looks messy

YBBQuicksort(A, *left*, *right*)

```
1  if right ⩽ left then return
2  P = A[left];   Q = A[right]
3  if P > Q then Swap P and Q end if
4  ℓ = left + 1;   g = right − 1;   k = ℓ
5  while k ⩽ g
6      if A[k] < P
7          Swap A[k] and A[ℓ] ; ℓ = ℓ + 1
8      else if A[k] ⩾ Q
9          while A[g] > Q and k < g
10             g = g − 1
11         end while
12         Swap A[k] and A[g] ;  g = g − 1
13         if A[k] < P
14             Swap A[k] and A[ℓ] ;  ℓ = ℓ + 1
15         end if
16     end if
17     k = k + 1
18 end while
19 ℓ = ℓ − 1;   g = g + 1
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23 YBBQuicksort(A, g + 1,  right )
```

Basic Algorithms

ClassicQuicksort(A, *left*, *right*)

```
1  if right ⩽ left then return
2  P = A[right]
3  k = left − 1;  g = right
4  while true
5      do k = k + 1 while A[k] < P end while
6      do g = g − 1 while A[g] > P end while
7      if k ⩾ g then break while end if
8      Swap A[k] and A[g]
9  end while
10 Swap A[k] and A[right]
11 ClassicQuicksort(A,  left  , k − 1)
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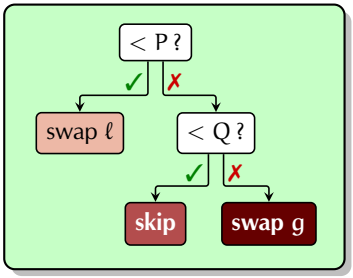
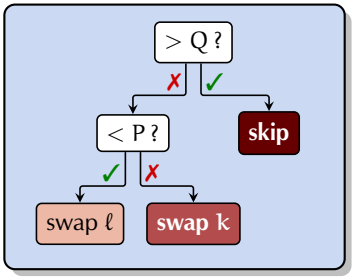
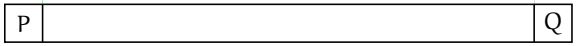
YBB Partitioning Example







Invariant:

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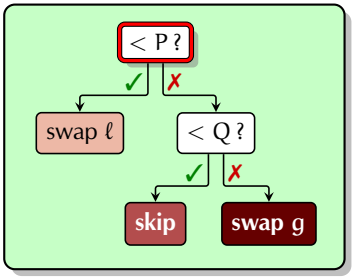
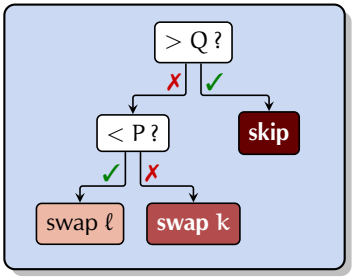
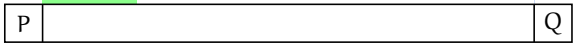






Invariant:

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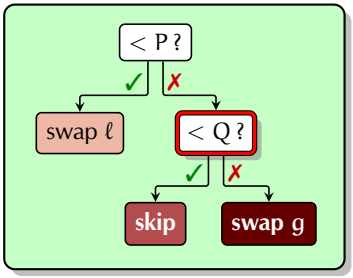
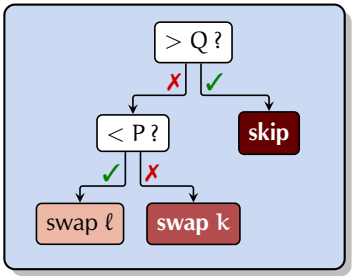
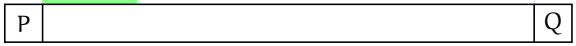
YBB Partitioning Example







Invariant:

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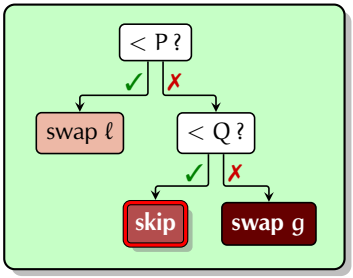
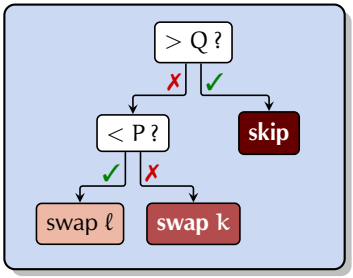
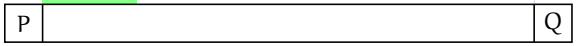






Invariant:

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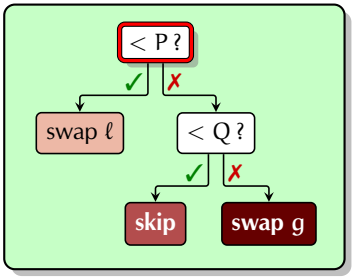
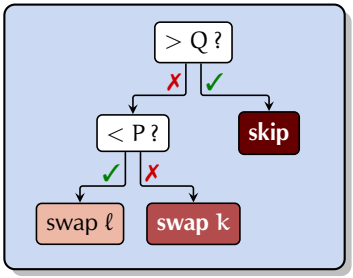
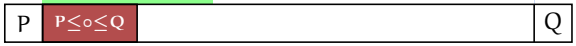






Invariant:

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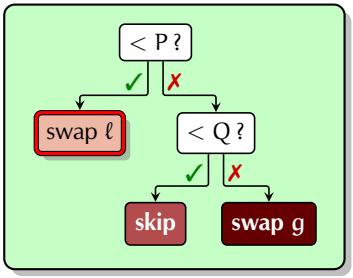

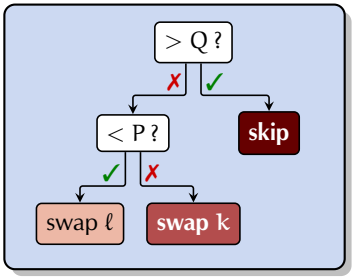



Invariant: P | P≤∘≤Q | Q

```
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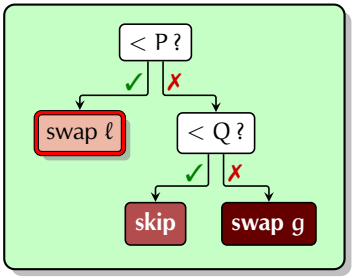

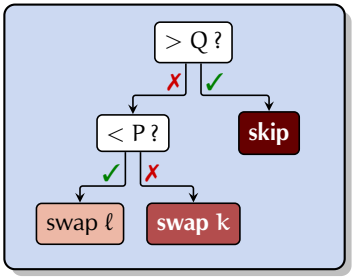



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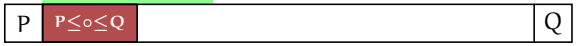


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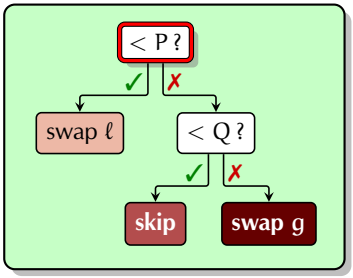

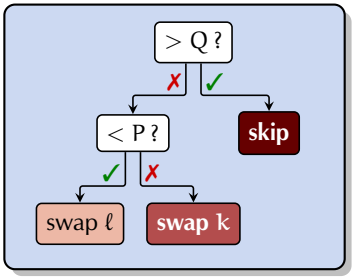


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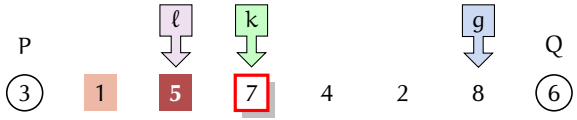

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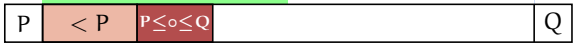

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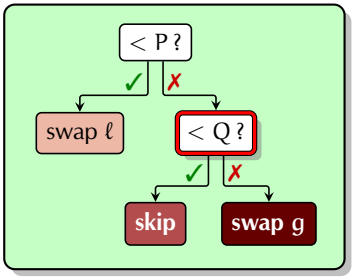

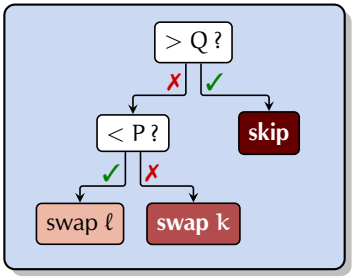


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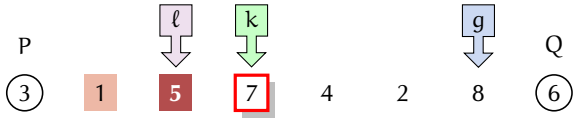
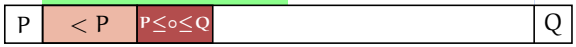
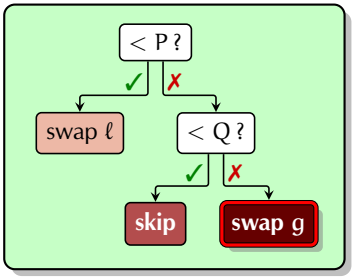
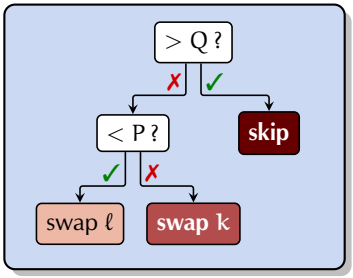


Invariant:



YBB Partitioning Example



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15          end if
16      end if
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19  ℓ = ℓ − 1;   g = g + 1
20  Swap A[left] and A[ℓ] ;   Swap A[right] and A[g]
21  YBBQuicksort(A, left , ℓ − 1)
22  YBBQuicksort(A, ℓ + 1, g − 1)
23  YBBQuicksort(A, g + 1, right )
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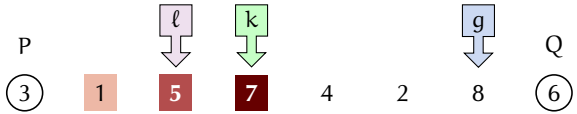

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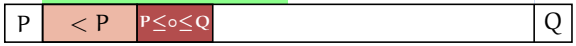

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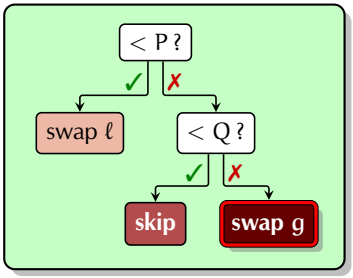

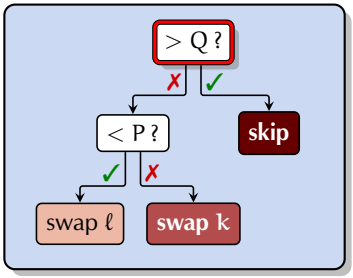

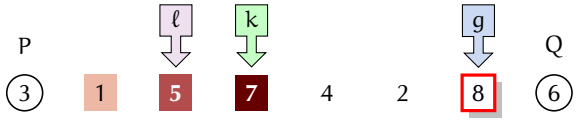

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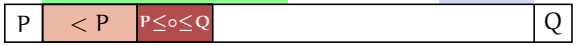


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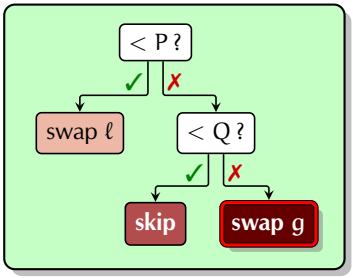

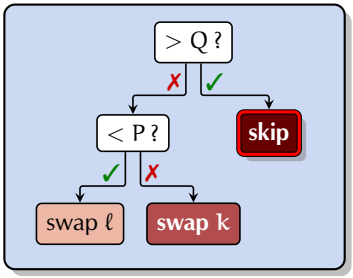


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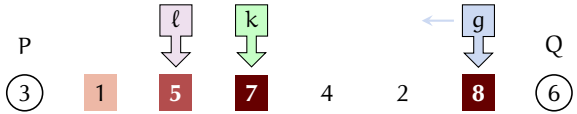
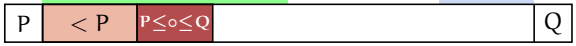


Invariant:



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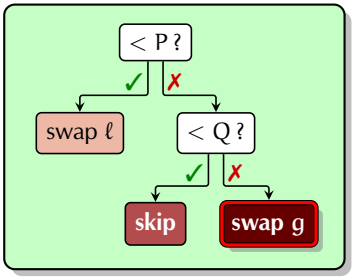

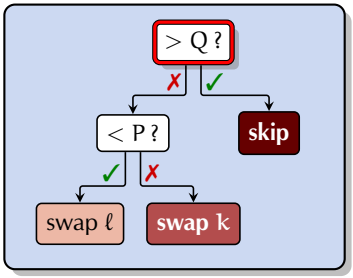


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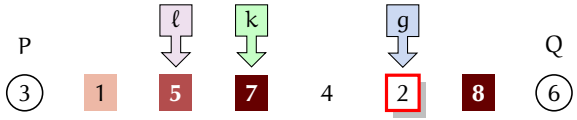

