Median-of-k Quicksort Is Optimal For Many Equal Keys

Sebastian Wild

wild@cs.uni-kl.de



originates from joint work with

Martin Aumüller, Martin Dietzfelbinger,
Conrado Martínez, and Markus Nebel

AofA 2017

28th International Meeting on Probabilistic,
Combinatorial and Asymptotic Methods
for the Analysis of Algorithms

Outline

0 Intro

1 Quicksort and Search Trees

2 Saturated Fringe-Balanced Trees

3 Back to Multiset Permutations

Don't we know everything about Quicksort by now?

- Extensive literature and results on Quicksort

Type of result	Analysis Techniques	Algorithm variants	Cost Measures

- **but:** most results consider **random permutations** as input!
- partly justified: we can (should!) randomize Quicksort, (not done in libraries...)
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Assumptions:

1. **Input:** (A)
   Multiset Model:
   Random permutation $U_1, \dots, U_n$ of **fixed multiset**
   $x_1, \dots, x_u$ number of occurrences of values $1, \dots, u$

   (B)
   Discrete i.i.d. Model:
   $U_1, \dots, U_n$ i.i.d. with $\Pr[U_1 = v] = q_v$
   $\vec{q} = (q_1, \dots, q_u)$ a **fixed** universe **distribution**

2. **fat-pivot** partitioning
   - all **duplicates** of pivots removed
   - $\rightsquigarrow$ subproblems of **same** type, (restricted to a sub-universe)

3. **Cost:** # ternary comparisons

Median-of-(2t+1) Quicksort:

- median-of-$(2t+1)$
- (extension to asymmetric sampling possible) not today

Example:
$t = 3$

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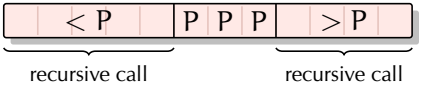

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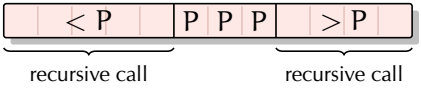

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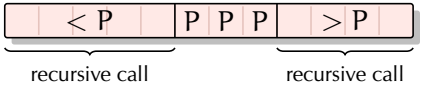

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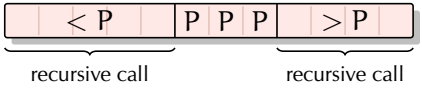

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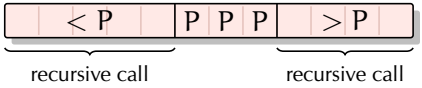
$U_1, \dots, U_n$ i.i.d. with $\Pr[U_1 = v] = q_v$ ⇝ random profile $\vec{X} \stackrel{\mathcal{D}}{=} \mathrm{Mult}(n, \vec{q})$
$\vec{q} = (q_1, \dots, q_u)$ a **fixed** universe **distribution**

2. **fat-pivot** partitioning
   - **all duplicates** of pivots removed

   ⇝ subproblems of **same** type, (restricted to a sub-universe)



3. **Cost:** # ternary comparisons

Median-of-(2t+1) Quicksort:

- median-of-$(2t+1)$
- (extension to asymmetric sampling possible)

not today

Example:
$t = 3$

Setup

Assumptions:

1. **Input:** **(A)** 

Multiset Model:
Random permutation $U_1, \dots, U_n$ of **fixed multiset**
$x_1, \dots, x_u$ number of occurrences of values $1, \dots, u$
profile $\vec{x}$ of input

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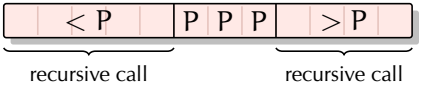

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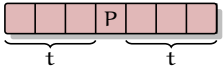

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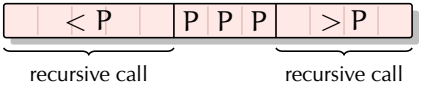

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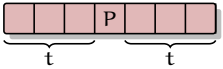

Previous work on equal keys

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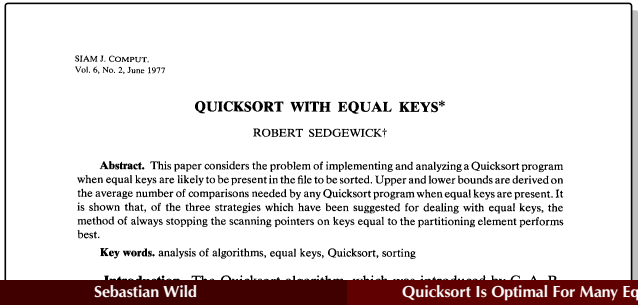

SIAM J. COMPUT.
Vol. 6, No. 2, June 1977

QUICKSORT WITH EQUAL KEYS*

ROBERT SEDGEWICK†

Abstract. This paper considers the problem of implementing and analyzing a Quicksort program when equal keys are likely to be present in the file to be sorted. Upper and lower bounds are derived on the average number of comparisons needed by any Quicksort program when equal keys are present. It is shown that, of the three strategies which have been suggested for dealing with equal keys, the method of always stopping the scanning pointers on keys equal to the partitioning element performs best.

Key words. analysis of algorithms, equal keys, Quicksort, sorting

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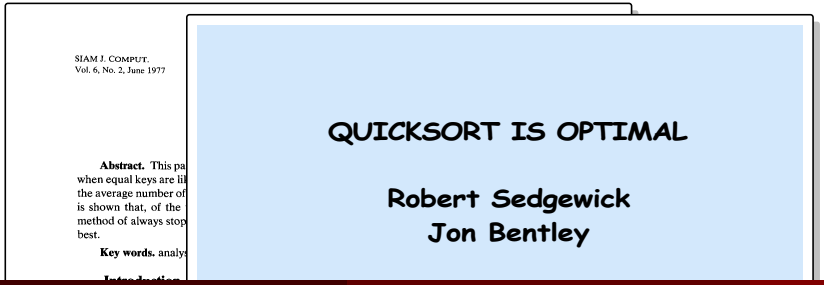

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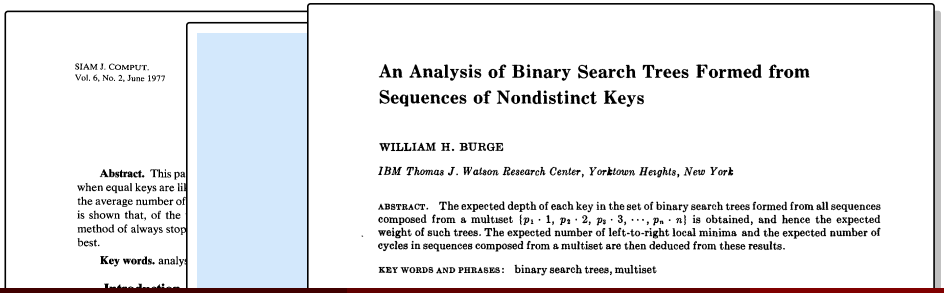

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An Analysis of Binary Search Trees Formed from Sequences of Nondistinct Keys

WILLIAM H. BURGE

IBM Thomas J. Watson Research Center, Yorktown Heights, New York

ABSTRACT. The expected depth of each key in the set of binary search trees formed from all sequences composed from a multiset $\{p_1 \cdot 1,\ p_2 \cdot 2,\ p_3 \cdot 3, \cdots, p_n \cdot n\}$ is obtained, and hence the expected weight of such trees. The expected number of left-to-right local minima and the expected number of cycles in sequences composed from a multiset are then deduced from these results.

