Polyamorous Scheduling

Sebastian Wild

joint work with Leszek Gąsieniec and Ben Smith

UNIVERSITY OF LIVERPOOL

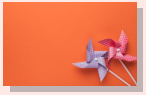
FUN with Algorithms 2024

Outline











1 Introduction

2 Hardness

3 Approximation

4 Density

5 Conclusion



1 Introduction

Polyamory

polyamory
society

Written by Alison Eldridge
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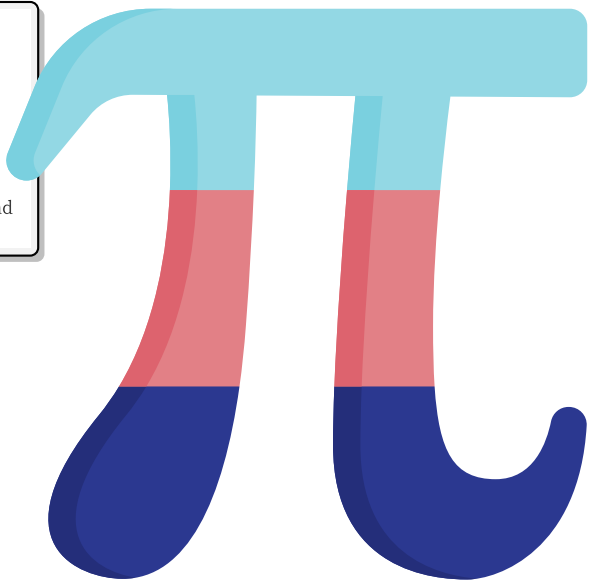
Last Updated: Apr 30, 2024 • Article History

Polyamory, having or desiring multiple intimate relationships at the same time with the full knowledge and consent of all parties involved.



Polyamory

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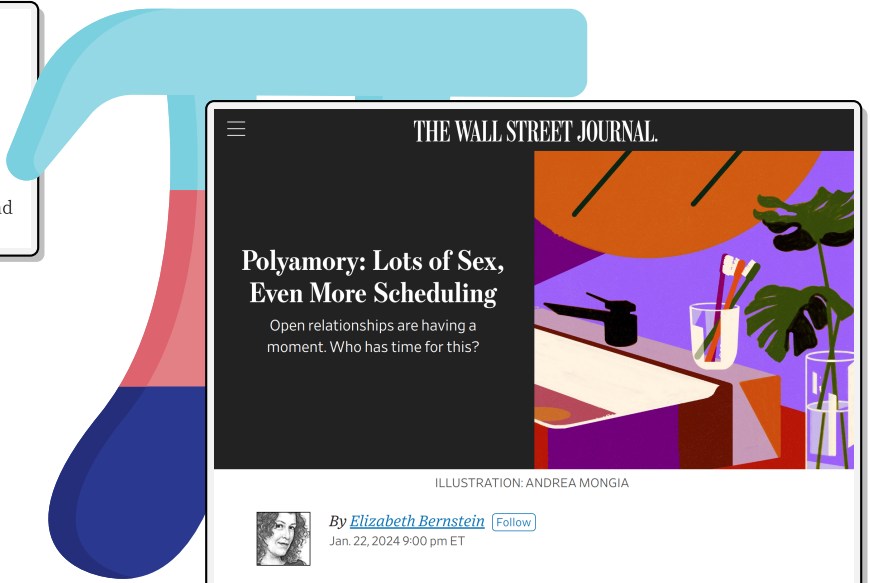

THE WALL STREET JOURNAL.

Polyamory: Lots of Sex, Even More Scheduling

Open relationships are having a moment. Who has time for this?

ILLUSTRATION: ANDREA MONGIA

By Elizabeth Bernstein Follow
Jan. 22, 2024 9:00 pm ET

Polyamory

polyamory
society

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Polyamory, having or desiring multiple intimate relationships at the same time with the full knowledge and consent of all parties involved.





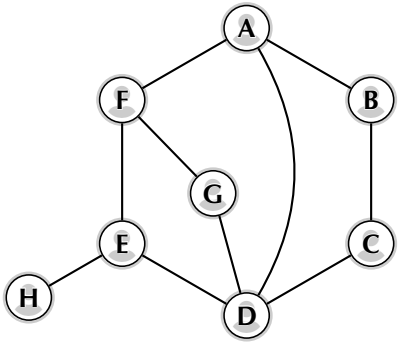
Poly Scheduling

Given:

- **People** and pairwise **Relationships**
  - ⇝ undirected graph (P, R)
- each relationship has a *"desire **growth rate**"*
  - ⇝ edge weights $g : R \to \mathbb{R}_{>0}$
    pain of separation grows by $g(e)$ each day

Goal:

- **perpetual schedule** $S : \mathbb{N}_0 \to 2^R$ that minimizes *"heat"*:
  maximal desire ever felt *(minimax objective)*
- each person can engage in **one relationship per day**
  - ⇝ every day t, schedule a **matching** $S(t)$



Example polycule with 8 persons:

Adam, Brady, Charlie, Daisy,
Eli, Frankie, Grace, and Holly

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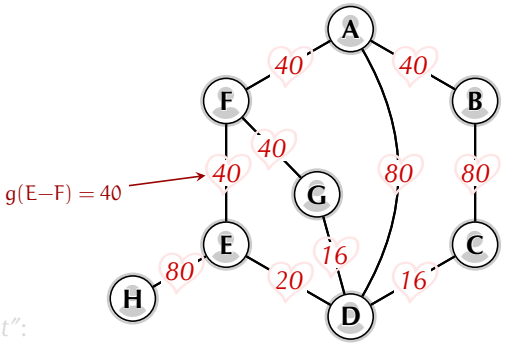