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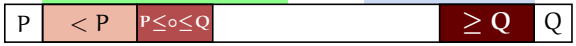
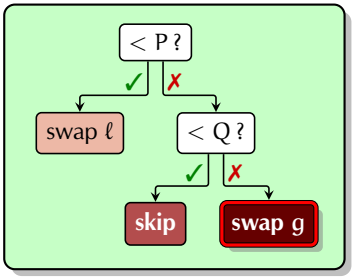
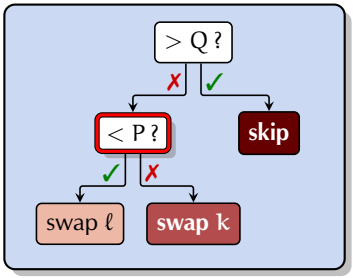
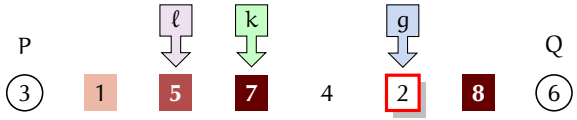


YBB Partitioning Example







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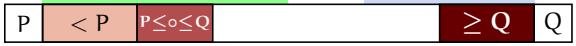


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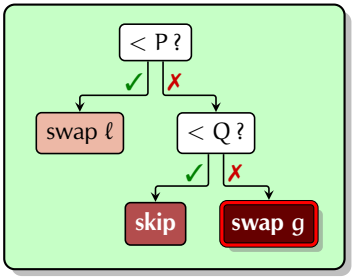
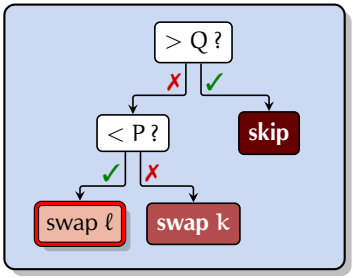
YBB Partitioning Example



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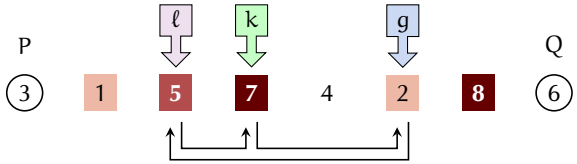
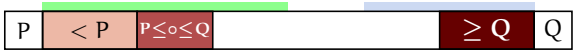


Invariant:



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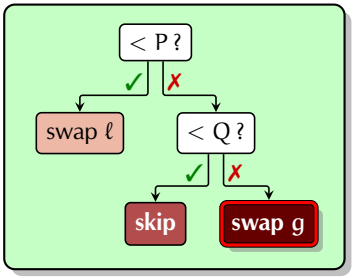

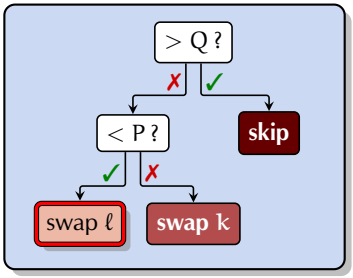

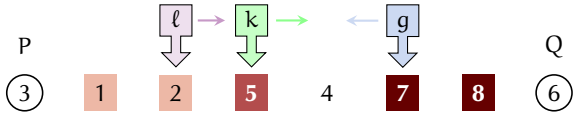

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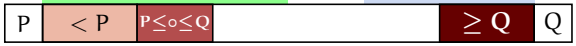


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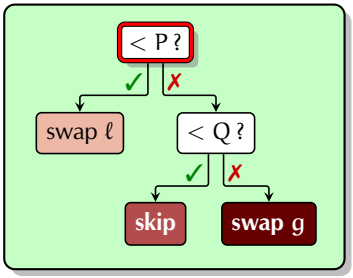

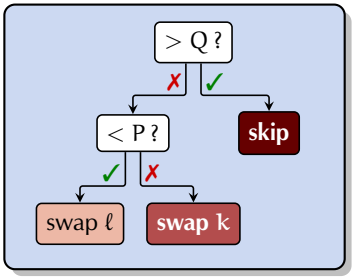

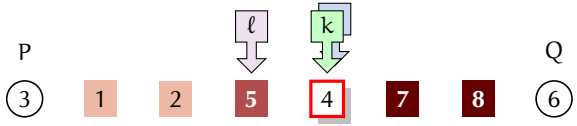

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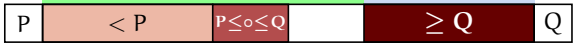


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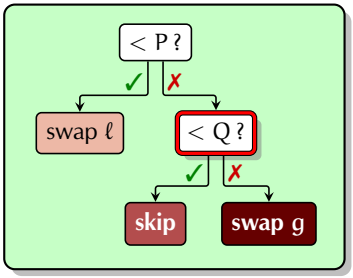

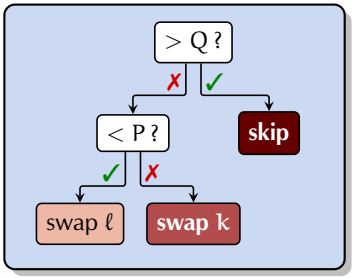

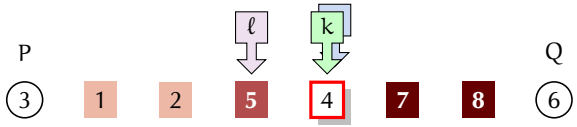

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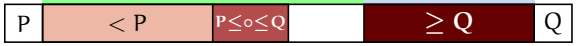


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 7          Swap A[k] and A[ℓ] ; ℓ = ℓ + 1
 8      else if A[k] ⩾ Q
 9          while A[g] > Q and k < g
10              g = g − 1
11          end while
12          Swap A[k] and A[g] ; g = g − 1
13          if A[k] < P
14              Swap A[k] and A[ℓ] ; ℓ = ℓ + 1
15          end if
16      end if
17      k = k + 1
18  end while
19  ℓ = ℓ − 1;   g = g + 1
20  Swap A[left] and A[ℓ] ;   Swap A[right] and A[g]
21  YBBQuicksort(A, left , ℓ − 1)
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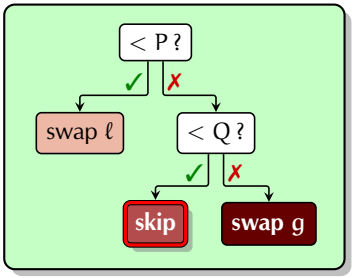
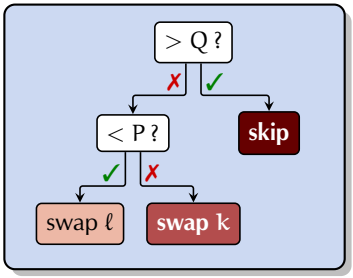
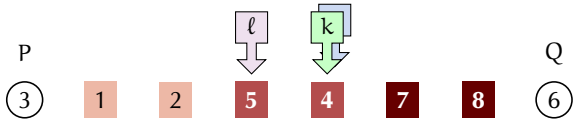
YBB Partitioning Example