KEY WORDS AND PHRASES: binary search trees, multiset

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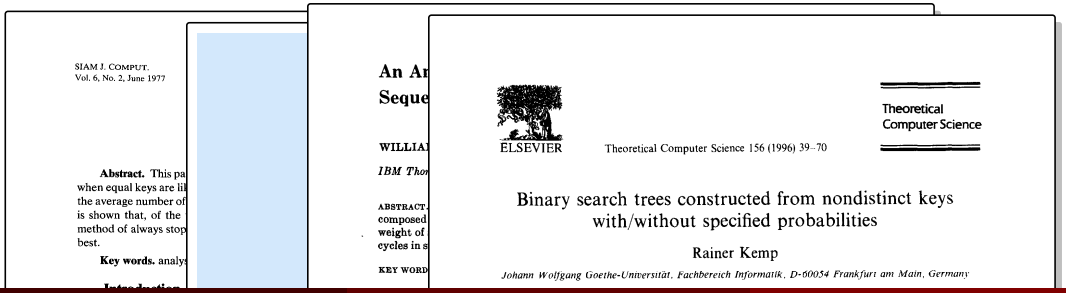
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Theoretical Computer Science 156 (1996) 39–70

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Binary search trees constructed from nondistinct keys with/without specified probabilities

Rainer Kemp

Johann Wolfgang Goethe-Universität, Fachbereich Informatik, D-60054 Frankfurt am Main, Germany

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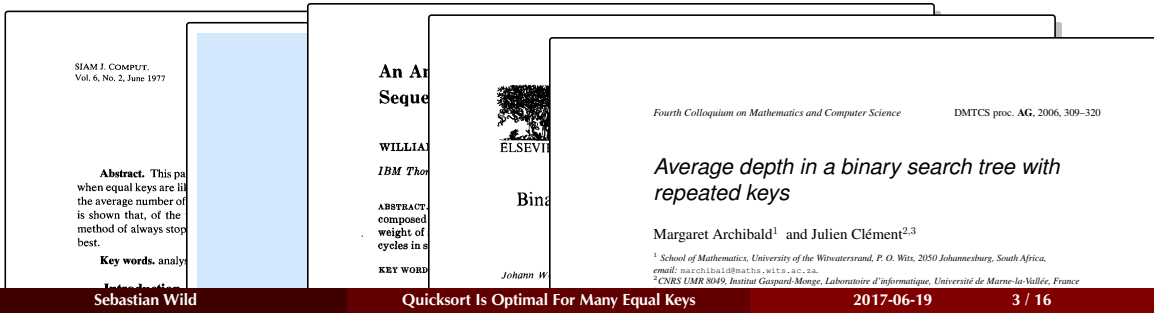

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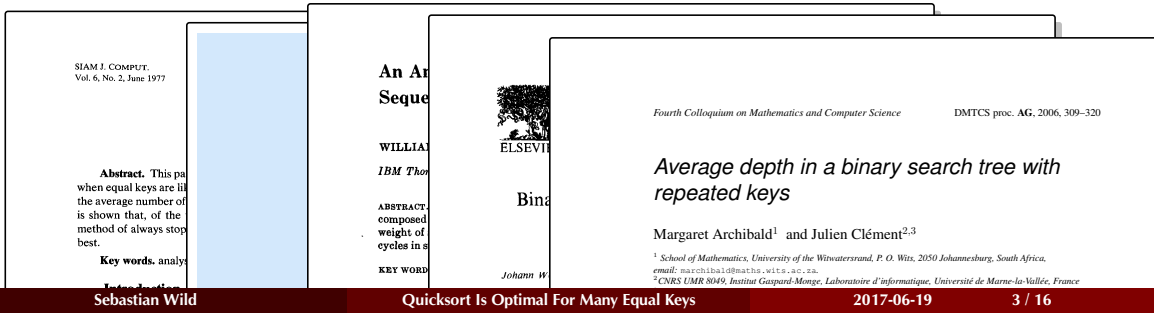
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Fourth Colloquium on Mathematics and Computer Science DMTCS proc. **AG**, 2006, 309–320

Average depth in a binary search tree with repeated keys

Margaret Archibald[1] and Julien Clément[2,3]

[1] *School of Mathematics, University of the Witwatersrand, P. O. Wits, 2050 Johannesburg, South Africa, email:* marchibald@maths.wits.ac.za
[2] *CNRS UMR 8049, Institut Gaspard-Monge, Laboratoire d'informatique, Université de Marne-la-Vallée, France*

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All these works only consider **classic** Quicksort:

- **No sampling** to choose pivots.
- (No multiway partitioning.)

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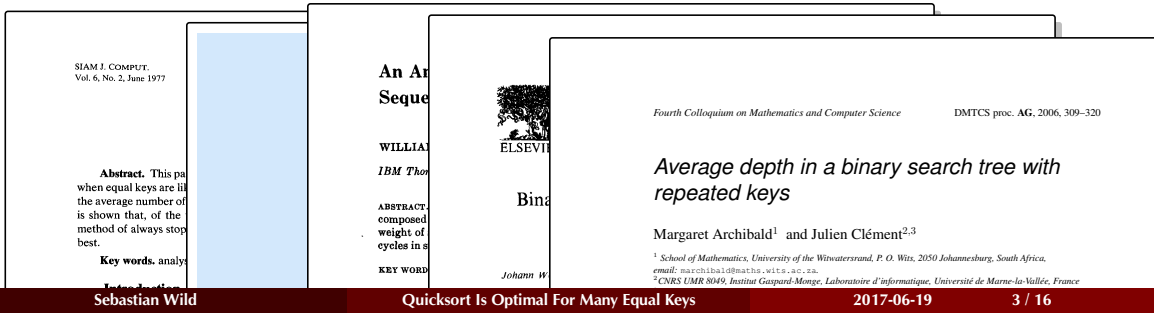

Sedgewick's analysis for classic Quicksort

Classic Quicksort:

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Classic Quicksort:

Analysis of Quicksort with equal keys

1. Define $C(x_1,\ldots,x_n) = C(1,n)$ to be the mean # compares to sort the file

$$C(1,n) = N - 1 + \frac{1}{N}\sum_{1\le j\le n} x_j(C(1,j-1) + C(j+1,n))$$

2. Multiply both sides by $N = x_1 + \ldots + x_n$

$$NC(1,n) = N(N-1) + \sum_{1\le j\le n} x_j C(1,j-1) + \sum_{1\le j\le n} x_j C(j+1,n)$$

3. Subtract same equation for $x_2,\ldots,x_n$ and let $D(1,n) = C(1,n) - C(2,n)$

$$(x_1 + \ldots + x_n)D(1,n) = x_1^2 - x_1 + 2x_1(x_2 + \ldots + x_n) + \sum_{2\le j\le n} x_j D(1,j-1)$$

4. Subtract same equation for $x_1,\ldots,x_{n-1}$

$$(x_1 + \ldots + x_n)D(1,n) - (x_1 + \ldots + x_{n-1})D(1,n-1) = 2x_1x_n + x_nD(1,n-1)$$

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Analysis of Quicksort with equal keys (cont.)

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6. Telescope (twice)

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THEOREM. Quicksort (with 3-way partitioning, randomized) uses

$N - n + 2QN$ compares (where $Q = \sum_{1\le k<j\le n} \frac{p_kp_j}{p_k + \dots + p_j}$, with $p_i = x_i/N$)

to sort an $(x_1,\dots,x_n)$–file, on the average .

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Classic Quicksort: Expected comparisons expressible **exactly**.

Analysis of Quicksort with equal keys

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Quicksort is optimal

The average number of compares per element C/N is always within a constant factor of the entropy H

lower bound: $C > NH - N$ (information theory)
upper bound: $C < 2\ln 2 NH + N$ (Burge analysis, Melhorn bound)

No comparison-based algorithm can do better.

Conjecture: With sampling, $C / N \to H$ as sample size increases.

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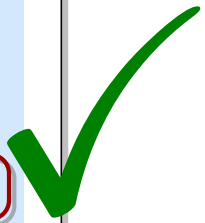

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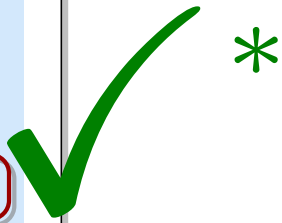

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Outline

0 Intro

1 Quicksort and Search Trees

2 Saturated Fringe-Balanced Trees

3 Back to Multiset Permutations

Quicksort & search trees

Classic Fact:

- Recursion Tree of Quicksort = Naturally grown BST from input
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- How about inputs with duplicates?