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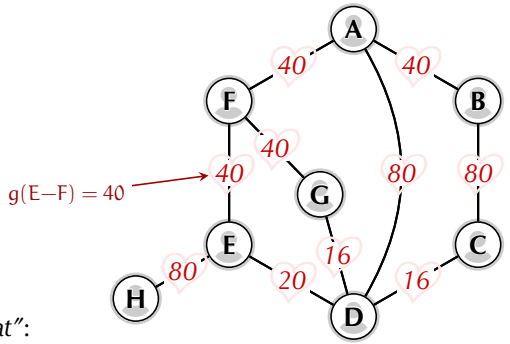


Example polycule with 8 persons:

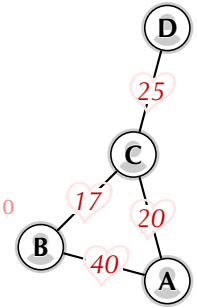
Adam, Brady, Charlie, Daisy,
Eli, Frankie, Grace, and Holly

Poly Scheduling – Small Example

Day 0

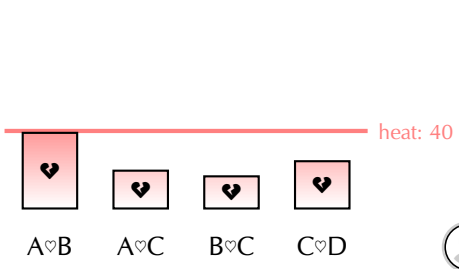
heat: 0

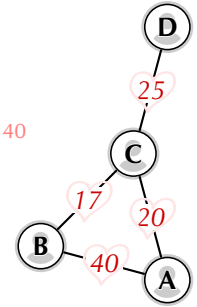
A♡B A♡C B♡C C♡D



Poly Scheduling – Small Example

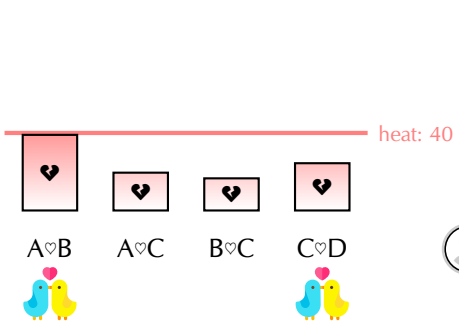
Day 0

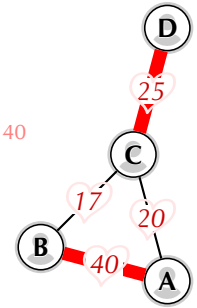




Poly Scheduling – Small Example

Day 0



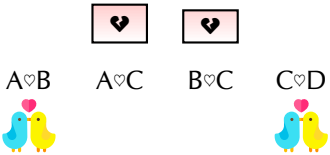


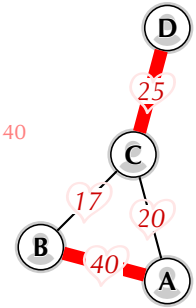
Day	A♡B	A♡C	B♡C	C♡D
0	❤			❤

Poly Scheduling – Small Example

Day 0

heat: 40

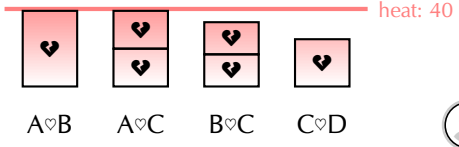


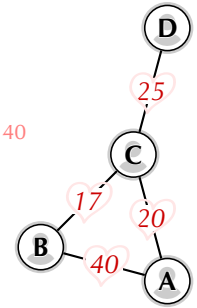


Day	A♡B	A♡C	B♡C	C♡D
0	❤			❤

Poly Scheduling – Small Example

Day 1

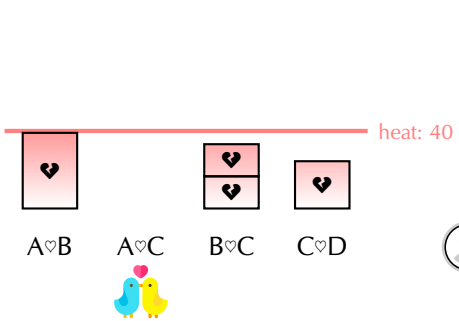


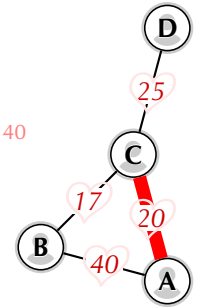


Day	A♡B	A♡C	B♡C	C♡D
0	❤			❤

Poly Scheduling – Small Example

Day 1

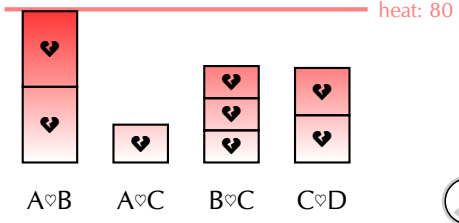


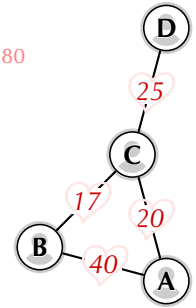


Day	A♡B	A♡C	B♡C	C♡D
0	♥			♥
1		♥		

Poly Scheduling – Small Example

Day 2

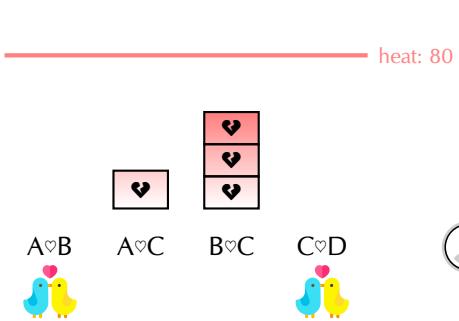


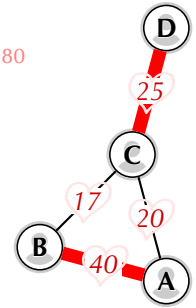


Day	A♡B	A♡C	B♡C	C♡D
0	♥			♥
1		♥		

Poly Scheduling – Small Example

Day 2

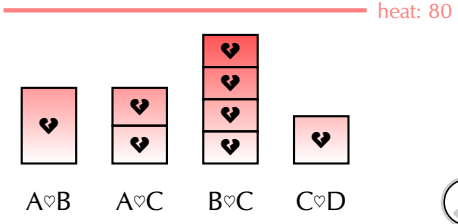