Invariant:




```
YBBQuicksort(A, left, right)
 1  if right ⩽ left then return
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 4  ℓ = left + 1;   g = right − 1;   k = ℓ
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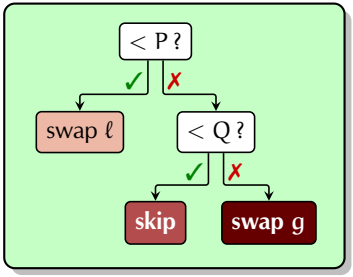

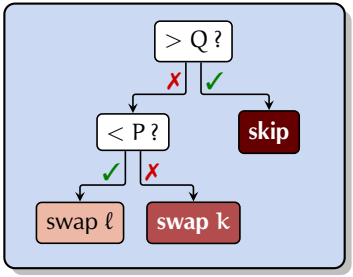



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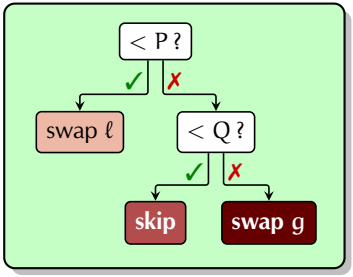

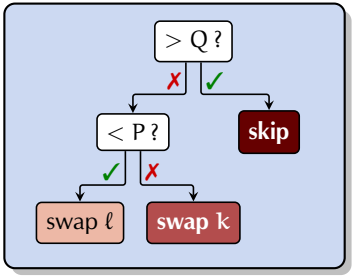




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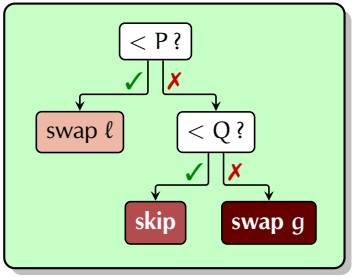

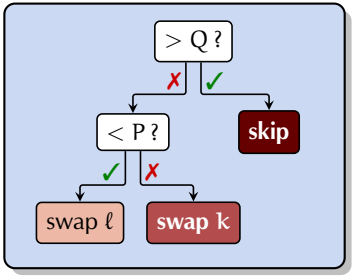



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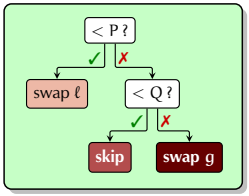

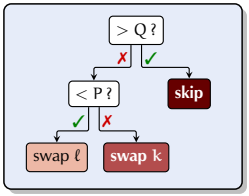

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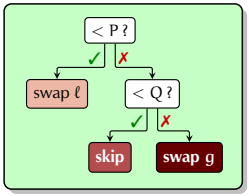

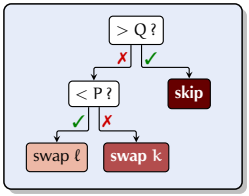

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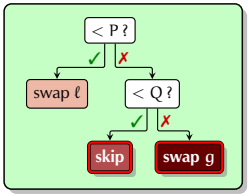

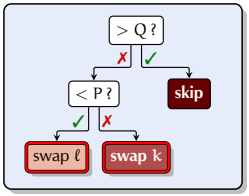

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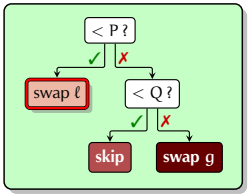

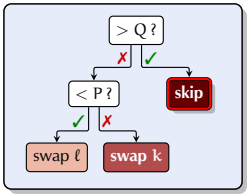

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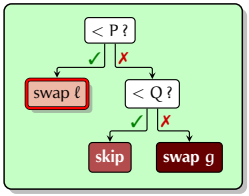

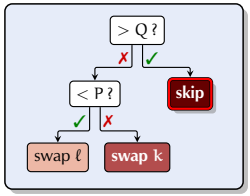

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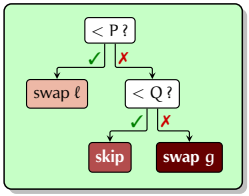

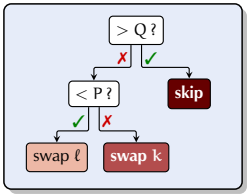

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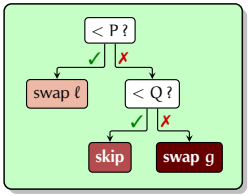

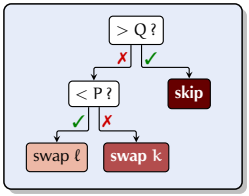

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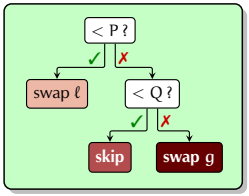

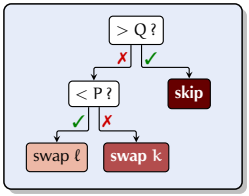

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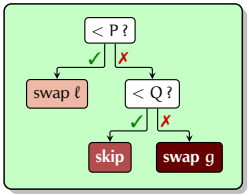

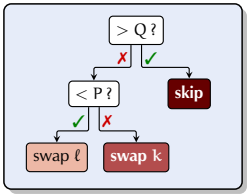

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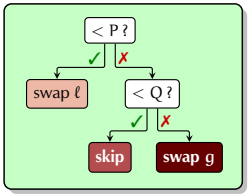

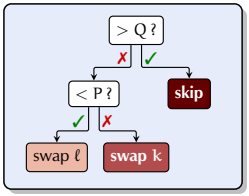

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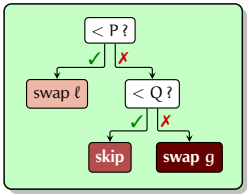

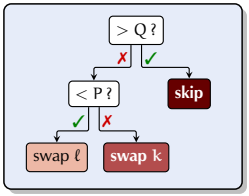

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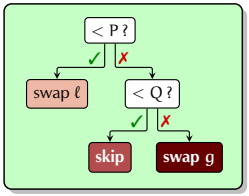

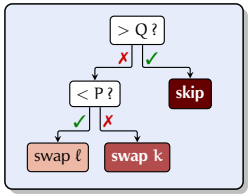

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- Sounds plausible? . . . but it's wrong!

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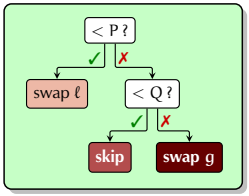

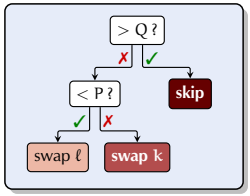

Problem: There are correlations.

- many small elements ⇝ k's range big
- many large elements ⇝ g's range big

⇝ $> \frac{1}{3}$ cheap elements on average.

How many? . . . stay tuned.

Comparisons

How many comparisons on average?

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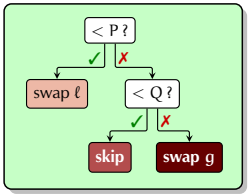

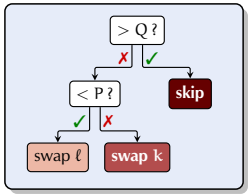

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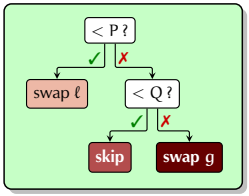

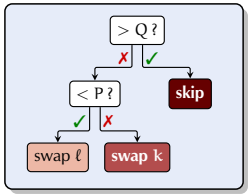

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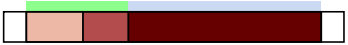
- 

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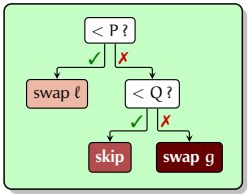

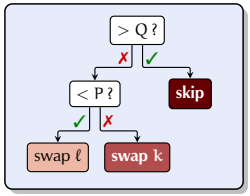

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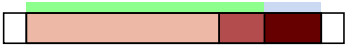
- 

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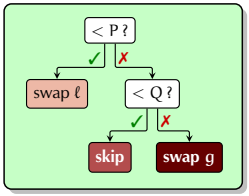

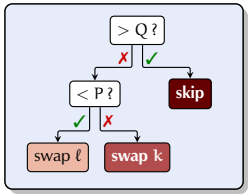

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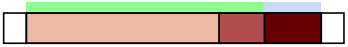
- 

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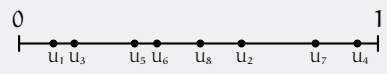

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- relative ranking is a **random permutation**
- $\rightsquigarrow$ equivalent to classic model

- For **dual-pivot** Quicksort with **fixed** pivots $P \leqslant Q$
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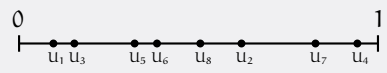

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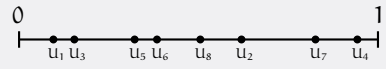

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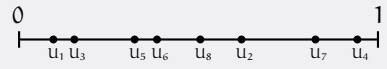

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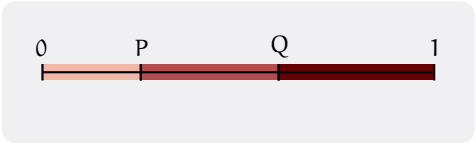

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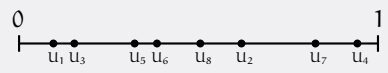

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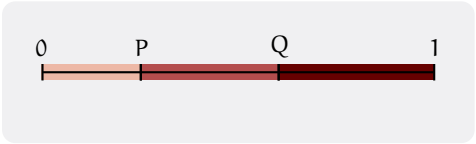

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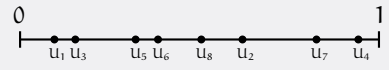

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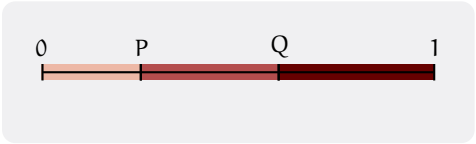

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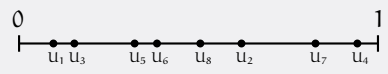

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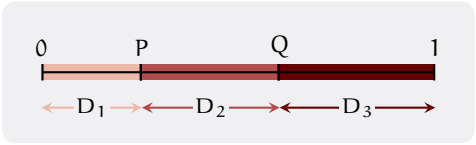

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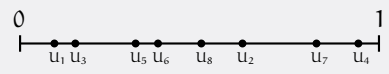

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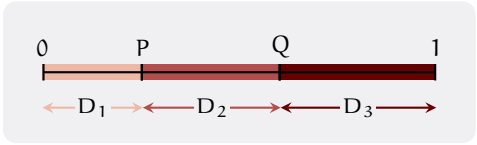

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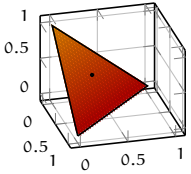

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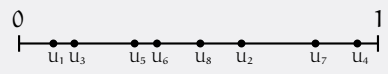

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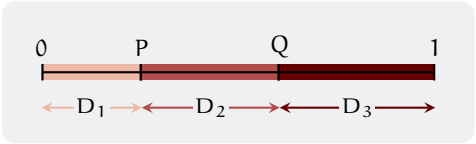

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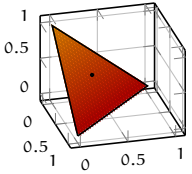

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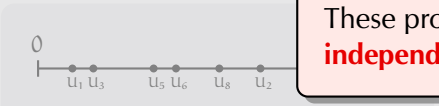

These probabilities hold for all elements U, **independent** of all other elements!

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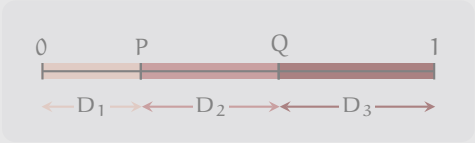

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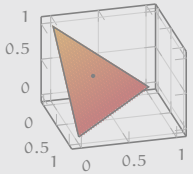

Comparisons in YBB Partitioning