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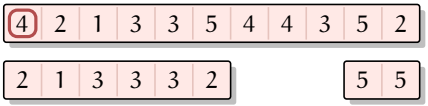

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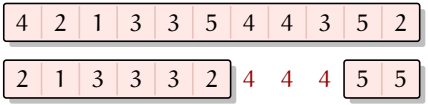

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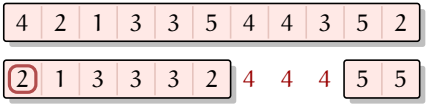

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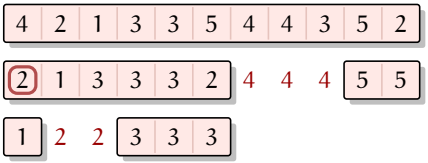

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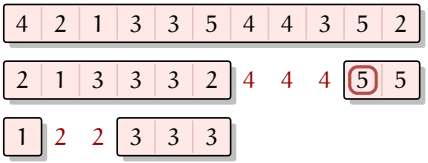

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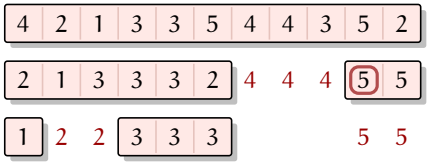

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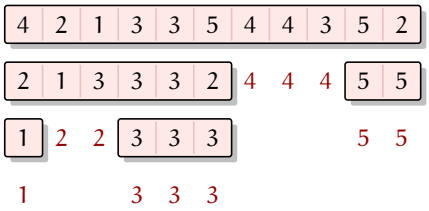

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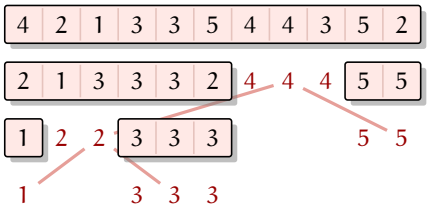

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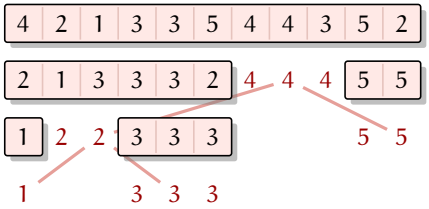

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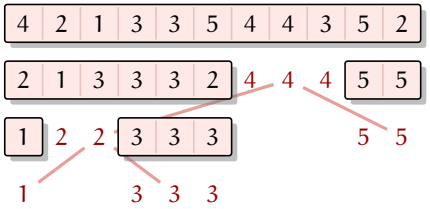

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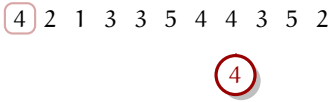

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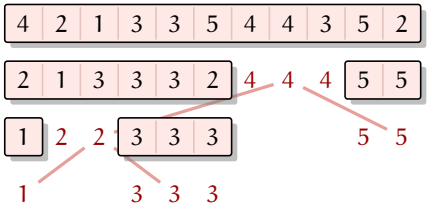

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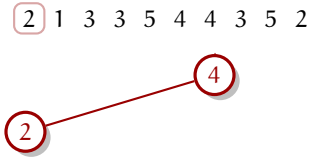

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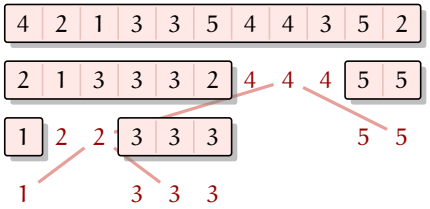

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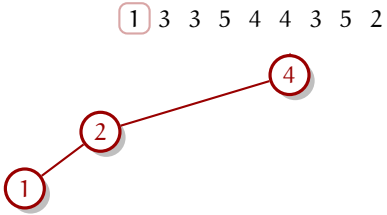

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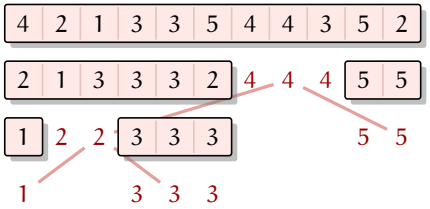

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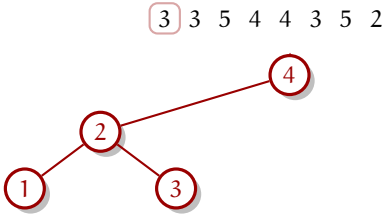

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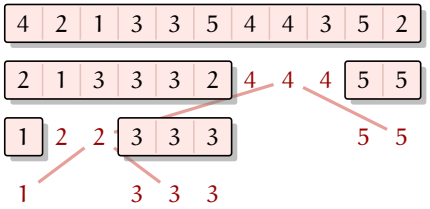

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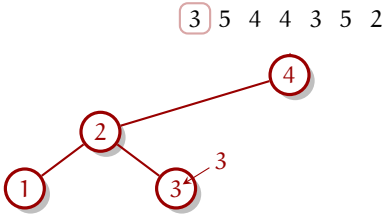

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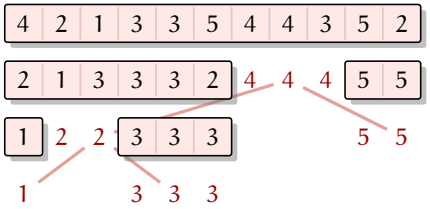

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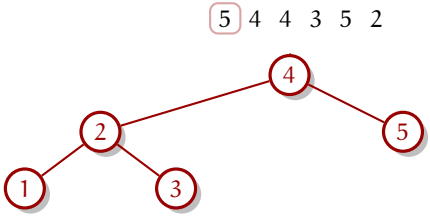

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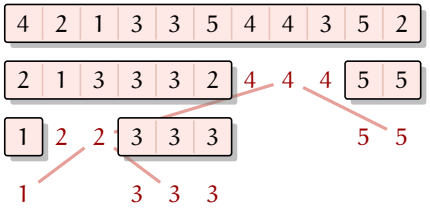

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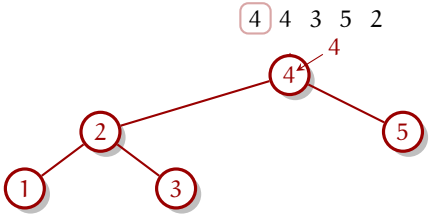

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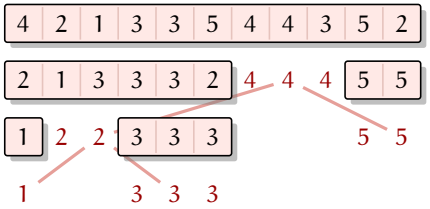

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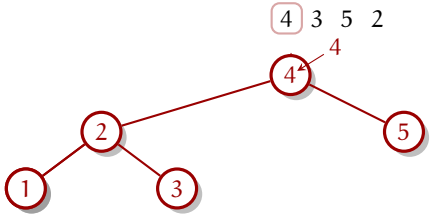

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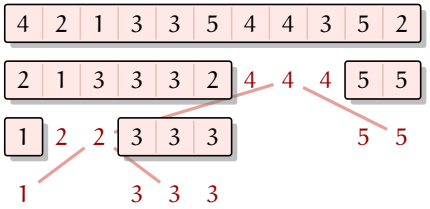

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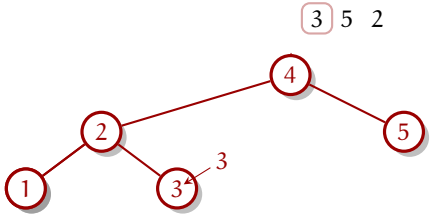

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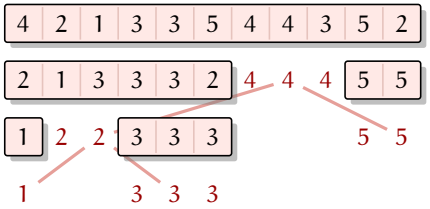

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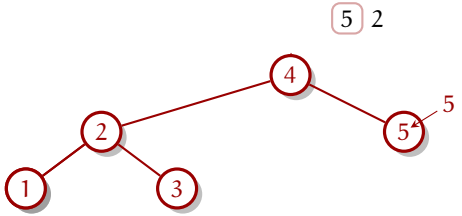

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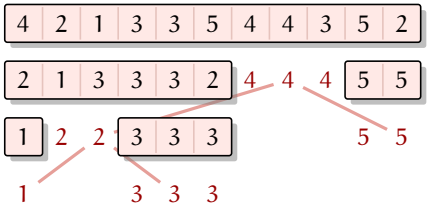

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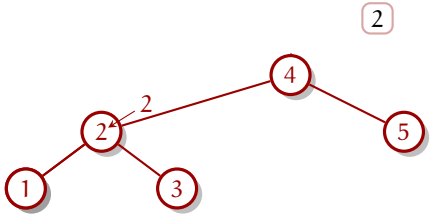

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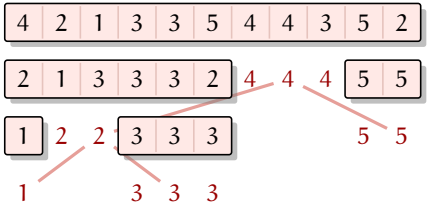

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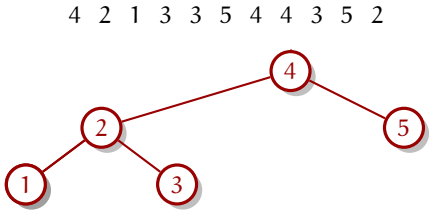

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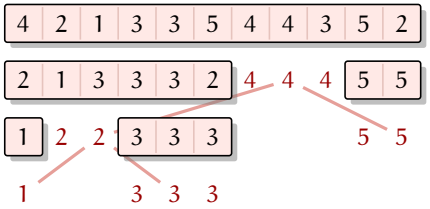

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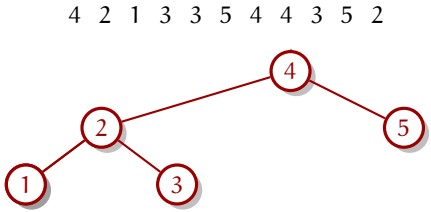

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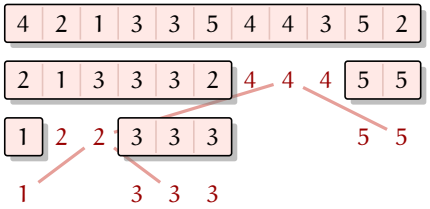

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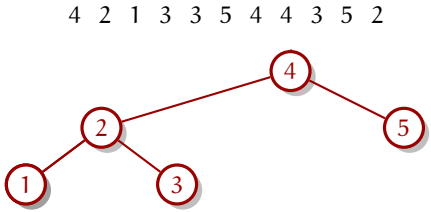

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- Leaves **buffer** $k = 2t + 1$ elements.
- If buffer is full, leaf is **split** $\rightsquigarrow$ new internal node with **chosen pivot**.