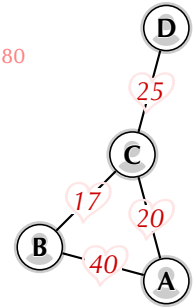


Day	A♡B	A♡C	B♡C	C♡D
0	❤			❤
1		❤		
2	❤			❤

Poly Scheduling – Small Example

Day 3

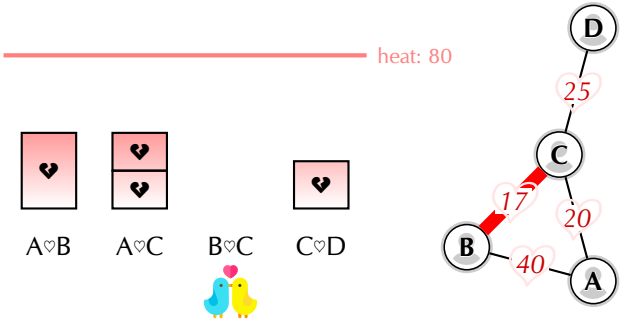




Day	A♡B	A♡C	B♡C	C♡D
0	♥			♥
1		♥		
2	♥			♥

Poly Scheduling – Small Example

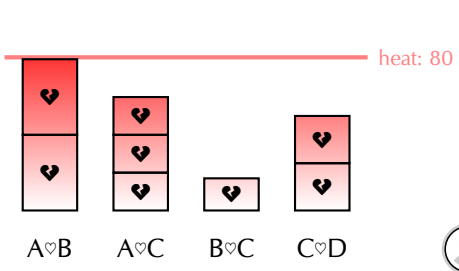
Day 3

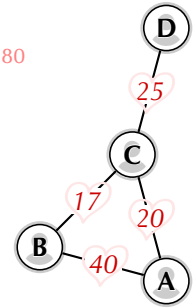


Day	A♡B	A♡C	B♡C	C♡D
0	♥			♥
1		♥		
2	♥			♥
3			♥	

Poly Scheduling – Small Example

Day 4

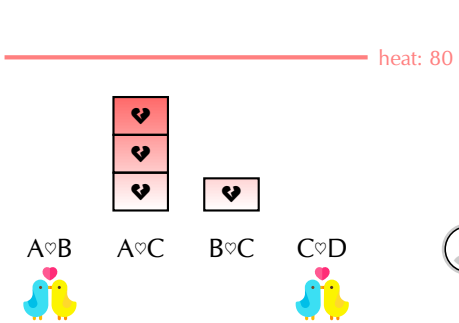


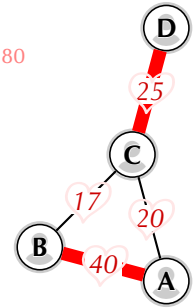


Day	A♡B	A♡C	B♡C	C♡D
0	♥			♥
1		♥		
2	♥			♥
3			♥	

Poly Scheduling – Small Example

Day 4

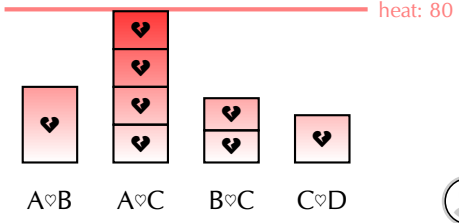


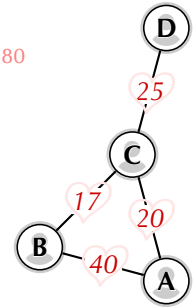


Day	A♡B	A♡C	B♡C	C♡D
0	♥			♥
1		♥		
2	♥			♥
3			♥	
4	♥			♥

Poly Scheduling – Small Example

Day 5

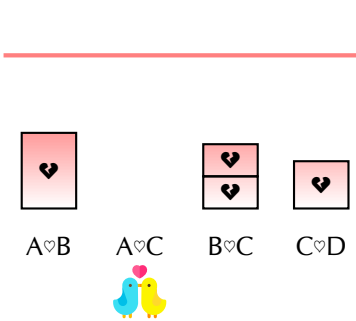


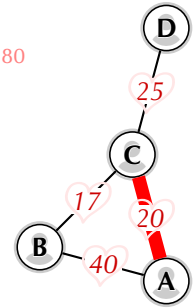


Day	A♡B	A♡C	B♡C	C♡D
0	♥			♥
1		♥		
2	♥			♥
3			♥	
4	♥			♥

Poly Scheduling – Small Example

Day 5

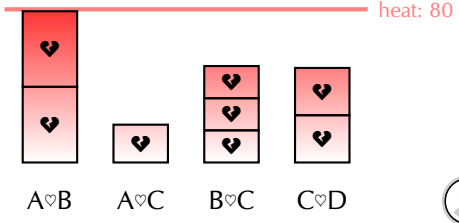


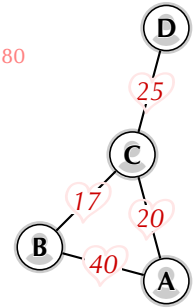


Day	A♡B	A♡C	B♡C	C♡D
0	♥			♥
1		♥		
2	♥			♥
3			♥	
4	♥			♥
5		♥		

Poly Scheduling – Small Example

Day 6

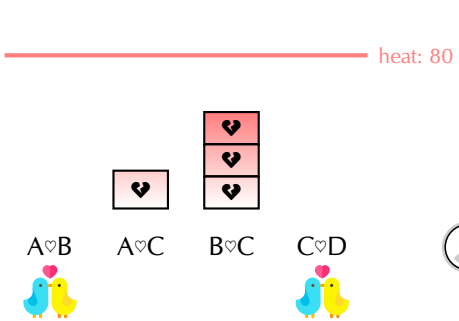


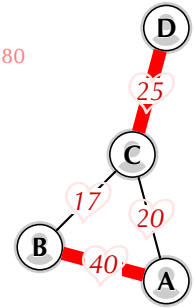


Day	A♡B	A♡C	B♡C	C♡D
0	♥			♥
1		♥		
2	♥			♥
3			♥	
4	♥			♥
5		♥		

Poly Scheduling – Small Example

Day 6