- $t(n)$: $\mathbb{E}[\#\text{cmps}]$ in **first partitioning step**
- I_1: number of small elements; I_2, I_3 similar
- $\mathcal{K}$: #elements scanned by k; $\mathcal{K} \sim I_1 + I_2$
- $s@\mathcal{K}$: # small elements in k's range; $l@\mathcal{G}$ similar

$\rightsquigarrow\ t(n) \sim 2n - \mathbb{E}[s@\mathcal{K}] - \mathbb{E}[l@\mathcal{G}]$

…what is $\mathbb{E}[s@\mathcal{K}]$?

- Consider **fixed** $\vec{D}$,
  $\vec{I} = (I_1, I_2, I_3) \overset{\mathcal{D}}{=} \text{Mult}(n-2, \vec{D})$
$\rightsquigarrow\ \vec{I} = \vec{D}n \pm O(n^{1/2+\varepsilon})$ whp

$\rightsquigarrow\ \mathbb{E}[l@\mathcal{G}] \sim \mathbb{E}\left[D_3 \cdot D_3\right] \cdot n, \qquad \mathbb{E}[s@\mathcal{K}] \sim \mathbb{E}\left[D_1 \cdot (D_1 + D_2)\right] \cdot n$

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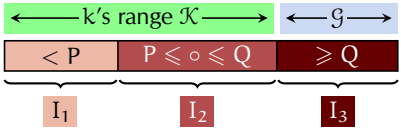

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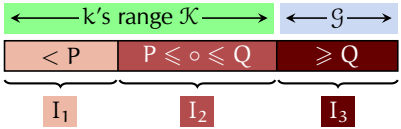

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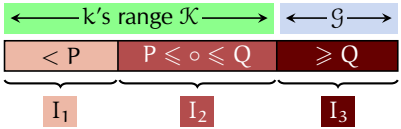

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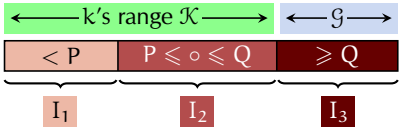

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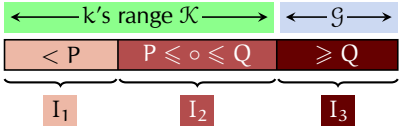

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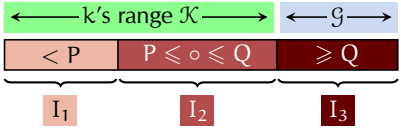

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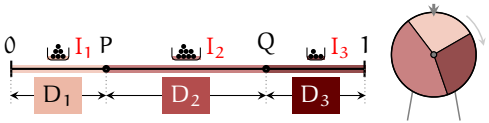

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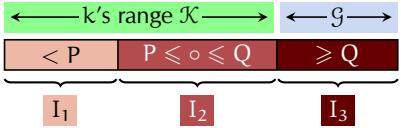

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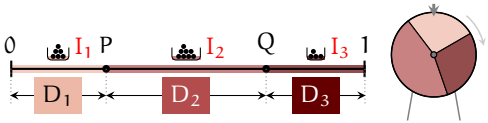

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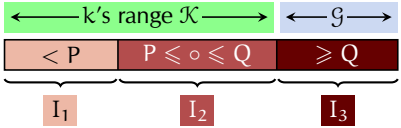

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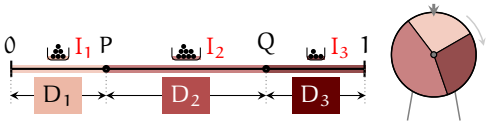

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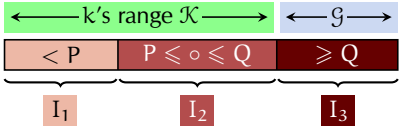

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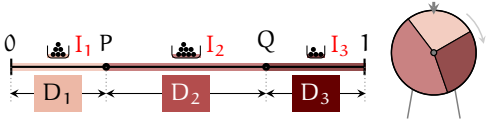

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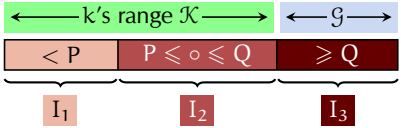

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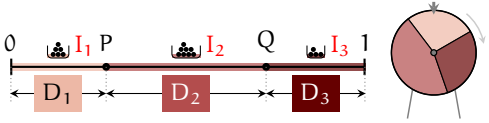

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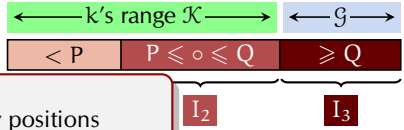

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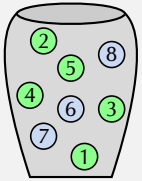

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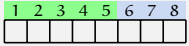

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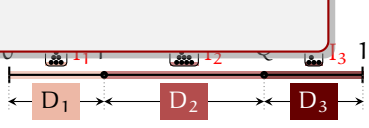

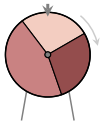

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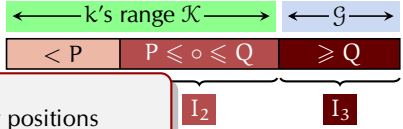

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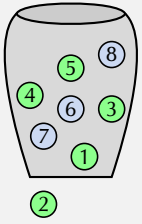

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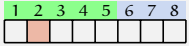

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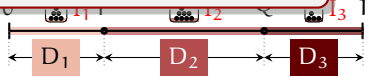

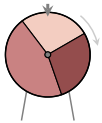

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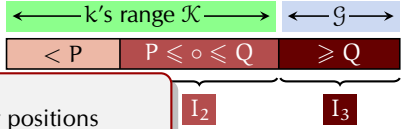

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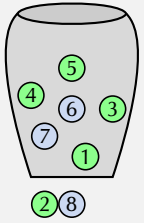

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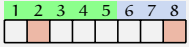

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fraction of elements in $\mathcal{K}$

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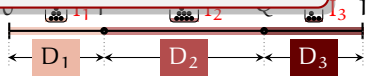

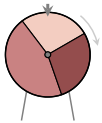

Comparisons in YBB Partitioning

- $t(n)$: $\mathbb{E}[\#\text{cmps}]$ in **first partitioning step**
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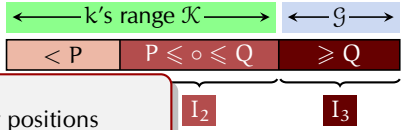
- $\mathcal{J}$
- s



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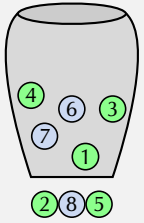
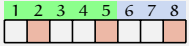


Draw positions of large elements



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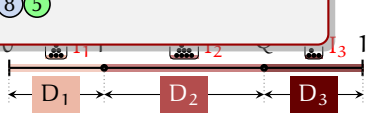

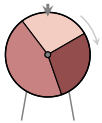

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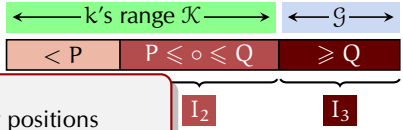

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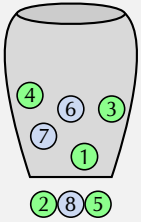
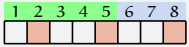


Draw positions of large elements



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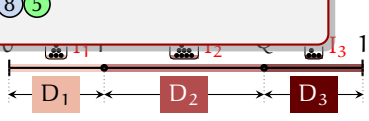

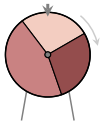

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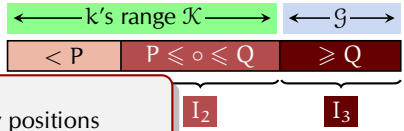

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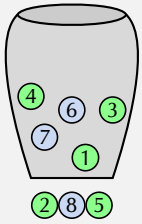

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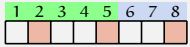

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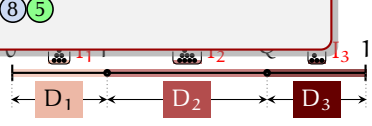

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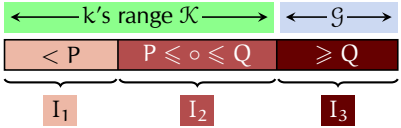

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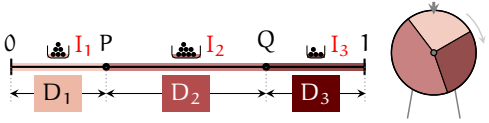

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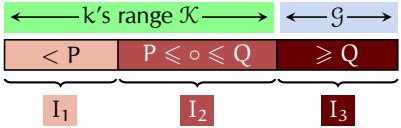

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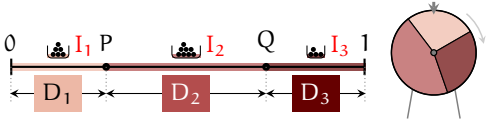

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$= \frac{1}{6} > \frac{1}{3} \cdot \frac{1}{3}$

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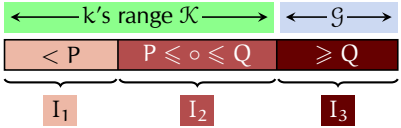

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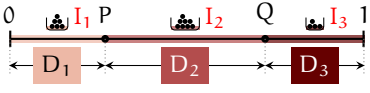

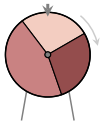

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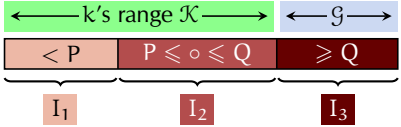

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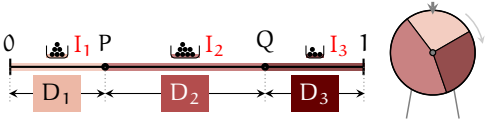

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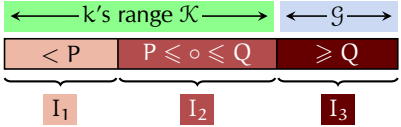