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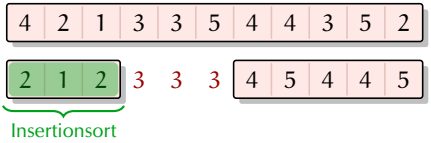

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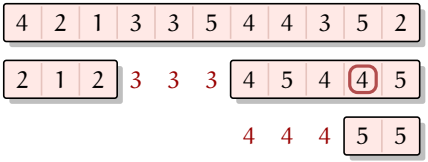

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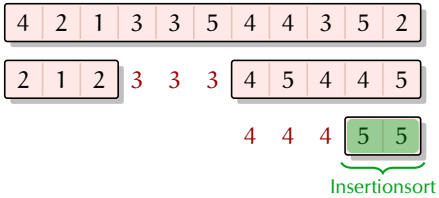

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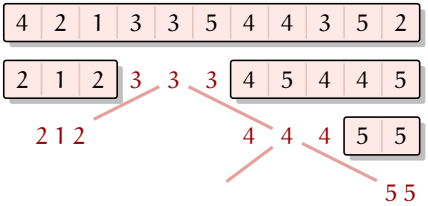

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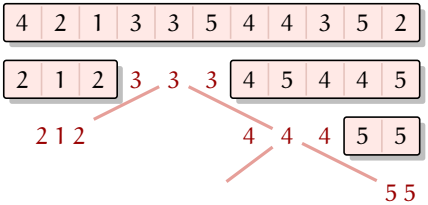

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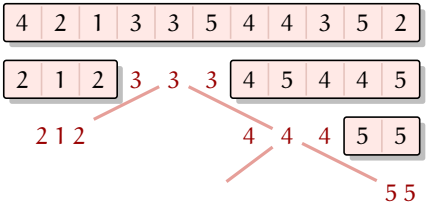

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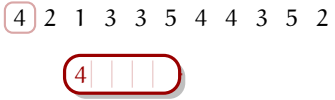

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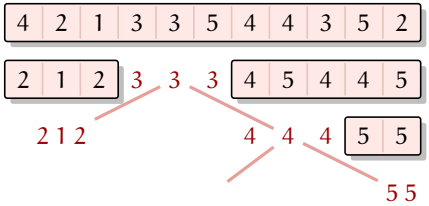

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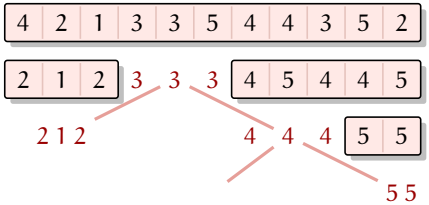

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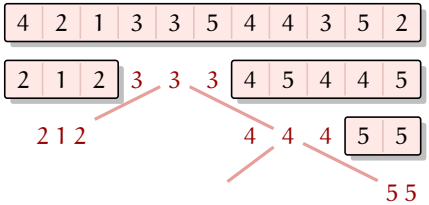

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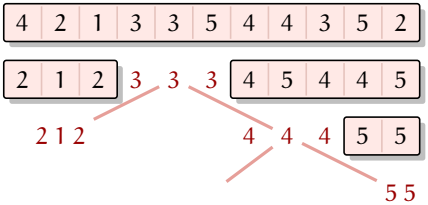

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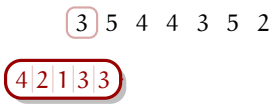

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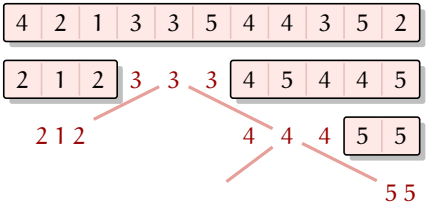

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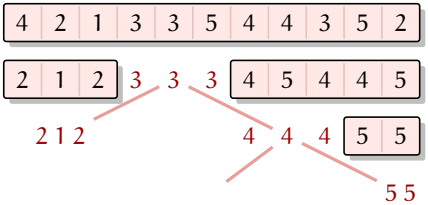

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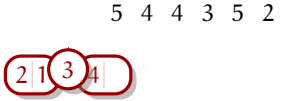

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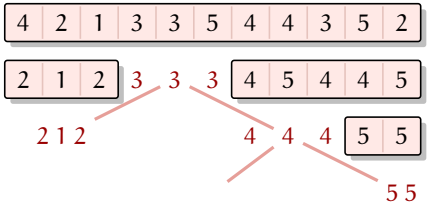

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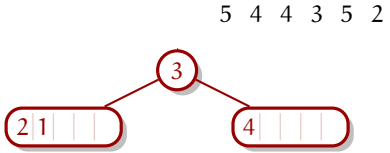

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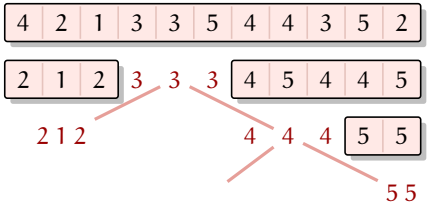

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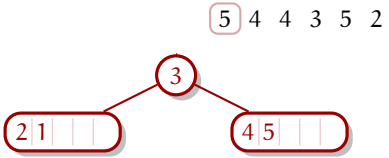

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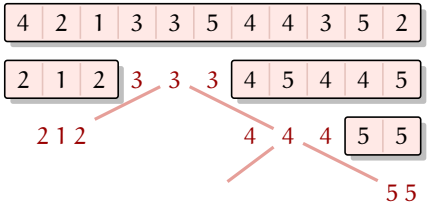

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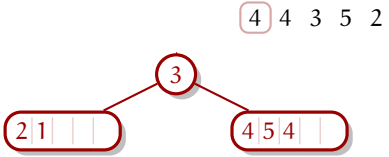

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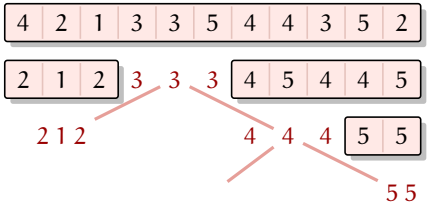

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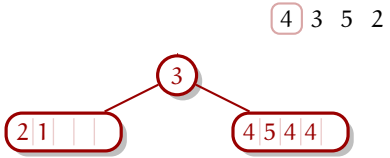

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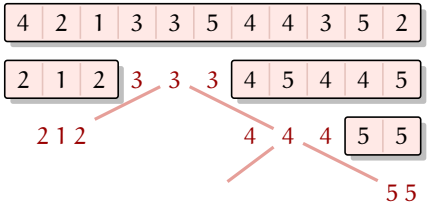

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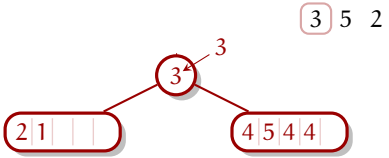

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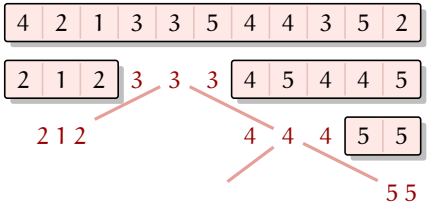

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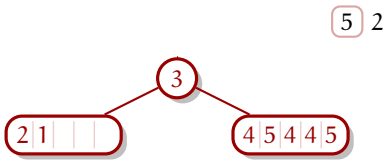