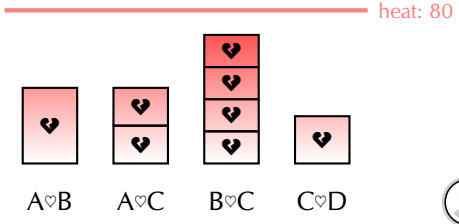


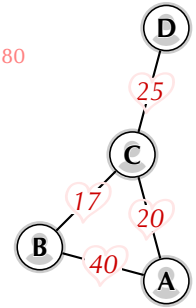


Day	A♡B	A♡C	B♡C	C♡D
0	❤			❤
1		❤		
2	❤			❤
3			❤	
4	❤			❤
5		❤		
6	❤			❤

Poly Scheduling – Small Example

Day 7

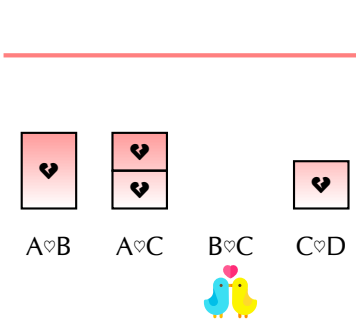


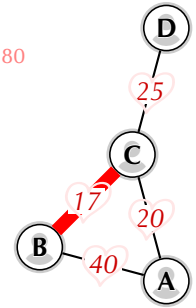


Day	A♡B	A♡C	B♡C	C♡D
0	♥			♥
1		♥		
2	♥			♥
3			♥	
4	♥			♥
5		♥		
6	♥			♥

Poly Scheduling – Small Example

Day 7

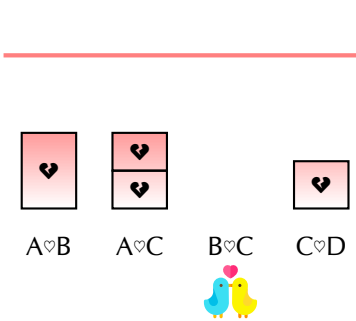


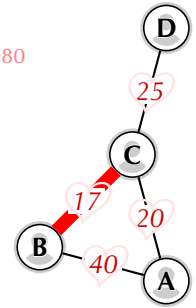


Day	A♡B	A♡C	B♡C	C♡D
0	♥			♥
1		♥		
2	♥			♥
3			♥	
4	♥			♥
5		♥		
6	♥			♥
7			♥	

Poly Scheduling – Small Example

Day 7





Day	A♡B	A♡C	B♡C	C♡D
0	♥			♥
1		♥		
2	♥			♥
3			♥	
4	♥			♥
5		♥		
6	♥			♥
7			♥	

$S^* = (\{AB, CD\}, \{AC\}, \{AB, CD\}, \{BC\})$
with heat $h(S) = 80$

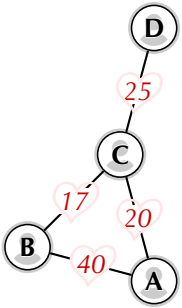
follows from finite state space

Observation: schedules without loss of generality **periodic**

$\exists T \forall t\ S(t) = S(t+T)$

Decision problem

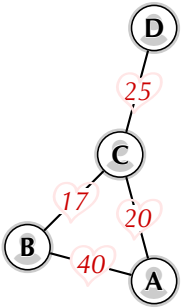
Optimization input:



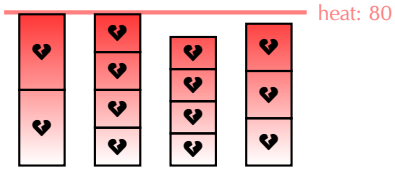
Optimization Poly Scheduling

Decision problem

Optimization input:

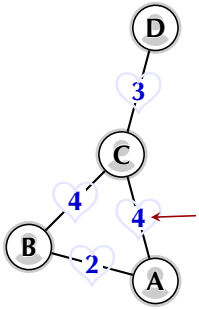


Maximal frequencies
to remain below **heat 80**:



$$\text{frequency} = \left\lfloor \frac{\text{heat}}{\text{growth rate}} \right\rfloor$$

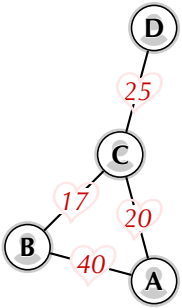
Decision instance:



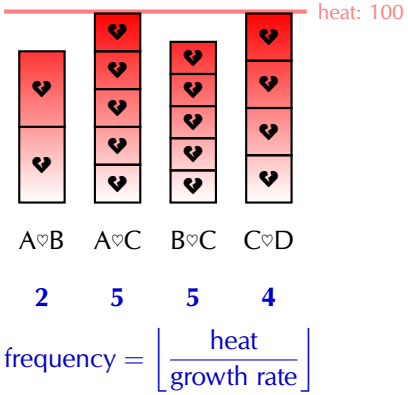
A and C must meet (at least) once in every 4-day period

Decision problem

Optimization input:



Maximal frequencies
to remain below **heat 100**:



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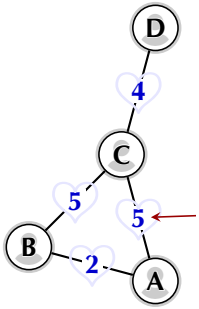
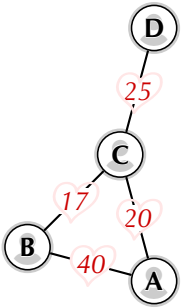
Decision instance:



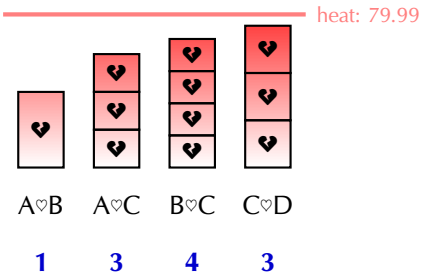
A and C must meet (at least) once in every 5-day period

Decision problem

Optimization input:



Maximal frequencies
to remain below **heat 79.99**:



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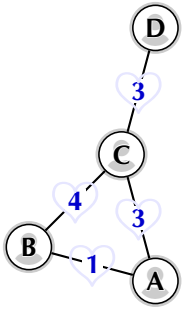

Decision problem

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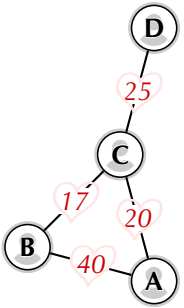

Optimization Poly Scheduling

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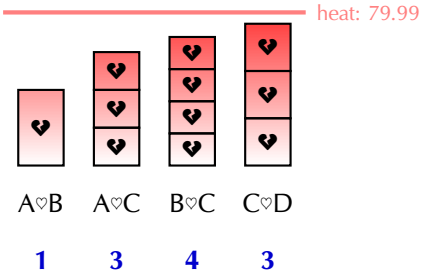

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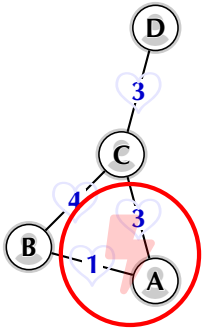

Decision Poly Scheduling

Decision problem

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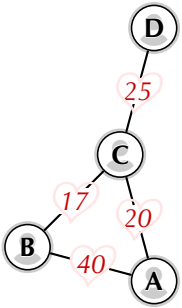

Maximal frequencies

For this instance:

- schedule S^* has heat 80
  $S^* = (\{AB, CD\}, \{AC\}, \{AB, CD\}, \{BC\})$
- heat $80 - \varepsilon$ leaves A unschedulable

$\rightsquigarrow$ optimal heat $h^* = 80$

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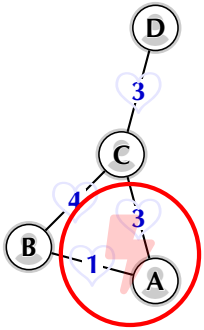

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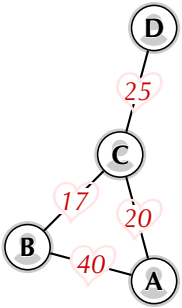

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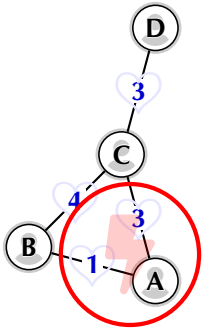

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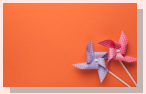
1 Introduction



2 Hardness



3 Approximation



4 Density



5 Conclusion



2 Hardness

Unweighted case

Do poly people just make their life complicated – or are they facing a ↓ *computationally hard problem?*

- Let's consider a simple special case: **Non-hierarchical Poly Scheduling**
  - (optimization): all edges have weight 1
  - (decision): all edges have the same frequency **f**

⇝ feasible schedule for decision problem $\iff$ schedule of heat $\leqslant$ f in optimization problem

⇝ Equivalent to **Edge Coloring** (a. k. a. **Chromatic Index**)

minimal #colors χ_1 to color all edges s.t. no 2 incident to same vertex have same color

- **Edge Coloring** $\Longrightarrow$ **Poly Schedule**:
  Schedule = Round Robin of χ_1 color classes
  heat of schedule χ_1.
- **Edge Coloring** $\Longleftarrow$ **Poly Schedule**:
  Color = first day edge is scheduled
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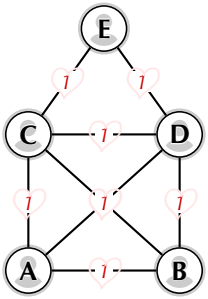

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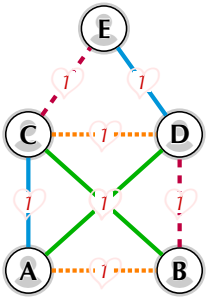

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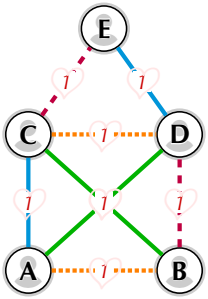