$\rightsquigarrow\ t(n) \sim 2n - \mathbb{E}[s@\mathcal{K}] - \mathbb{E}[l@\mathcal{G}] \sim \frac{19}{12}n$ …what is $\mathbb{E}[s@\mathcal{K}]$?

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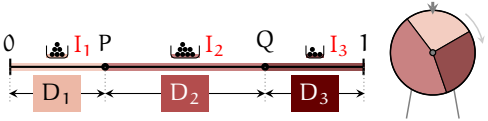

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Recurrence

- $c(n)$: $\mathbb{E}[\#\text{cmps}]$ for sorting

$$\rightsquigarrow \quad c(n) \;=\; t(n) + \mathbb{E}\Big[c(I_1)\Big] + \mathbb{E}\Big[c(I_2)\Big] + \mathbb{E}\Big[c(I_3)\Big]$$

Solve recurrence **without** error term by ***ansatz:***

$$c(n) = \alpha n \ln(n)$$

$$\alpha n \ln(n) \;=\; \tfrac{19}{12} n \;+\; \alpha n \ln(n) \sum_{r=1}^{3} \mathbb{E}[D_r] + \alpha n \sum_{r=1}^{3} \mathbb{E}[D_r \ln(D_r)]$$

- indeed (by induction): $c(n) = \frac{19}{10} n \ln n \pm O(n)$.

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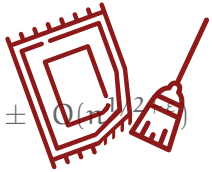

Solve recurrence **without** error term by ***ansatz:***

$$c(n) = \alpha n \ln(n)$$

$$\alpha n \ln(n) = \tfrac{19}{12}n + \alpha n \ln(n) \sum_{r=1}^{3} \mathbb{E}[D_r] + \alpha n \sum_{r=1}^{3} \mathbb{E}[D_r \ln(D_r)]$$

- indeed (by induction): $c(n) = \frac{19}{10} n \ln n \pm O(n)$.

Recurrence

- $c(n)$: $\mathbb{E}[\#\text{cmps}]$ for sorting

$$\rightsquigarrow \quad c(n) = t(n) + \mathbb{E}\Big[c(I_1)\Big] + \mathbb{E}\Big[c(I_2)\Big] + \mathbb{E}\Big[c(I_3)\Big]$$
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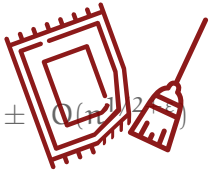

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($\log_e$)

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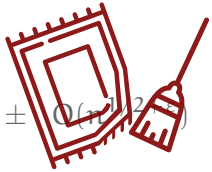



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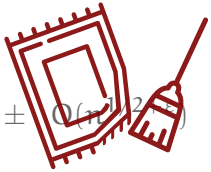



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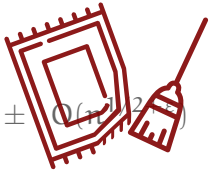

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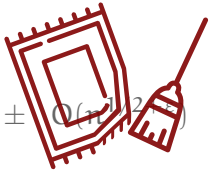

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base e Shannon entropy of $\vec{D}$

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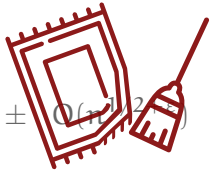

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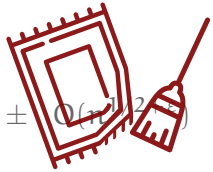

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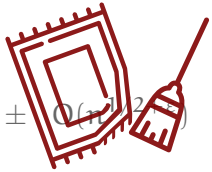

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$$\iff \quad \alpha = \frac{\frac{19}{12}}{\mathbb{E}[\mathcal{H}(\vec{D})]} \overset{\leftarrow \frac{5}{6}}{=} \frac{19}{10}$$

- indeed (by induction): $c(n) = \frac{19}{10} n \ln n \pm O(n)$.

Recurrence

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$$\iff \alpha = \frac{\frac{19}{12}}{\mathbb{E}[\mathcal{H}(\vec{D})]} = \tfrac{19}{10} \quad \left(\mathbb{E}[\mathcal{H}(\vec{D})] \leftarrow \tfrac{5}{6}\right)$$

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Recurrence

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$\mathbb{E}[\mathcal{H}(\vec{D})] \leftarrow \frac{5}{6}$

$\rightsquigarrow$ Salvador Roura's *Continuous Master Theorem*

- indeed (by induction): $c(n) = \frac{19}{10} n \ln n \pm O(n)$.

Recurrence

Why does Shannon entropy show up here?

- Must learn $\mathrm{ld}(n!) \sim n \,\mathrm{ld}\, n$ **bits of information** to sort.
- For fixed $\vec{D}$, one classification (small / medium / large)
  - yields $\mathcal{H}_{\mathrm{ld}}(\vec{D}) = -\sum_{r=1}^{3} D_r \,\mathrm{ld}(D_r)$ bits of information
  - costs $a = 2 - \mathbb{E}[D_1 + D_3] - \mathbb{E}[D_1 \,(D_1 + D_2)] = \frac{19}{12}$ cmps

⇝ Average ***information yield***: $\mathcal{I} = \mathbb{E}[\mathcal{H}_{\mathrm{ld}}(\vec{D})] \,/\, a$ bits per comparison

⇝ Need $\dfrac{\mathrm{ld}(n!)}{\mathcal{I}} \sim \dfrac{a}{\mathbb{E}[\mathcal{H}_{\mathrm{ld}}(\vec{D})]} n \,\mathrm{ld}\, n = \dfrac{a}{\mathbb{E}[\mathcal{H}(\vec{D})]} n \ln n$ cmps!

$$\iff \quad \alpha = \frac{\frac{19}{12}}{\mathbb{E}[\mathcal{H}(\vec{D})]} = \frac{19}{10}$$

$\mathbb{E}[\mathcal{H}(\vec{D})] = \frac{5}{6}$

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Recurrence

Why does Shannon entropy show up here?

- Must learn $\log_2 \mathrm{ld}(n!) \sim n \,\mathrm{ld}\, n$ **bits of information** to sort.
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$\iff$ $\alpha = \dfrac{\frac{19}{12}}{\mathbb{E}[\mathcal{H}(\vec{D})]} = \dfrac{19}{10}$ ($\mathbb{E}[\mathcal{H}(\vec{D})] = \frac{5}{6}$)

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$r=1$ $r=1$

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- Must learn $\mathrm{ld}(n!) \sim n \,\mathrm{ld}\, n$ **bits of information** to sort.
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binary entropy

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definition of entropy!

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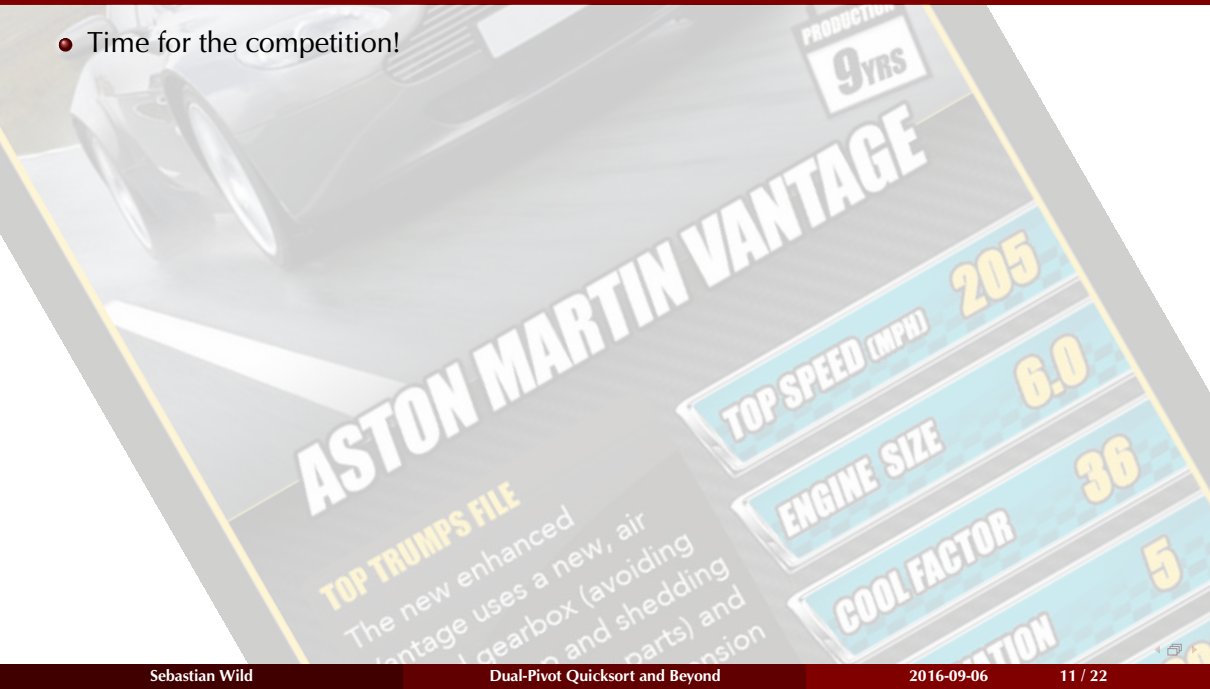
Classic vs. YBB Quicksort

- Time for the competition!



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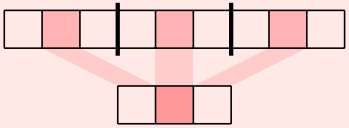

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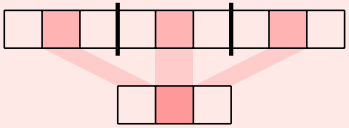

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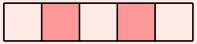

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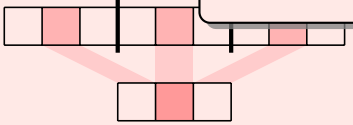

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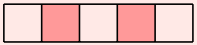

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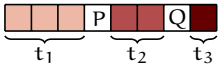

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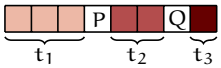

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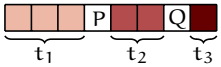

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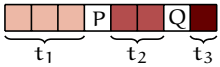

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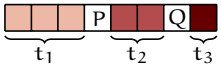