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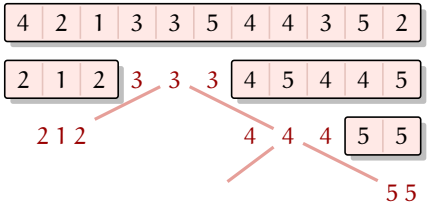

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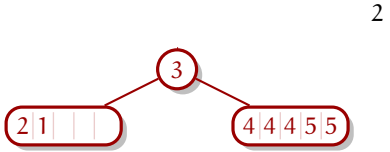

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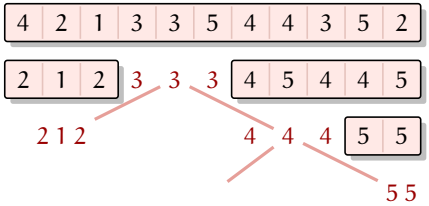

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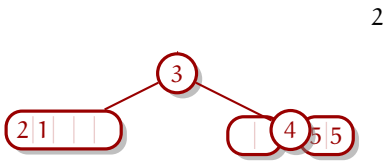

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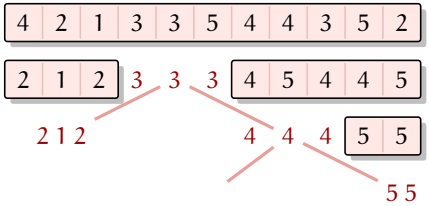

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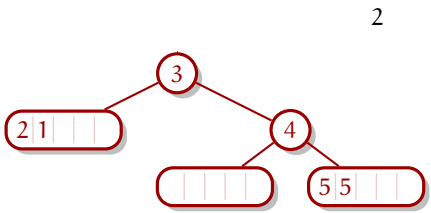

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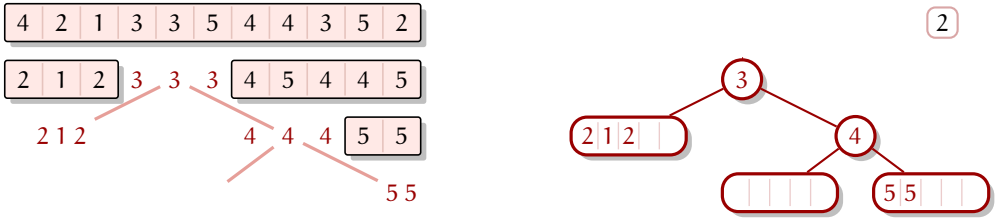

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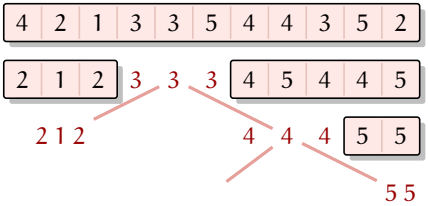

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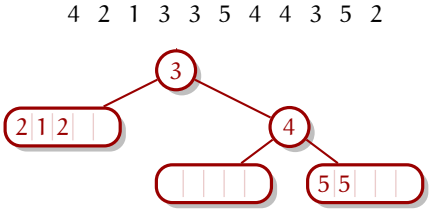

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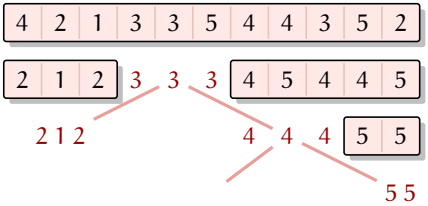

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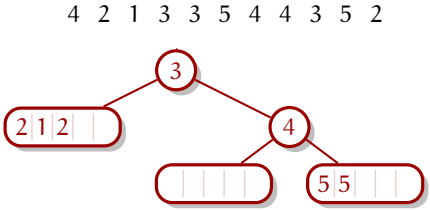

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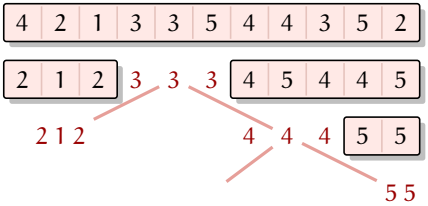

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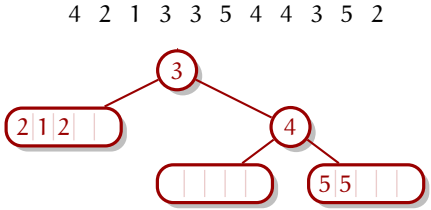

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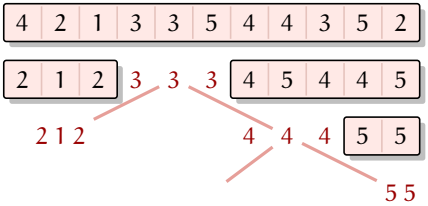

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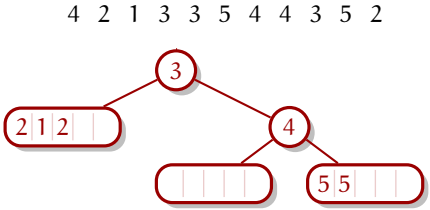

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- $\mathcal{T}$ fixed $\rightsquigarrow$ search cost depends only on **profile** $\vec{X} = (X_1, \dots, X_u)$
- **but:** $\mathcal{T}$ also depends on $\vec{U}$ (Recall: $\mathcal{T}$ is *built* from $\vec{U}$!)

$\rightsquigarrow$ direct analysis no simpler than for Quicksort

$(k = 1)$

Observation: $\mathcal{T}$ becomes **stationary** after each value was inserted!

Split input into **tree-growing** part and **searching** part:

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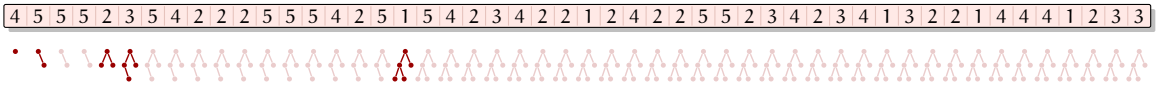

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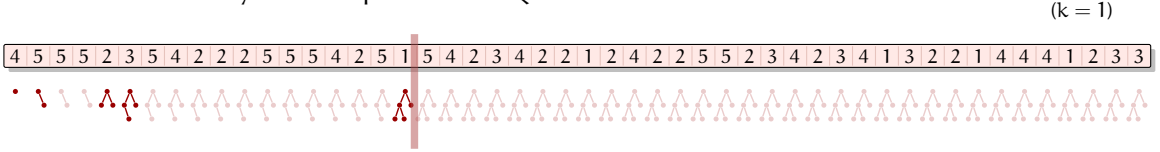

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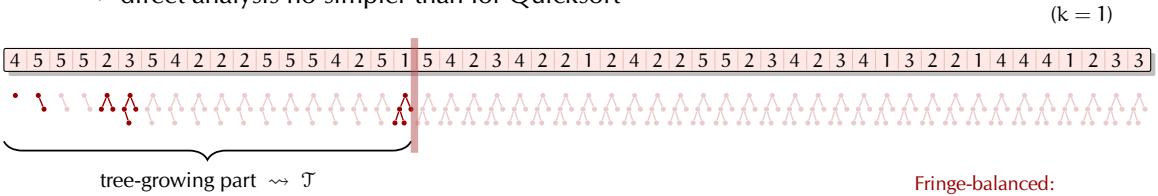

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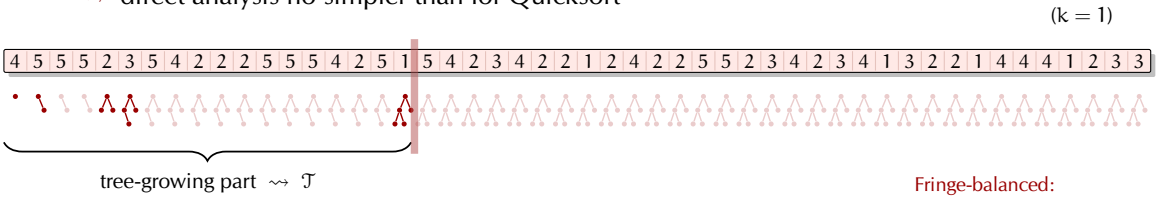

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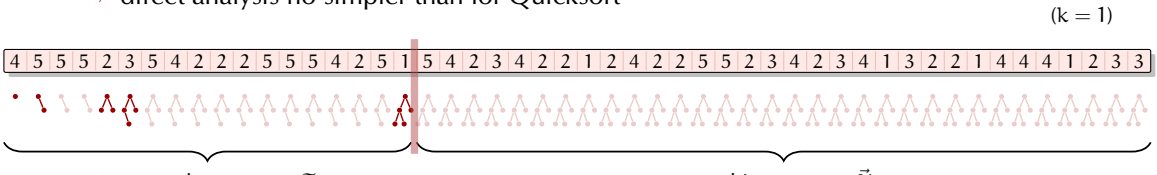

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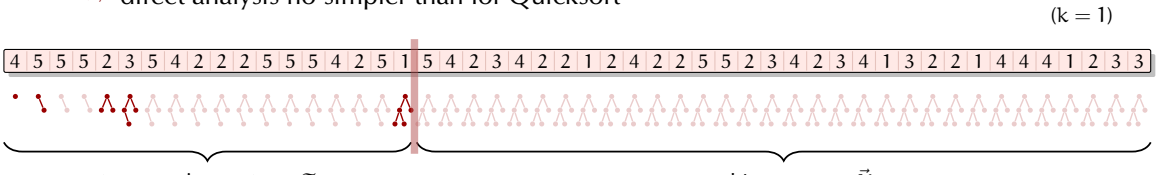

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Two parts of input always dependent! (profiles must **sum to** $\vec{x}$)

Two parts are **independent** (i.i.d.!)