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Chromatic Index Problem

- Recall: χ_1 = chromatic index of graph G
  = #colors in proper edge coloring

$\rightsquigarrow$ Obivous: $\chi_1 \geqslant \Delta$ ← maximum degree in G

- Vizing's Theorem (1965): $\chi_1 \leqslant \Delta + 1$

1. $\Delta + 1$ **coloring** can always be found in **efficiently**.
   running time very recently improved to $\tilde{O}(mn^{1/3})$
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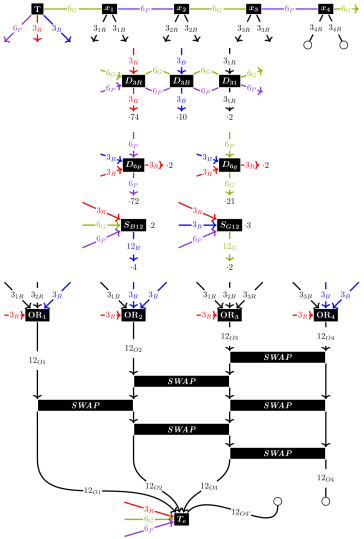
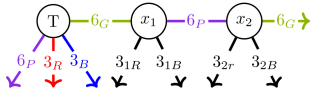
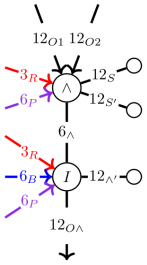
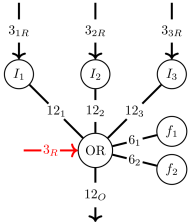
Bonus: SAT-Reduction

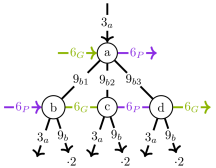


- Alternative hardness of approximation: Reduction from (decision version of) MAX-3SAT
- Given formula φ over $\mathfrak{m}$ clauses and $\mathfrak{n}$ variables, can $\geqslant k$ clauses be satisfied simultaneously?
- Key idea: relation scheduled on odd (even) days = True (False)
- doesn't quite work,
  but with slots module 6 it does.









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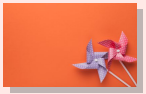

4 Density



5 Conclusion



3 Approximation

Edge-Coloring Approximation

- **Recall:** Can compute $(\Delta+1)$-coloring efficiently

$\rightsquigarrow$ $\dfrac{\Delta+1}{\Delta}$-approx for ***unweighted*** instances

- any valid schedule must reach heat $h \geqslant \Delta$ on a degree-Δ vertex

$\rightsquigarrow$ in general: $\dfrac{\Delta+1}{\Delta} \cdot \dfrac{g_{\max}}{g_{\min}}$-approx and (simultaneously) $(\Delta+1)$-approx

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could be $\Omega(n)$ ...

- any schedule reaches heat $h \geqslant \Delta \cdot g_{\min}$ on a degree-Δ vertex
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Layering approximation

- Partition edges into exponential **"layers"** by growth rate
  - Layers $i = 0, \dots L-1$ contain edges with $\frac{g_{\max}}{2^{i+1}} \leqslant g(e) \leqslant \frac{g_{\max}}{2^i}$
    Layer L the rest, i.e., $g(e) \leqslant \frac{g_{\max}}{2^L}$
  - ⇝ within one layer $g_{\max}/g_{\min} \leqslant 2$
  - ⇝ Color Round Robin **within layer** gives good approximation.
- Also schedule **layers** Round Robin
- ⇝ Heat of overall schedule $\leqslant \overline{h}$ with Δ_i = max degree in layer i

$$\overline{h} = \max_{i \in [0..L]} (L+1) \cdot (\Delta_i + 1)\frac{g_{\max}}{2^i}$$

- ⇝ each layer (in isolation) induces lower bound, so $h^* \geqslant \underline{h}$ with

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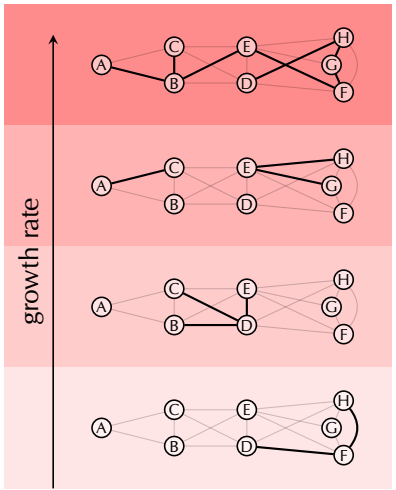

Layering approximation

- Partition edges into exponential **"layers"** by growth rate
  - Layers $i = 0, \dots L-1$ contain edges with $\frac{g_{max}}{2^{i+1}} \leqslant g(e) \leqslant \frac{g_{max}}{2^i}$
    Layer L the rest, i.e., $g(e) \leqslant \frac{g_{max}}{2^L}$
  - ⇝ within one layer $g_{max}/g_{min} \leqslant 2$
  - ⇝ Color Round Robin **within layer** gives good approximation.
- Also schedule **layers** Round Robin

⇝ Heat of overall schedule $\leqslant \overline{h}$ with Δ_i = max degree in layer i