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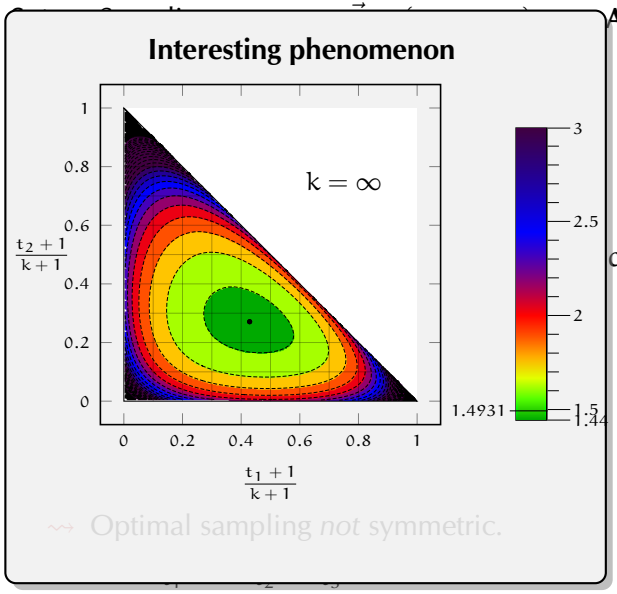

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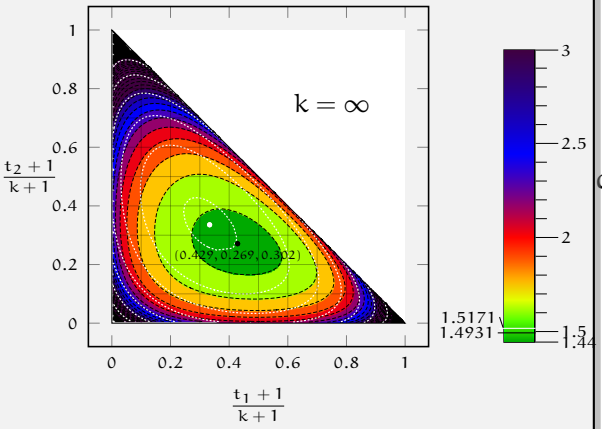

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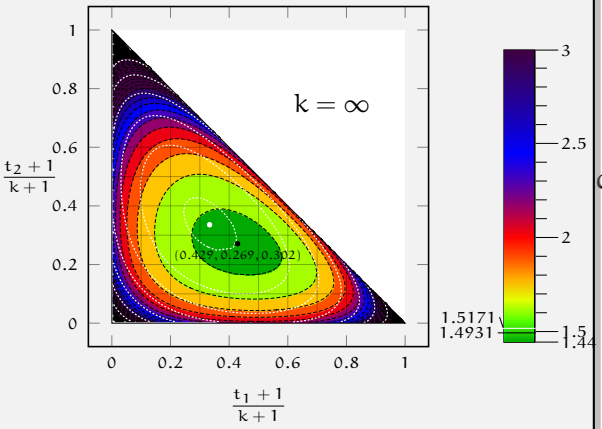

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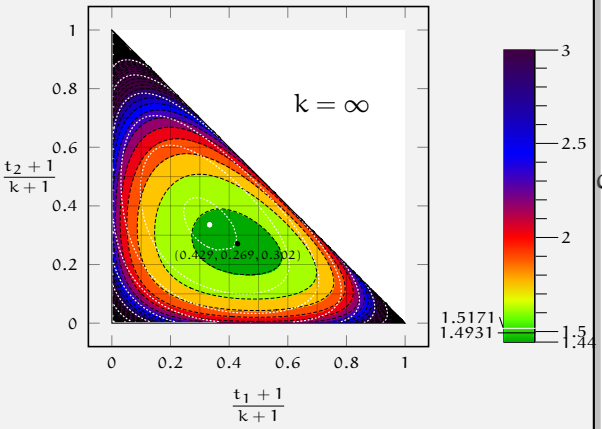

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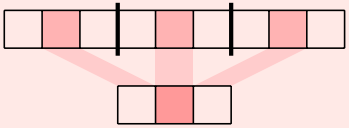

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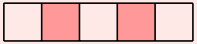

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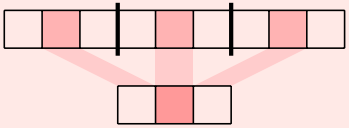

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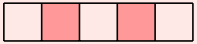

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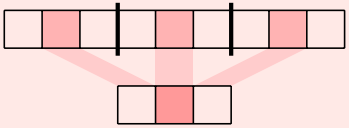

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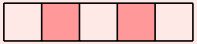

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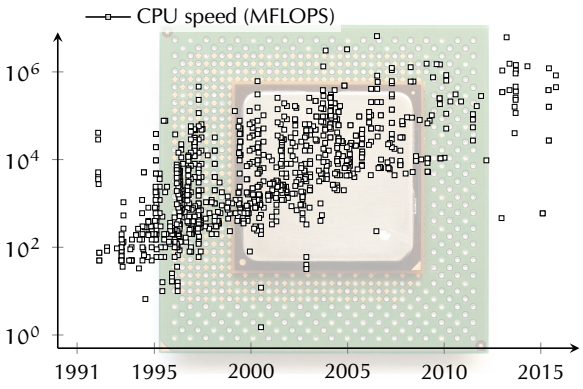

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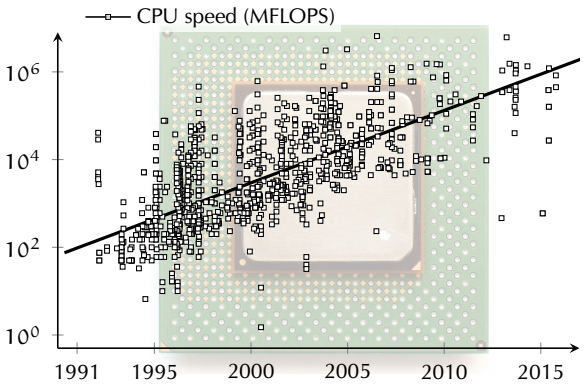

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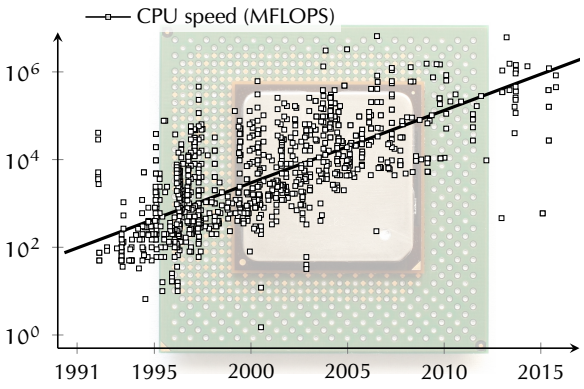

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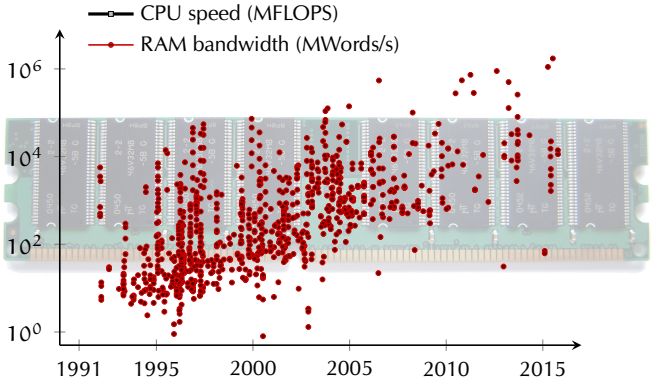

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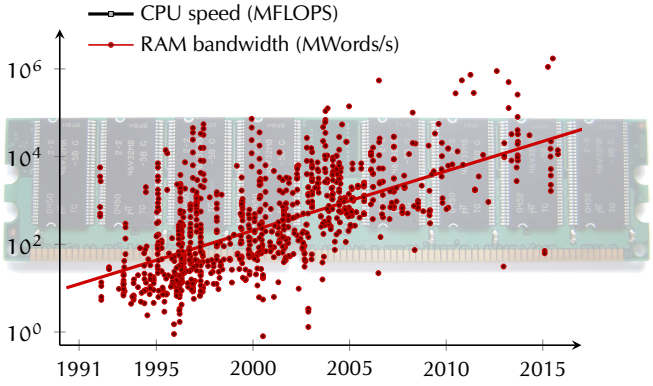

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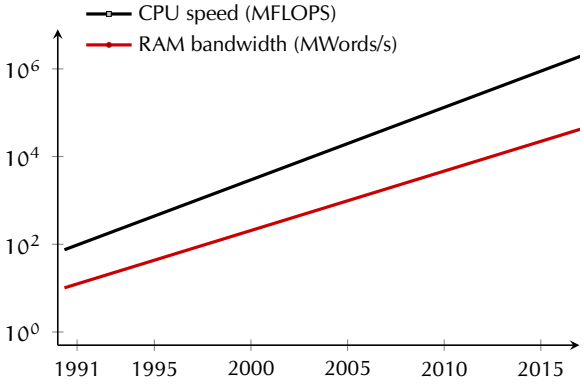

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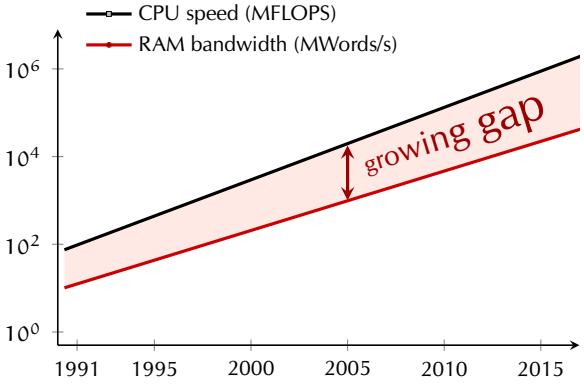

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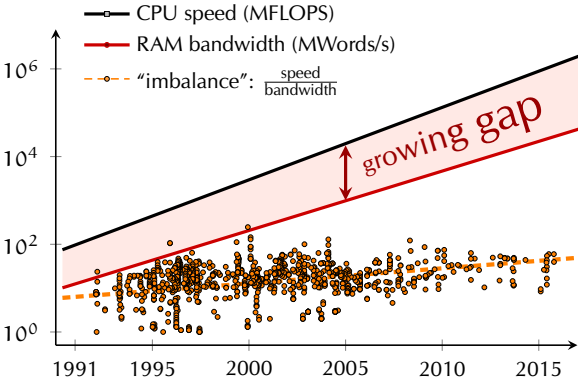

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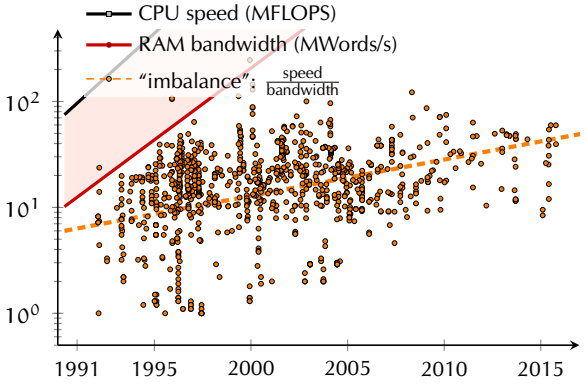

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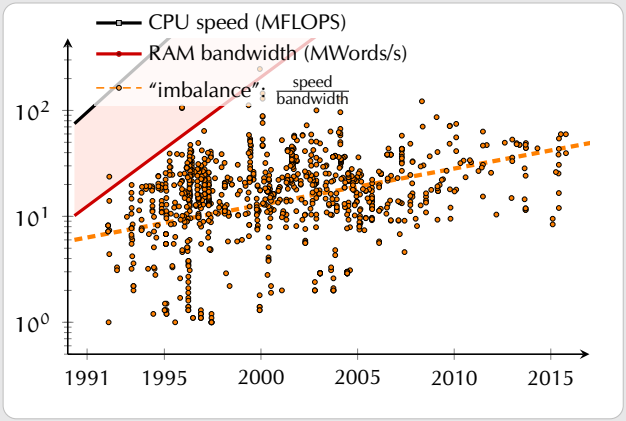

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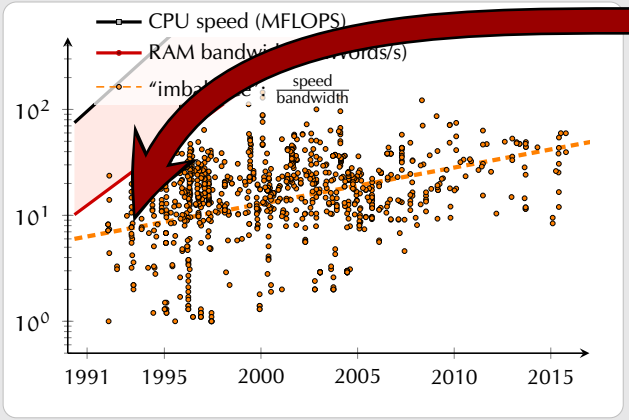

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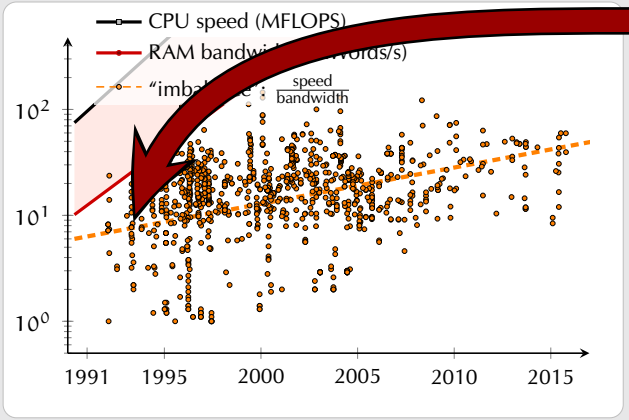

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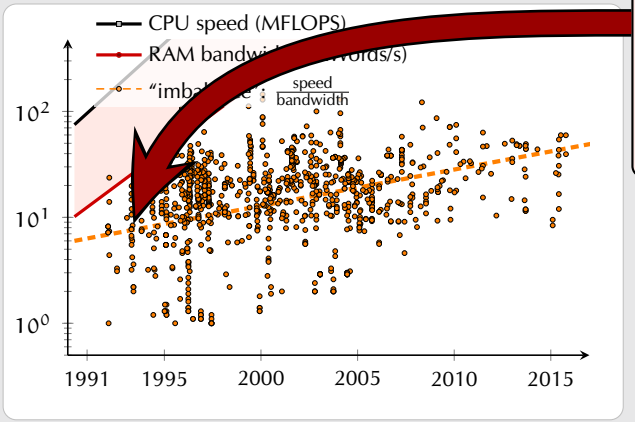

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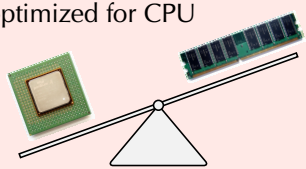

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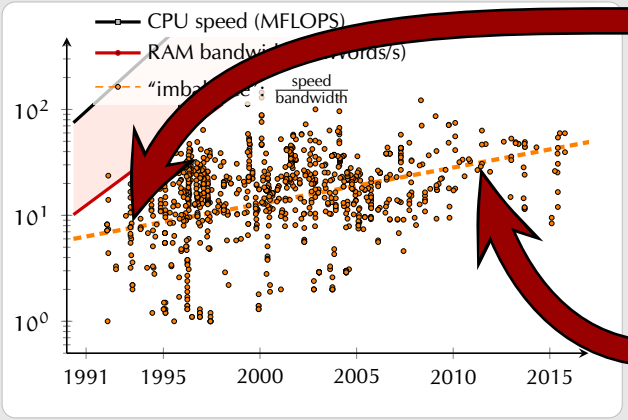

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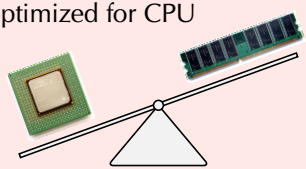

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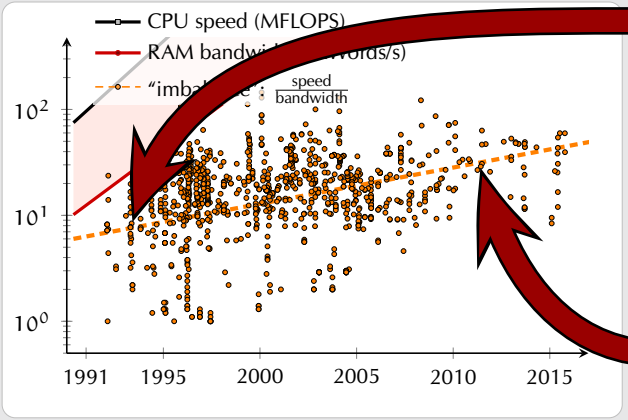

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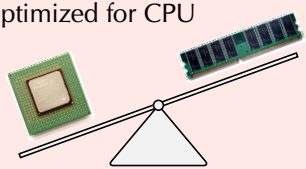

Processor speed: 40%

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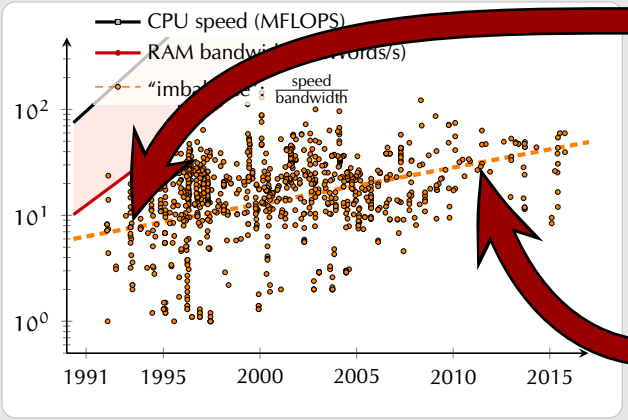

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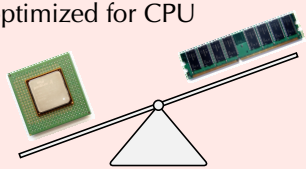

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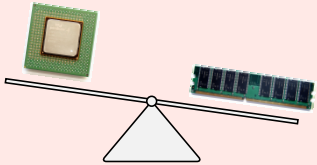

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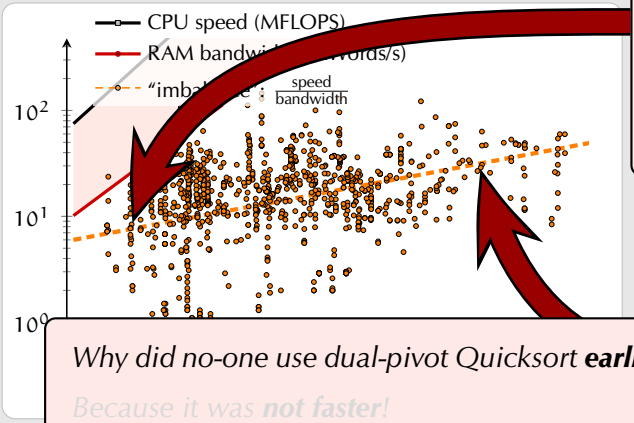

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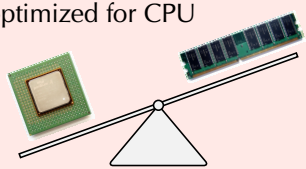

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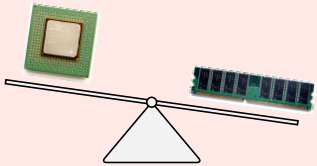

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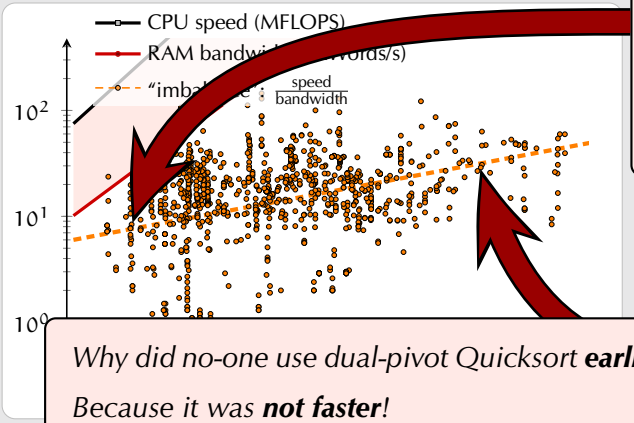

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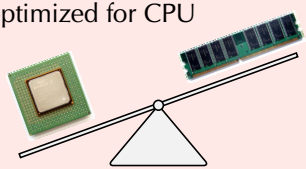

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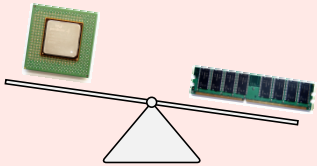

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Need good cost measure!

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- *I/Os* in *external-memory model*
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  - typically: $O(\cdot)$, **only** I/Os count
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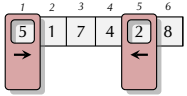

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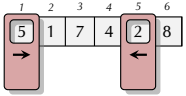

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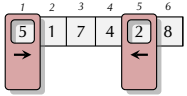

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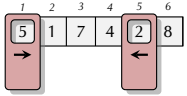

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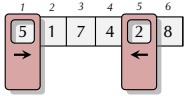

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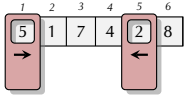

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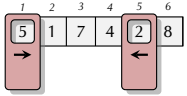

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≈ **cache misses**, but hardware independent

Why reinvent the wheel?

- ***I/Os* in *external-memory model***
  - additional parameters M and B
  - typically: $O(\circ)$, **only** I/Os count
- ***cache misses***
  - hardware-specific
  - misses vs. bandwidth

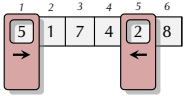
Scanned Elements

How to analyze bandwidth consumption?

Need good cost measure!

Proposal: count ***scanned elements***

- Access to array only through **iterators**
  (no random access!)



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Costs of Partitioning

Classic Quicksort

⇝ n scanned elements
(array scanned once)

YBB Quicksort

⇝ $1.\overline{3}n$ scanned elements
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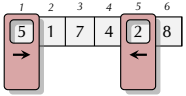

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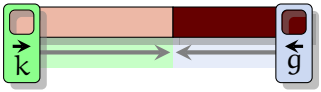

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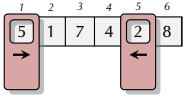

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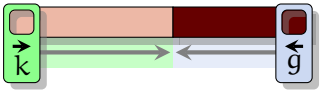

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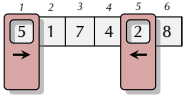

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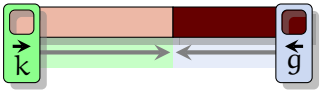