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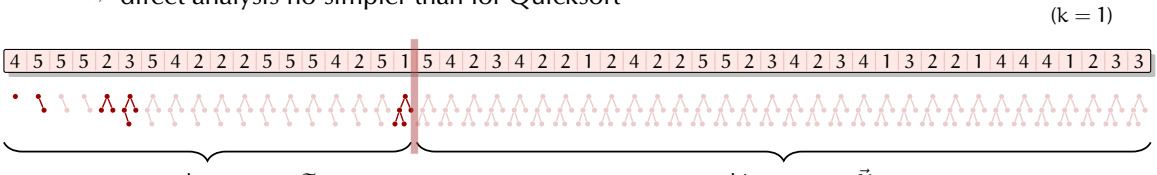

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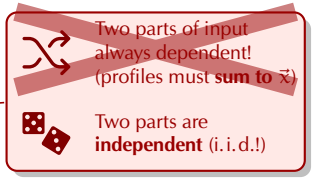

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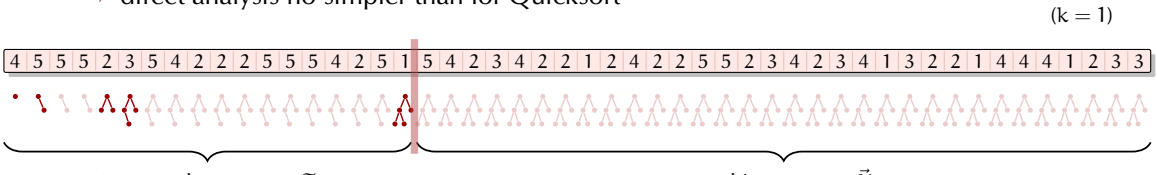

Observation: $\mathcal{T}$ becomes **stationary** after each value was inserted!

Fringe-balanced: $\rightsquigarrow$ stationary after each value inserted $k = 2t + 1$ **times** (up to k duplicates in buffer)

hopefully a **short** prefix!

Split input into **tree-growing** part and **searching** part:

1. We built $\mathcal{T}$ until it is stationary, **ignoring costs**.
2. Determine costs of **searching** remaining elements.

$\rightsquigarrow$ profile $\vec{X}_S$ of search part **independent** of $\mathcal{T}$

~~Two parts of input always dependent! (profiles must **sum to** $\vec{x}$)~~

Two parts are **independent** (i.i.d.!)

Bounding the tree-growing part

Goal: *ignore tree-growing for analysis.*



$\rightsquigarrow$ Allow only first n_T elements for tree growing.

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Outline

0 Intro

1 Quicksort and Search Trees

2 Saturated Fringe-Balanced Trees

3 Back to Multiset Permutations

Search Costs in Saturated Trees

Recall: $\alpha(\vec{q}) = \sum_{v=1}^{u} q_v \cdot depth_{\mathcal{T}}(v)$ $\mathcal{T}$ from inserting i.i.d. $\mathcal{D}(\vec{q})$ elements **until saturation**

***Warmup:* Ordinary BSTs** ($t = 0$)

Old result: **Allen & Munro** 1978:
Self-Organizing Binary Search Trees

$\alpha(\vec{q}) = 2\mathcal{H}_Q(\vec{q}) + 1$ with

$$\mathcal{H}_Q(\vec{q}) = \sum_{1 \leqslant i < j \leqslant u} \frac{q_i q_j}{q_i + \cdots + q_j}$$

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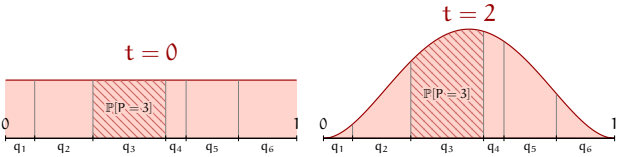

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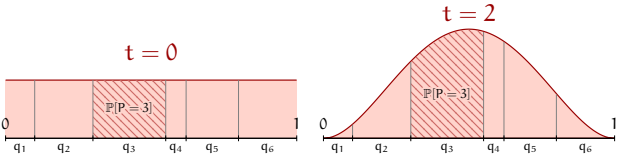

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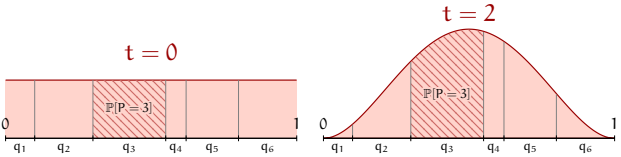

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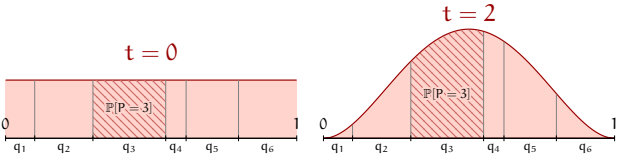

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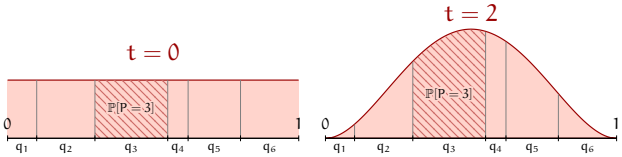

⇝ prob. that i inserted first among $i, \dots j$??

old approach does not work

Search Costs in Saturated Trees

Recall: $\alpha(\vec{q}) = \sum_{v=1}^{u} q_v \cdot depth_{\mathcal{T}}(v)$

$\mathcal{T}$ from inserting i.i.d. $\mathcal{D}(\vec{q})$ elements **until saturation**

(distribution with prob. weights $q_1, \dots, q_u$)

Warmup: Ordinary BSTs ($t = 0$)

Old result: **Allen & Munro** 1978: *Self-Organizing Binary Search Trees*

$$\alpha(\vec{q}) = 2\mathcal{H}_Q(\vec{q}) + 1 \text{ with}$$

$$\mathcal{H}_Q(\vec{q}) = \sum_{1 \leqslant i < j \leqslant u} \frac{q_i q_j}{q_i + \dots + q_j}$$

- **Proof sketch:**
  Sum prob. that i is ancestor of j over all i, j
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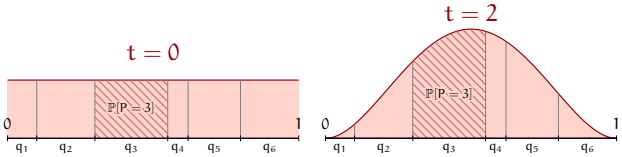

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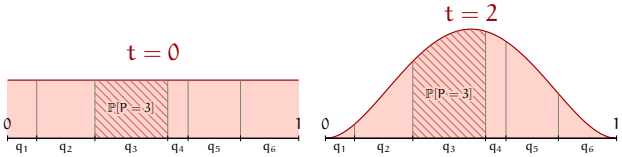

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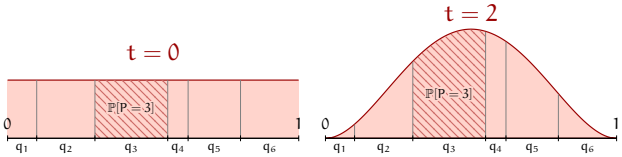

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$\rightsquigarrow$ Try to generalize this!