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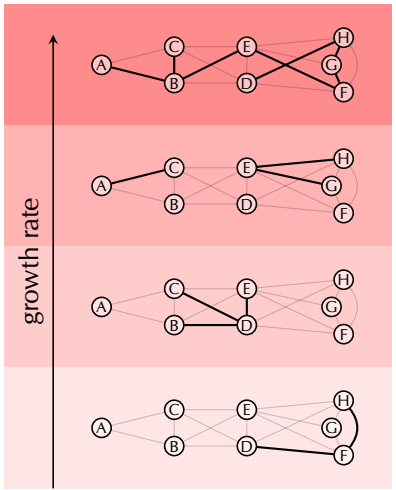

g_{max}

$g_{max}/2$

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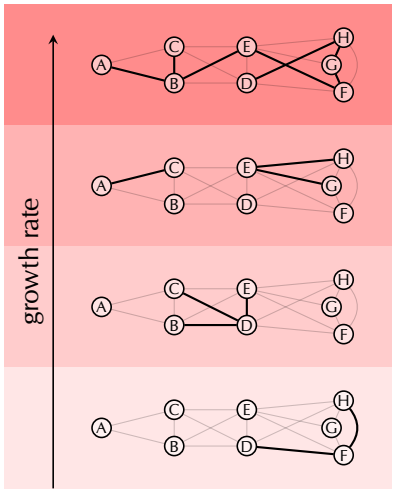

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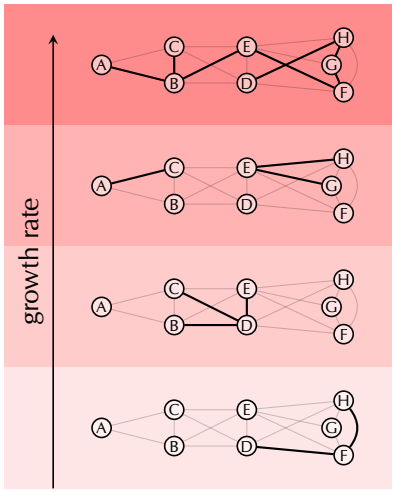

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Layering approximation [2]

- Distinguish cases where max in $\overline{h}$ is attained.

1. **Layer $i < L$ dominates**: $\overline{h} = (L+1)(\Delta_i)\frac{g_{\max}}{2^i}$
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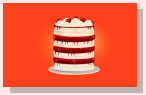
Outline



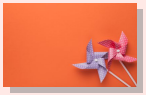
1 Introduction



2 Hardness



3 Approximation



4 Density



5 Conclusion

4 Density



Special Case: Pinwheel Scheduling

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  - Given k tasks (each with **frequency** $a_i \in \mathbb{N}$)
    Is there a perpetual schedule for a (single) agent who can perform one task per day such that every section of length a_i contains at least one instance of task i?
- Example:
  - $A = \underbrace{[3, 4, 5, 8]}_{k=4}$

Task	1	2	3	4
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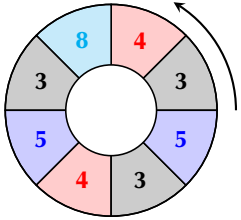

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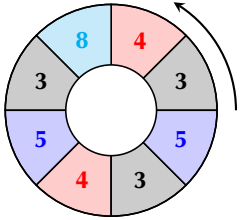

Decision version of
Bamboo Garden Trimming
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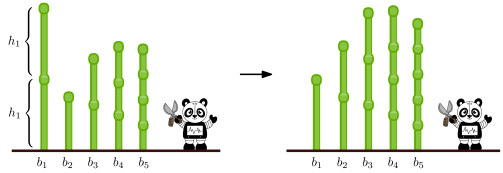
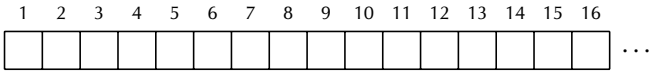


The 5/6 Density Conjecture

- Density $d(A) = \sum_{i=1}^{k} 1/a_i$
- $[2, 3, M]$



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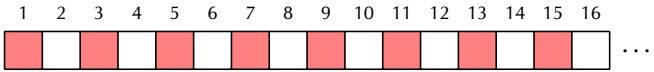

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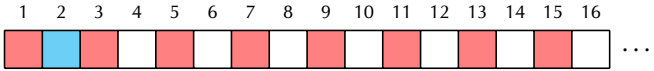

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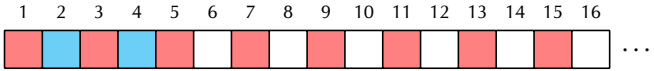

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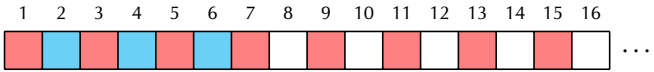

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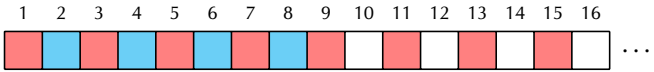

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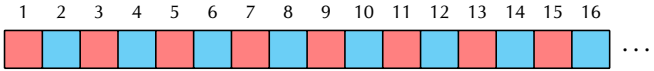

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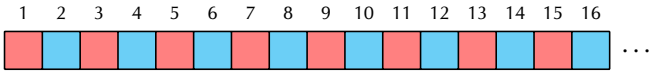

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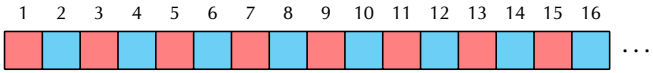

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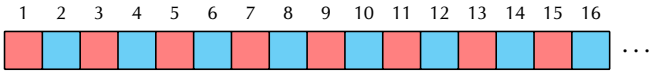

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$\rightsquigarrow$ Similar result possible for Poly Scheduling?

Fractional Scheduling

- lower-bound tool: **fractional relaxation**
  - ⇝ we can spend a fraction y_M of each day on matching M
  - ⇝ without loss of generality: all days the same!

⇝ Best fractional solution given by LP:

$\mathcal{M}$ = set of maximal matchings

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  - ⇝ without loss of generality: all days the same!