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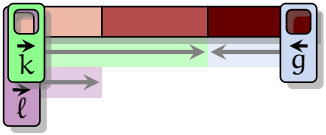

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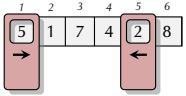

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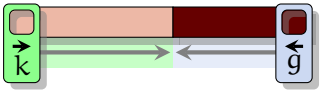

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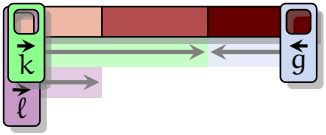

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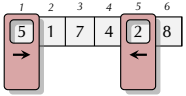

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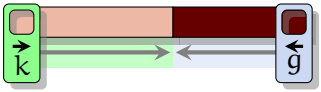

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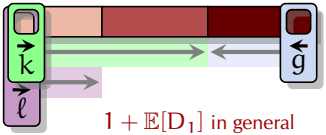

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Classic vs. YBB Quicksort

- Time for the competition!

	CQS	YQS	Relative
Running Time	(from various experiments)		$-10\pm2\%$
Comparisons	2	1.9	-5%
(with sampling as in JRE)	1.57	1.70	$+8.6\%$
Swaps	$0.\overline{3}$	0.6	$+80\%$
Bytecode Instructions	18	21.7	$+20.6\%$
MMIX oops υ	11	13.1	$+19.1\%$
MMIX mems μ	$2.\overline{6}$	2.8	$+5\%$
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$\cdot n \ln n + O(n)$, *average case results*

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$\cdot n \ln n + O(n)$, *average case results*

Only plausible explanation for running time: 20% less memory transfers in YQS.

Beyond Dual-Pivot: Multiway Partitioning

Can we do even better than YBB Quicksort?

Setup: *Generic s-way one-pass partitioning*

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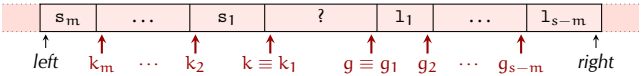

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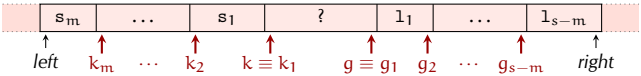

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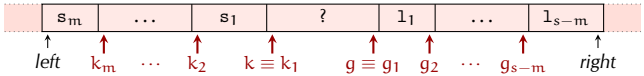

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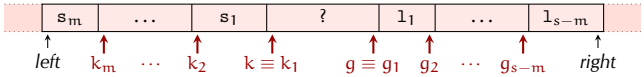

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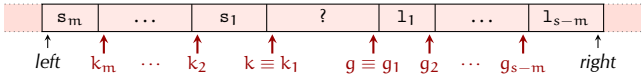

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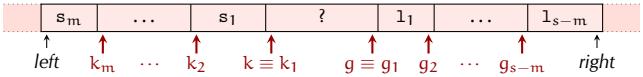

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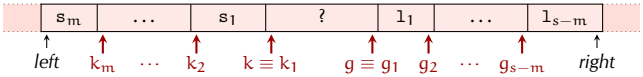

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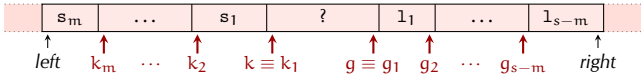

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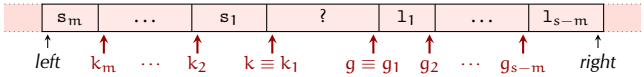

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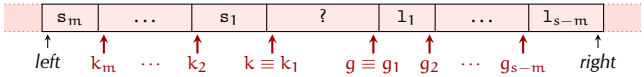

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Fair Competition?

Consider the #comparisons again.

4-way Quicksort

- Always need 2 cmps to classify.

$\rightsquigarrow \frac{24}{13} n \ln(n)$ cmps (without sampling)

≈ 1.846

Classic Quicksort

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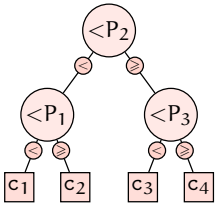
situation for YBB similar

4-way Quicksort



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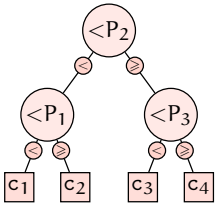

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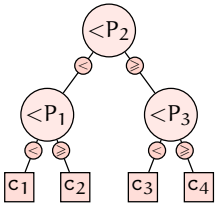

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$\rightsquigarrow \frac{24}{13} n \ln(n)$ cmps (without sampling)

≈ 1.846

Classic Quicksort

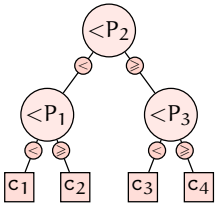
- $2 n \ln(n)$ comparisons (without sampling)

Fair Competition?

Consider the #comparisons again.

situation for YBB similar

4-way Quicksort



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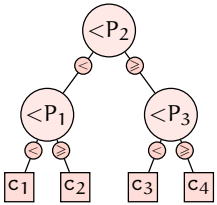

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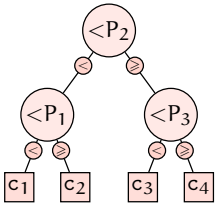

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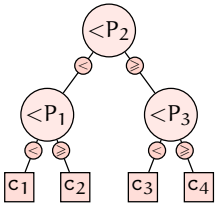

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- $2 n \ln(n)$ comparisons (without sampling)
- $\frac{12}{7} n \ln(n)$ comparisons (median-of-3)

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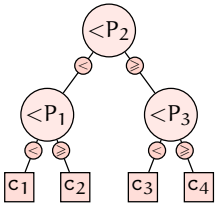

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…or not; it depends on **sampling**.

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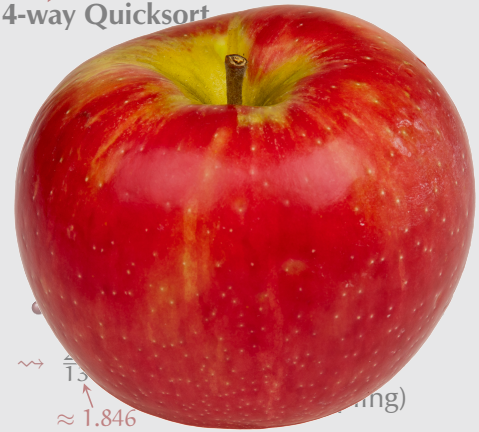

$\rightsquigarrow \frac{2}{13}$ ing)

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VS.

Classic Quicksort

- 2



Aha! Multiwa

. . . or not; it depe

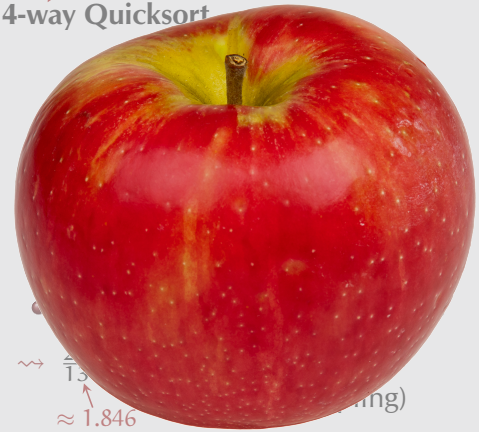
Fair Competition?

How to **separate** influence of multiway partitioning from that of sampling?

Conside

situation for YBB similar

4-way Quicksort



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Key Idea: Simulate by binary partitioning.

One round of 4-way partitioning ...

(without sampling)

... corresponds to three binary partitionings.

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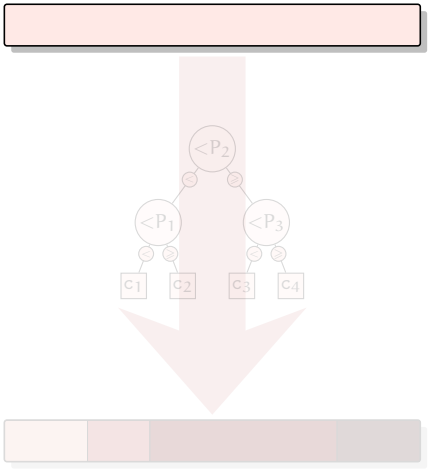

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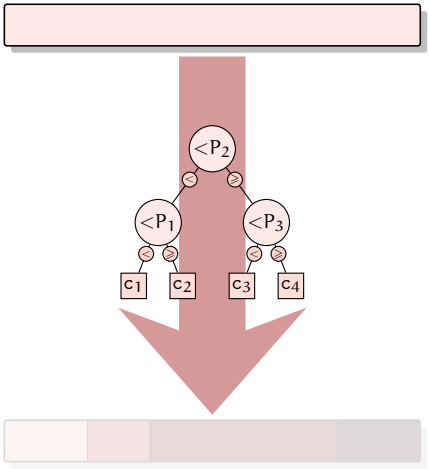

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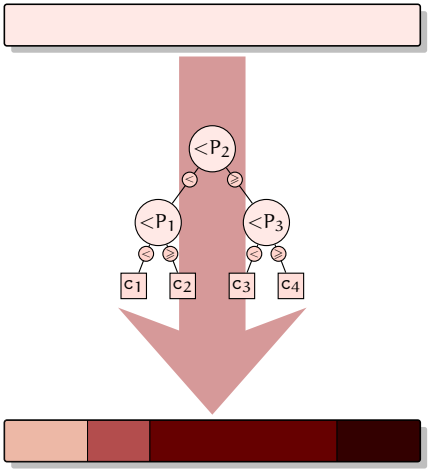

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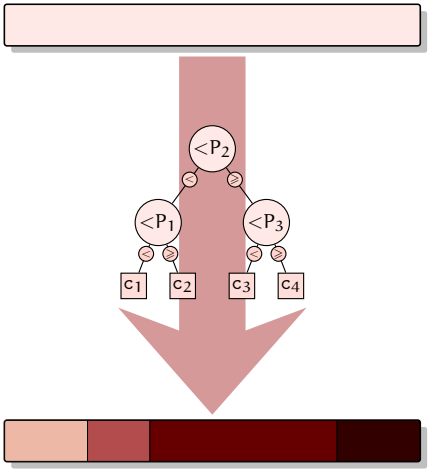

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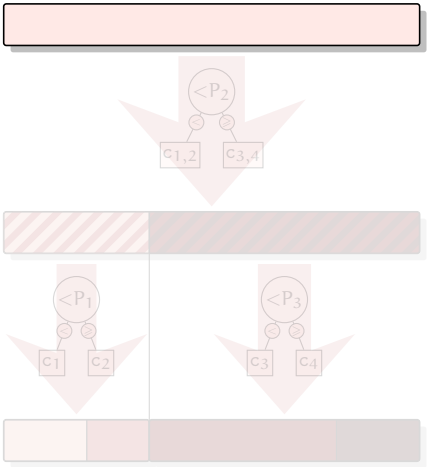

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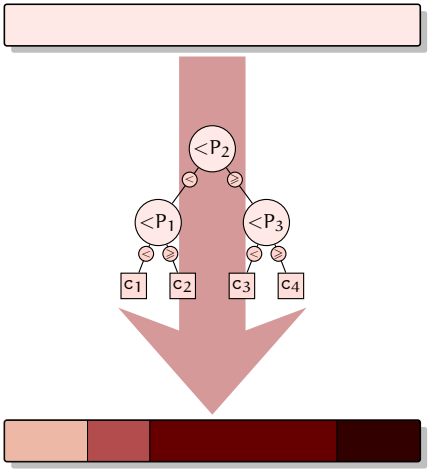

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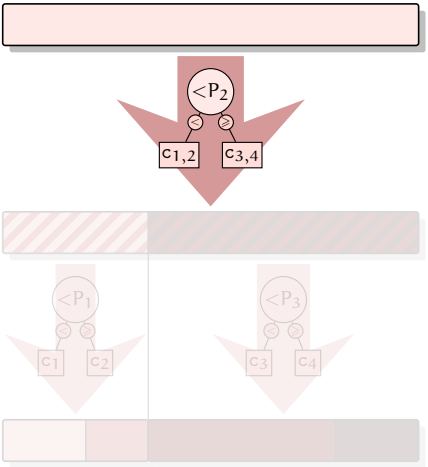

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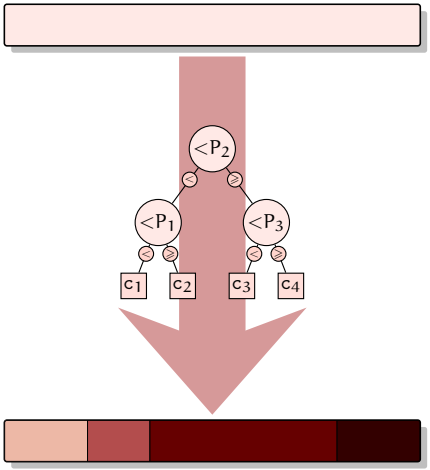

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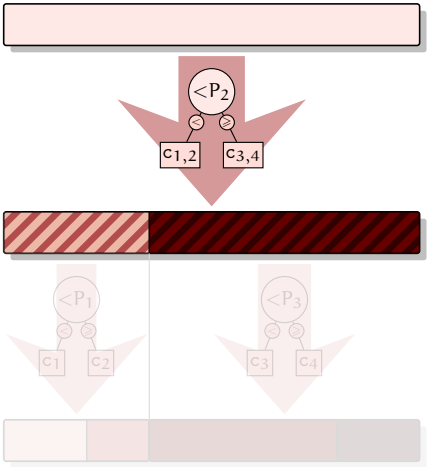

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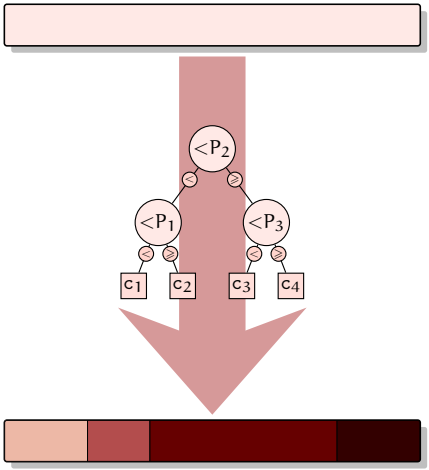

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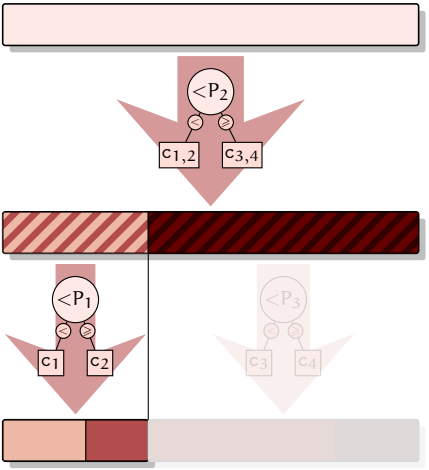

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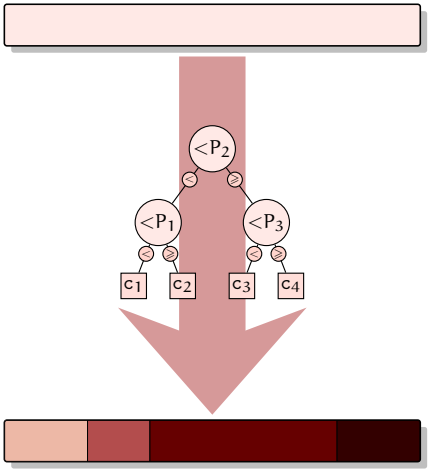

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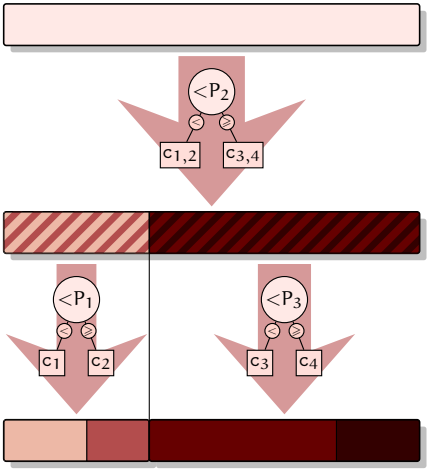