Aggregation of Entropy

- One of the defining properties of Shannon entropy: ***aggregation***

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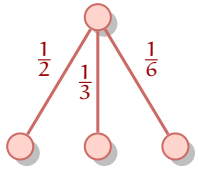

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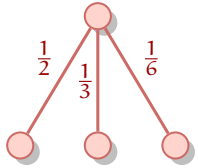

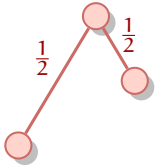

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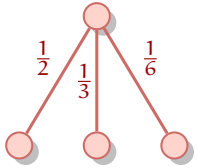

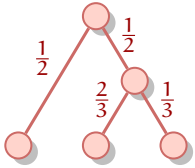

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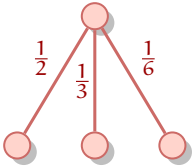
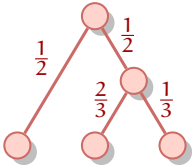


$\mathcal{H}(\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$



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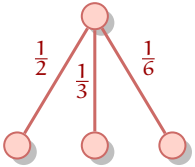

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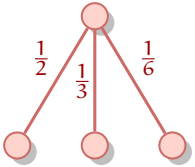

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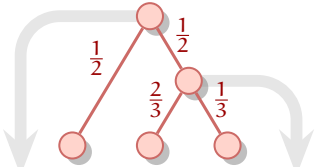

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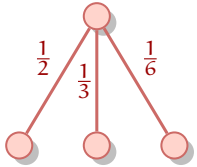

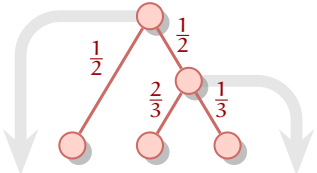

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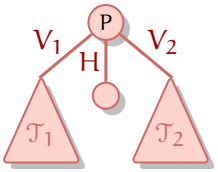

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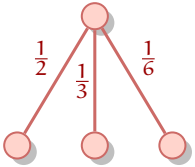

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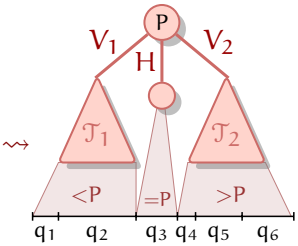

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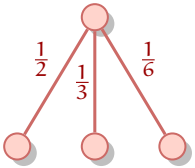

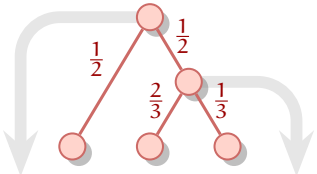

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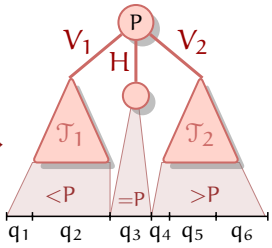

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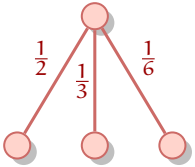

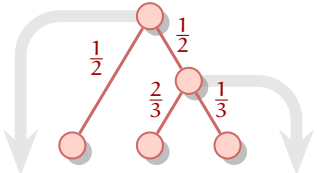

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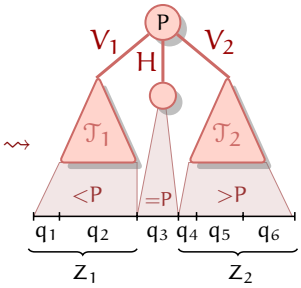

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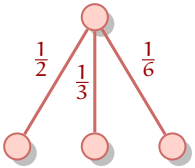

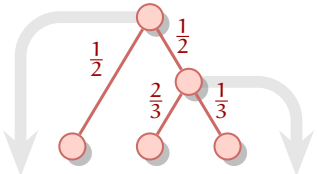

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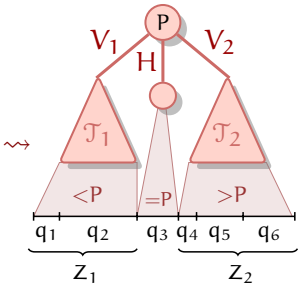

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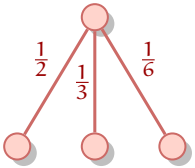

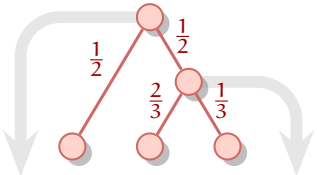

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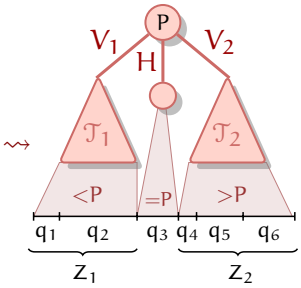

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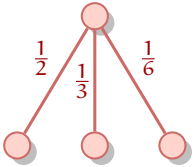

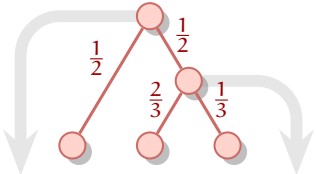

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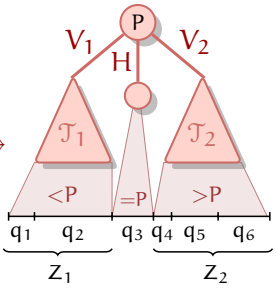

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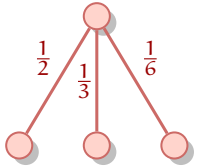

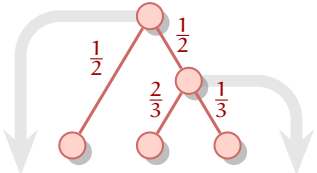

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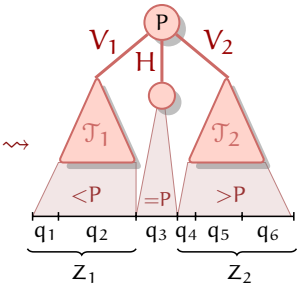

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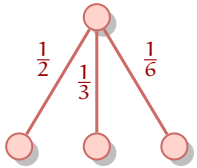

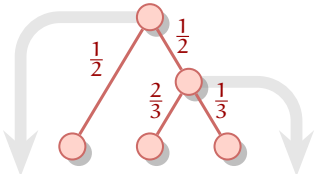

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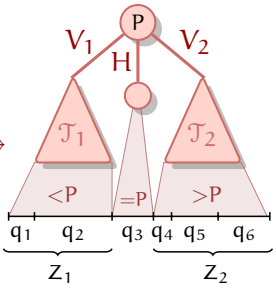

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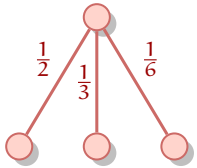

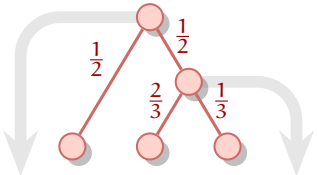

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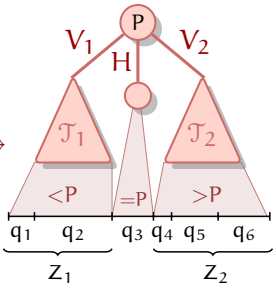

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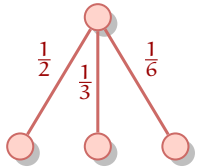

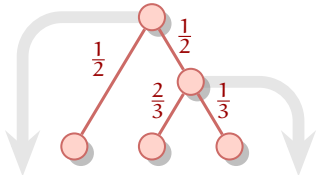

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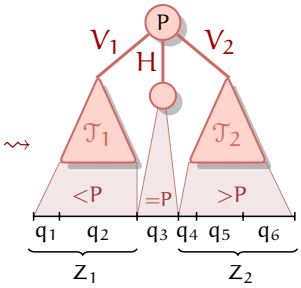

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Entropy Bounds for Search Costs

$\rightsquigarrow$ $\alpha(\vec{q}) = c \cdot \mathcal{H}(\vec{q})$ does **not** seem to hold for any constant c

- But we can show

$$\begin{aligned} \alpha(\vec{q}) &\leqslant c \cdot \mathcal{H}(\vec{q}) + d \\ \alpha_{\vec{q}} &\geqslant c' \cdot \mathcal{H}(\vec{q}) - d' \end{aligned}$$

  for family of constants (c, d) and (c', d').