⇝ Best fractional solution given by LP: after change of variable $\boxed{\bar{h} = \frac{1}{\ell}}$ and slight massaging

$\mathcal{M}$ = set of maximal matchings

$$
\begin{aligned}
\max \quad & \ell \\
\text{s.t.} \quad & \sum_{M \in \mathcal{M}} y_M \leqslant 1 \\
& \sum_{M \in \mathcal{M}: e \in M} y_M \cdot \frac{1}{g_e} \geqslant \ell && \forall e \in R \\
& y_M \in [0,1] && \forall M \in \mathcal{M}
\end{aligned}
$$

Dual LP $\rightarrow$ Density

- LP has exponentially many variables ... but few constraints

$\rightsquigarrow$ Consider the **dual LP**:

$$\begin{aligned} \min \quad & x \\ \text{s.t.} \quad & \sum_{e \in R} z_e \geqslant 1 \\ & \sum_{e \in M} \frac{z_e}{g_e} \leqslant x \quad \forall M \in \mathcal{M} \\ & z_e \geqslant 0 \quad \forall e \in R \end{aligned}$$

$\rightsquigarrow$ any feasible solution $(x, \vec{z})$ lower bounds optimal heat h^*!

Fractional Poly Scheduling LP

$$\begin{aligned} \max \quad & \ell \\ \text{s.t.} \quad & \sum_{M \in \mathcal{M}} y_M \leqslant 1 \\ & \frac{1}{g_e} \sum_{M \in \mathcal{M}: e \in M} y_M \geqslant \ell \quad \forall e \in R \\ & y_M \geqslant 0 \quad \forall M \in \mathcal{M} \end{aligned}$$

Theorem (Fractional lower bound)

For any $z_e \in [0,1]$, for $e \in R$, with $\sum_{e \in R} z_e = 1$, we have

$$h^* \geqslant \bar{h}(z) = \frac{1}{\max_{M \in \mathcal{M}} \sum_{e \in M} \frac{z_e}{g_e}}.$$

Outline



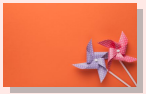
1 Introduction



2 Hardness



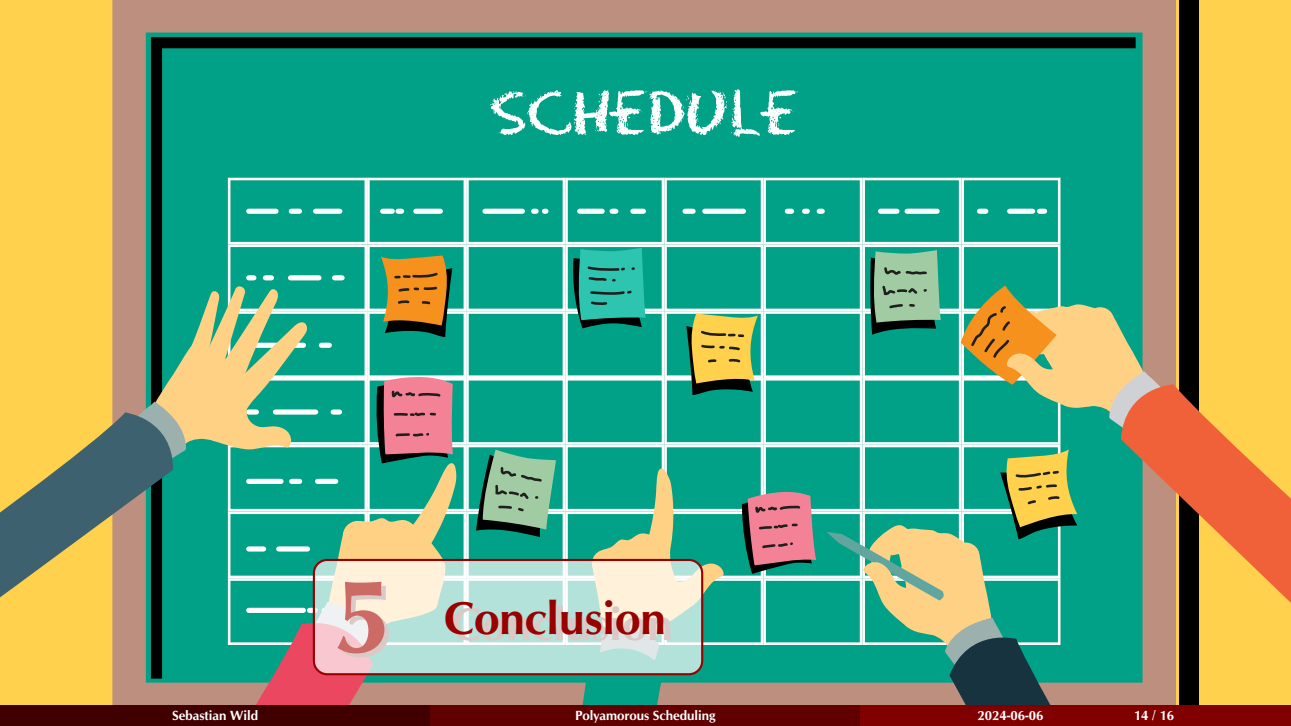
3 Approximation



4 Density



5 Conclusion



5 Conclusion

Conclusion

What you've seen:

- **Poly Scheduling** is a highly desired practical problem
- but it's hard
  - NP-hard
  - hard to approximate better than $4/3$
- best known polytime approximation only $O(\log \Delta) = O(\log n)$

Some questions:

1. Best possible approximation ratio?
2. Better results for restricted versions (e.g., special types of graphs)
3. Density threshold that guarantees schedule?


M T W T F S S

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