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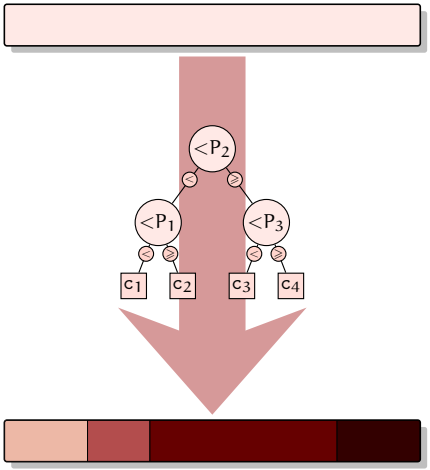

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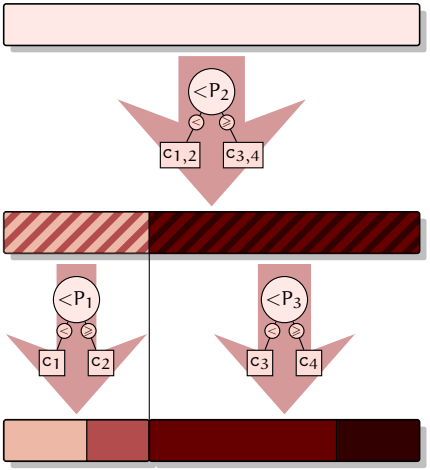
But how to choose pivots P_1, P_2, P_3?



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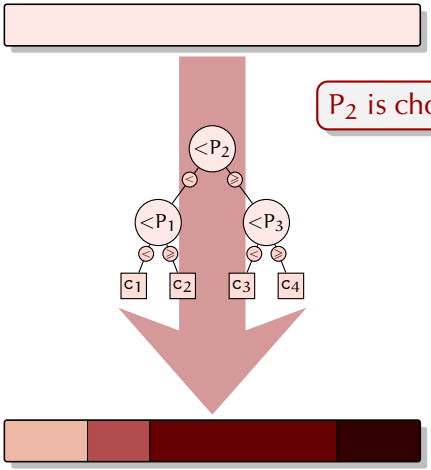

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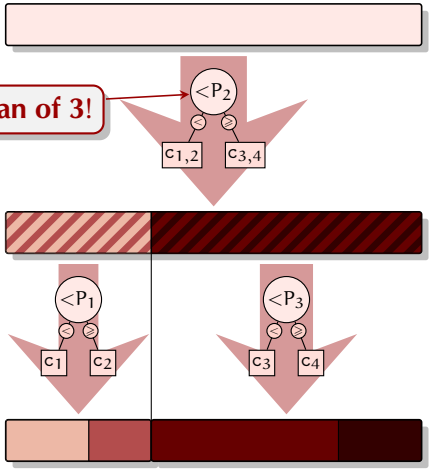
P_2 is chosen as **median of 3**!



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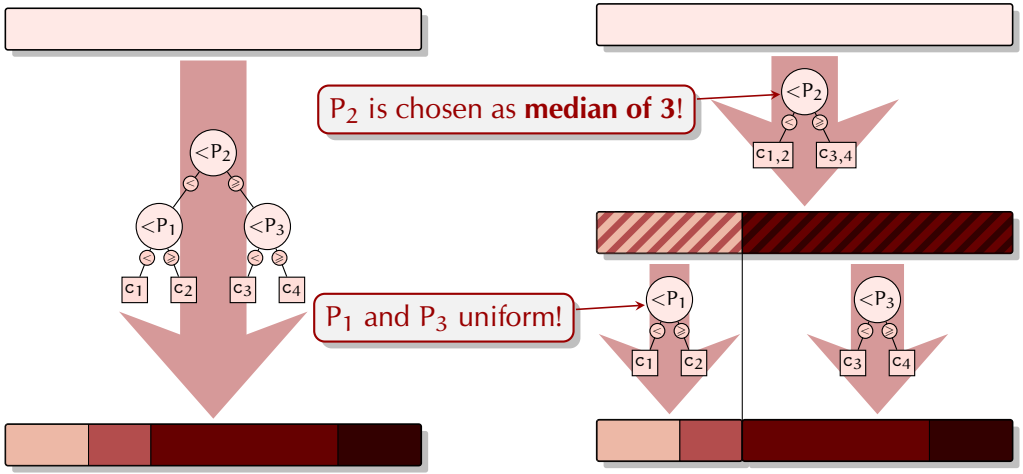

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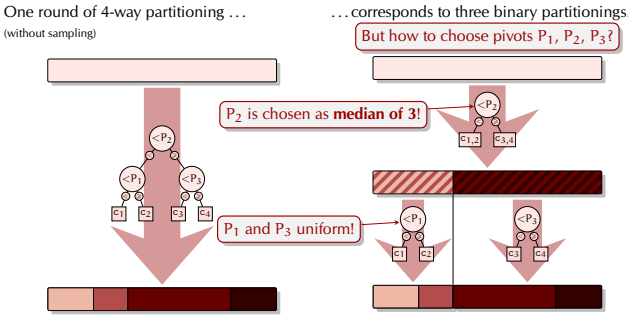

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- ⇝ **same on average**:
  1. flip a fair coin
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random-parameter pivot sampling

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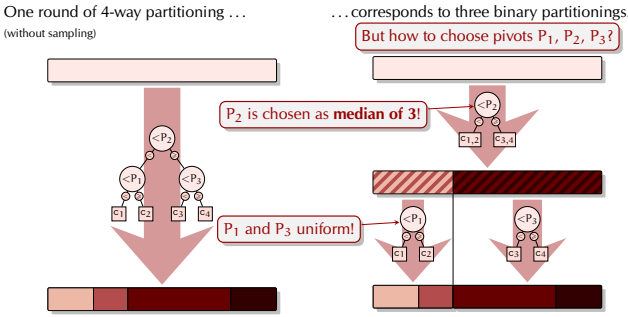

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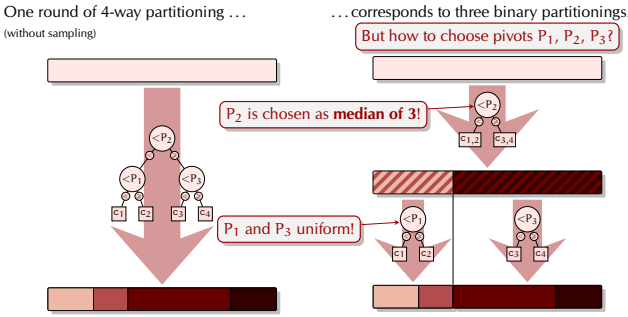

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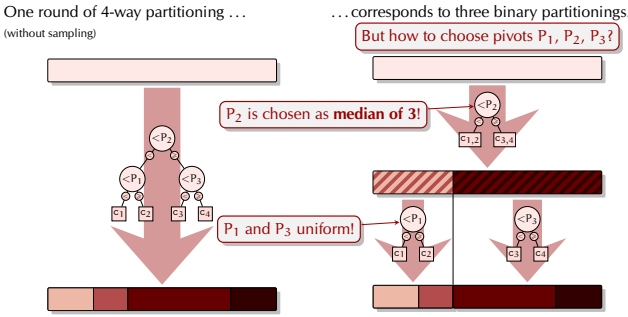

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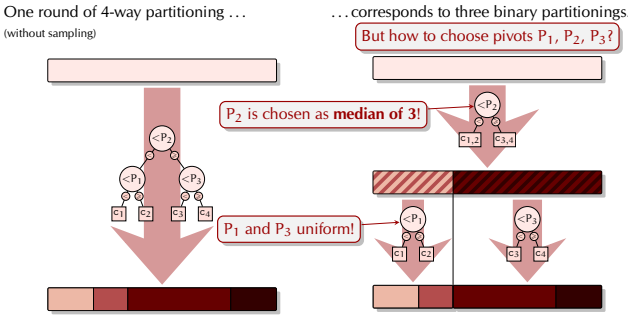

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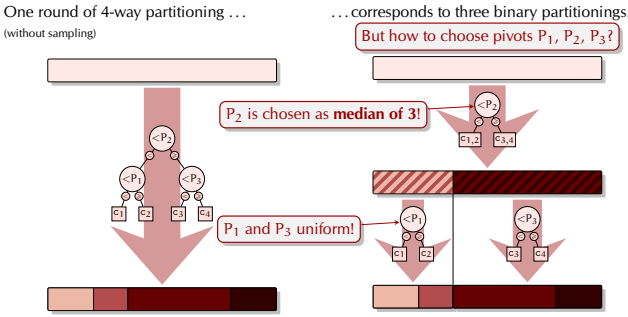

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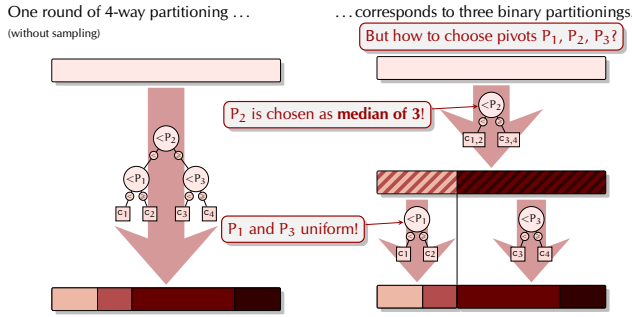

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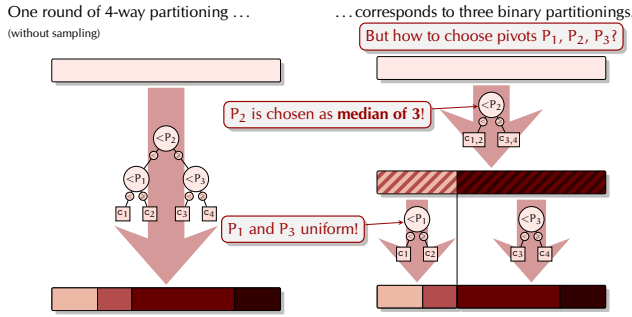
Theorem *(Entropy-Equivalent Sampling)*

For every s-way sampling parameter $\vec{t}$ there is a single-pivot random-parameter scheme with the same quality of pivots.

(formal meaning: same denominator)

Entropy-Equivalent Sampling

Key Idea: Simulate by binary partitioning.



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Theorem ***(Entropy-Equivalent Sampling)***

For every s-way sampling parameter $\vec{t}$ there is a single-pivot random-parameter scheme with the same quality of pivots.

(formal meaning: same denominator)

Example: $\vec{t} = (1,1,1)$ with 

$$\rightsquigarrow \vec{T} = \begin{cases} (3,1) & \text{with prob. } \frac{3}{5} \\ (1,1) & \text{otherwise} \end{cases}$$

Finally: The Fair Competition

Compare multiway method to **b rounds of binary partitioning**.
b = average number of binary partitioning rounds per element.

Consider **YBB Quicksort** with $\vec{t} = (1, 1, 1)$

$\rightsquigarrow$ 2 options

1. First around P, then around Q
   $b = 1 + \mathbb{E}\left[D_1 + D_2 \right]$
2. First around Q, then around P
   $b = 1 + \mathbb{E}\left[D_1' + D_2' \right]$

$\rightsquigarrow$ $b = 1 + \frac{2}{3} = 1.\overline{6}$ (in both cases here)

- but YBB costs
  - $\frac{34}{21} \approx 1.619$ cmps/element -2.9%
  - $\frac{4}{3} = 1.\overline{3}$ scans/element -20%

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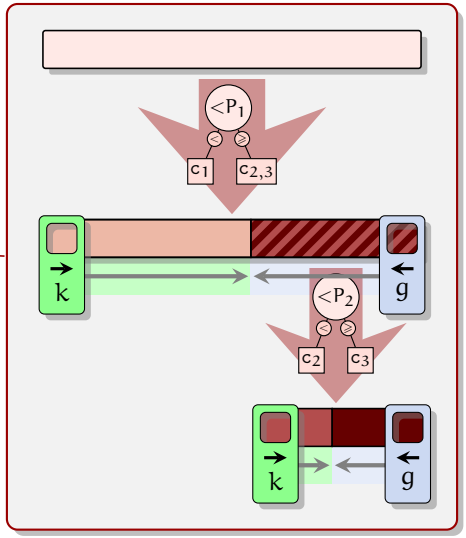

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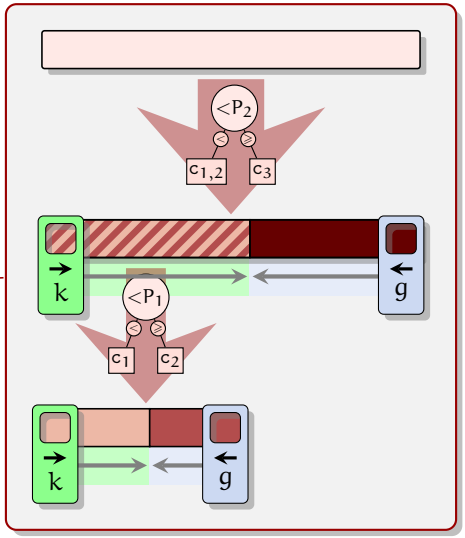

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Consider **YBB Quicksort** with $\vec{t} = (1, 1, 1)$

$\rightsquigarrow$ 2 options

1. First around P, then around Q
   $b = 1 + \mathbb{E}\left[D_2 + D_3 \right]$



2. First around Q, then around P
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$\rightsquigarrow$ $b = 1 + \frac{2}{3} = 1.\overline{6}$ (in both cases here)

- but YBB costs
  - $\frac{34}{21} \approx 1.619$ cmps/element $\Rightarrow$ +2.9%
  - $\frac{1}{3} = 1.\overline{3}$ scans/element $\Rightarrow$ +20%

Finally: The Fair Competition

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- but YBB costs
  - $\frac{34}{21} \approx 1.619$ cmps/element — −2.9%
  - $\frac{4}{3} = 1.\overline{3}$ scans/element — −20%

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⇝ 2 options

1. First around P, then around Q
   $b = 1 + \mathbb{E}\left[\boxed{D_2} + \boxed{D_3}\right]$



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1.34… vs. 1.40…

But: scanned-element-optimal: $\vec{t} = (0, 1, 2)$

⇝ Option 2 strictly better: $b = 1.5$

- YBB costs
  - $\frac{34}{21} \approx 1.619$ cmps/element +7.9%
  - $\frac{7}{6} = 1.1\overline{6}$ scans/element −22.2%

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1. $\rightsquigarrow b = 1.8\overline{3}$

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General Observation

- **Comparisons:** hardly any savings
  if k grows, often even costly
- **Scanned Elements:** significant savings
  large k helps, need asymmetric $\vec{t}$

But: scanned-element-optimal: $\vec{t} = (0, 1, 2)$

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Conclusion

You have seen:

1. tricky asymmetries in *YBB Quicksort*
   - ⇝ saves a few comparisons
     - **but:** savings vanish with pivot sampling
2. *scanned elements*
   - simple cost model for memory transfer
   - significant savings possible with multiway partitioning
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   - separate effects: better pivots vs. partitioning method
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You have *not* seen:

- Branch mispredictions

- Treatment of equal keys

- Intriguing interplay of parameters

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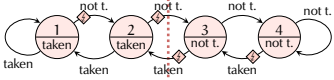

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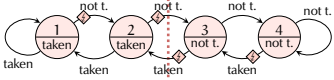

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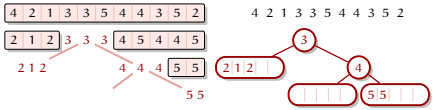

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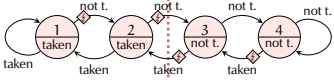

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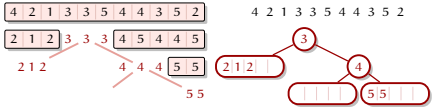

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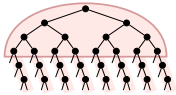



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