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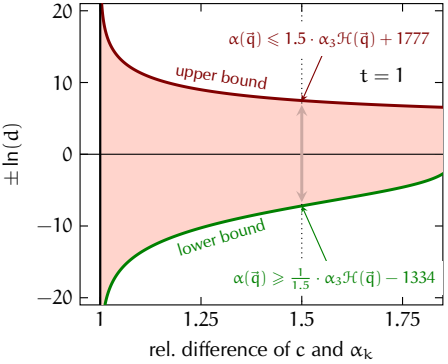

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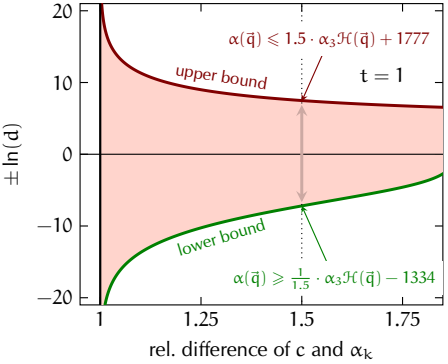

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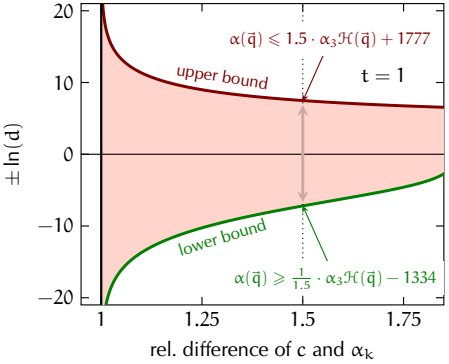

$\rightsquigarrow$ **Asymptotically matching** values for c and c'

$\rightsquigarrow$ $$\alpha(\vec{q}) = \alpha_k \cdot \mathcal{H}_{ld}(\vec{q}) \pm O\Big(\mathcal{H}(\vec{q})^{\frac{t+2}{t+3}} \log\big(\mathcal{H}(\vec{q})\big)\Big)$$

Results in the i.i.d. Model



Time to put the pieces together!

Separation Theorem:
Quicksort costs

- in the **i.i.d. model**
- with **"many duplicates"**
  ($\Omega(n^\varepsilon)$ duplicates of each value in expectation)

are given by (as $n \to \infty$, for any $\delta \in (0, \varepsilon)$)

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Average search costs
in saturated k-fringe-balanced trees:

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Quicksort Costs (i.i.d. model)

Under the assumptions above, we have for any $\delta \in \big(0, \frac{1}{t+3}\big)$
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Outline

0 Intro

1 Quicksort and Search Trees

2 Saturated Fringe-Balanced Trees

3 Back to Multiset Permutations

Back to the Multiset Model

How about the multiset model?

Many duplicates $\rightsquigarrow$ profile $\vec{X}$ **concentrated** around $\mathbb{E}[\vec{X}] = \vec{q}n$

1. Replace multiset model with profile $\vec{x}$ by i.i.d. model with $\vec{q} = \vec{x}/n$
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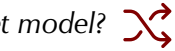
How about the multiset model? 



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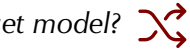
How about the multiset model? 



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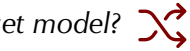
How about the multiset model? 



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The Annals of Statistics
1983, Vol. 11, No. 3, 896–904

THE EQUIVALENCE OF WEAK, STRONG AND COMPLETE CONVERGENCE IN L_1 FOR KERNEL DENSITY ESTIMATES[1]

BY LUC DEVROYE

McGill University

Let f be a density on R^d, and let f_n be the kernel estimate of f,

$$f_n(x) = (nh^d)^{-1} \sum_{i=1}^{n} K((x - X_i)/h)$$

where $h = h_n$ is a sequence of positive numbers, and K is an absolutely integrable function with $\int K(x)\,dx = 1$. Let $J_n = \int |f_n(x) - f(x)|\,dx$. We show

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The Annals of Statistics

LEMMA 3. (*A multinomial distribution inequality*). *Let* $(X_1, \cdots, X_k)$ *be a multinomial* $(n, p_1, \cdots, p_k)$ *random vector. For all* $\varepsilon \in (0, 1)$ *and all* k *satisfying* $k/n \leq \varepsilon^2/20$, *we have*

$$P(\textstyle\sum_{i=1}^{k} |X_i - E(X_i)| > n\varepsilon) \leq 3 \exp(-n\varepsilon^2/25).$$

McGill University

Let f be a density on R^d, and let f_n be the kernel estimate of f,

$$f_n(x) = (nh^d)^{-1} \textstyle\sum_{i=1}^{n} K((x - X_i)/h)$$

where $h = h_n$ is a sequence of positive numbers, and K is an absolutely integrable function with $\int K(x)\,dx = 1$. Let $J_n = \int |f_n(x) - f(x)|\,dx$. We show

Back to the Multiset Model

How about the multiset model? 



Many duplicates $\rightsquigarrow$ profile $\vec{X}$ **concentrated** around $\mathbb{E}[\vec{X}] = \vec{q}n$

1. Replace multiset model with profile $\vec{x}$ by i.i.d. model with $\vec{q} = \vec{x}/n$
2. Use Chernoff bounds to **bound difference** between costs.

$\rightsquigarrow$ Need Chernoff bound for **multinomial variables**.

The Annals of Statistics

LEMMA 3. *(A multinomial distribution inequality). Let $(X_1, \cdots, X_k)$ be a multinomial $(n, p_1, \cdots, p_k)$ random vector. For all $\varepsilon \in (0, 1)$ and all k satisfying $k/n \leq \varepsilon^2/20$, we have*

$$P(\textstyle\sum_{i=1}^{k} |X_i - E(X_i)| > n\varepsilon) \leq 3 \exp(-n\varepsilon^2/25).$$

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$\rightsquigarrow$ **Same result** holds for **multiset model** with **"many duplicates"**, i.e., $x_\nu = \Omega(n^\varepsilon)$.

Conclusion

Findings

- First analysis of **median-of-k Quicksort on equal keys** ... for **"many duplicates"**.
  - $\rightsquigarrow$ Same relative speedup as for random permutations.
- Partial Answer to **conjecture of Sedgewick & Bentley**:

  Median-of-k Quicksort approaches lower bound for $k \to \infty$.
- *Not in this talk:* For uniform $\vec{q} = (\frac{1}{u}, \dots, \frac{1}{u})$ with $u = O(n^{1-\varepsilon})$
  - better error bounds
  - extension for multiway partitioning

Open Problems

- Get rid of "many duplicates" restriction; $n^{1-\varepsilon}$ seems (to me) best possible so that
  - inputs are non-degenerate w.h.p.
  - insertsorting costs are still negligible
  - difference between U.D. model and multiset model is negligible
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- Get rid of "many duplicates" restriction: $n^{1-\varepsilon}$ seems (to me) best possible so that
  - Inputs are non-degenerate, $n \gg u$
  - Insertion sort and pivot sorting costs are still negligible
  - Difference between 1 and n is negligible
  - Uniform distribution is a lower bound

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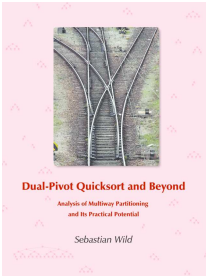

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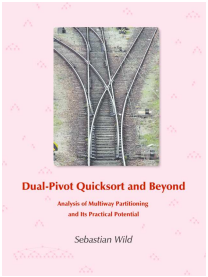

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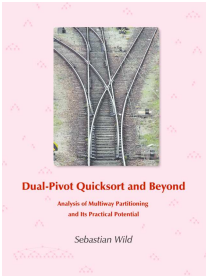

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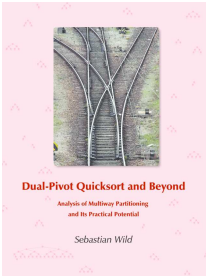

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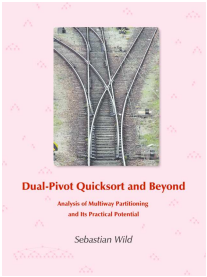

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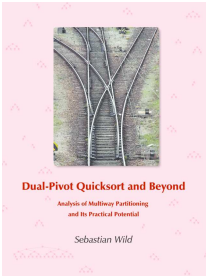

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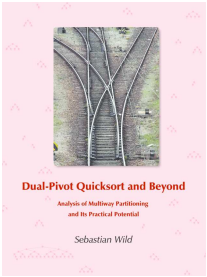

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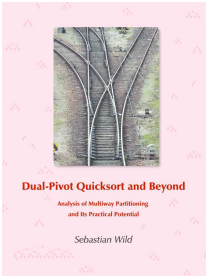

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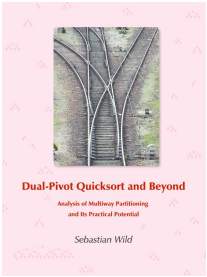

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NICKELS
$2
PENNIES
$.50
50EA
DIMES
$5
50EA

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