Practical Adaptive Sorting and Searching

Sebastian Wild

www.wild-inter.net

based on joint work with Conrado Martínez, Ian Munro, Markus Nebel, Ralph Neininger

Philipps Universität Marburg

AofA 2025

There and back again . . .



Sebastian Wild

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AG Algorithmik

Philipps Universität Marburg

since 01.09.2024 Professor @

Philipps Universität Marburg

2019 – 2024

(Senior) Lecturer @

UNIVERSITY OF LIVERPOOL

Liverpool, UK

2019

Marburg

Kaiserslautern

2017

2016 PhD

TECHNISCHE UNIVERSITÄT KAISERSLAUTERN

2017 – 2019

postdoc @

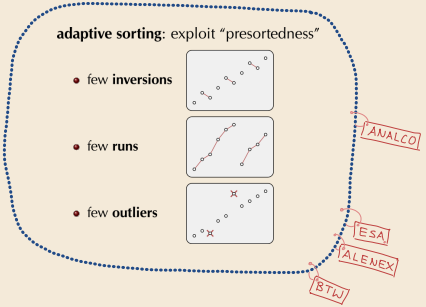
UNIVERSITY OF WATERLOO

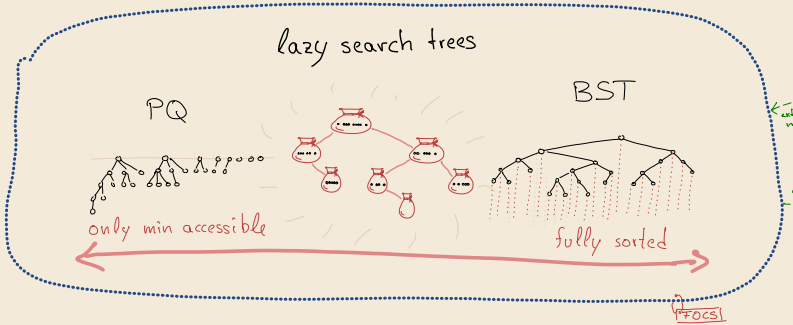
Waterloo, Canada

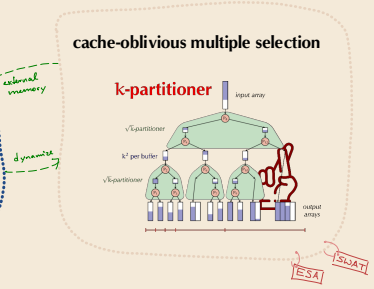
Research Interests: (1) Beyond worst-case algorithms

Goal:
Improve performance when input is “tame” (far form worst case)

My Focus:
Fundamental algorithms & data structures
Dictionaries, Sorting, Searching



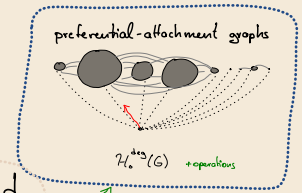


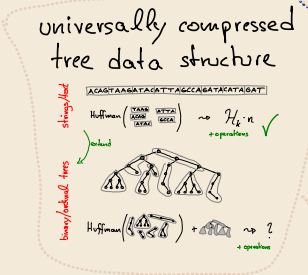


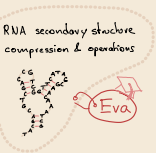
Research Interests: (2) Space-efficient Data Structures

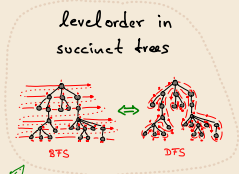
Goal:
Store data in compressed form
But retain efficient access
(without decompressing first)

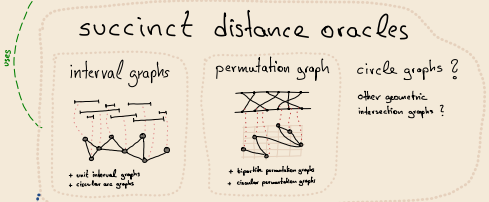
My Focus:
Trees, Graph-structured data

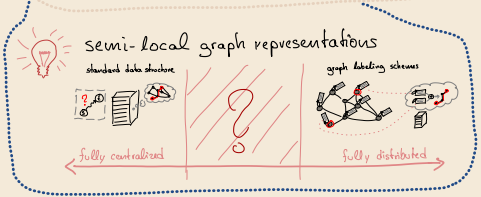








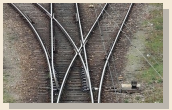




Outline



1 Context



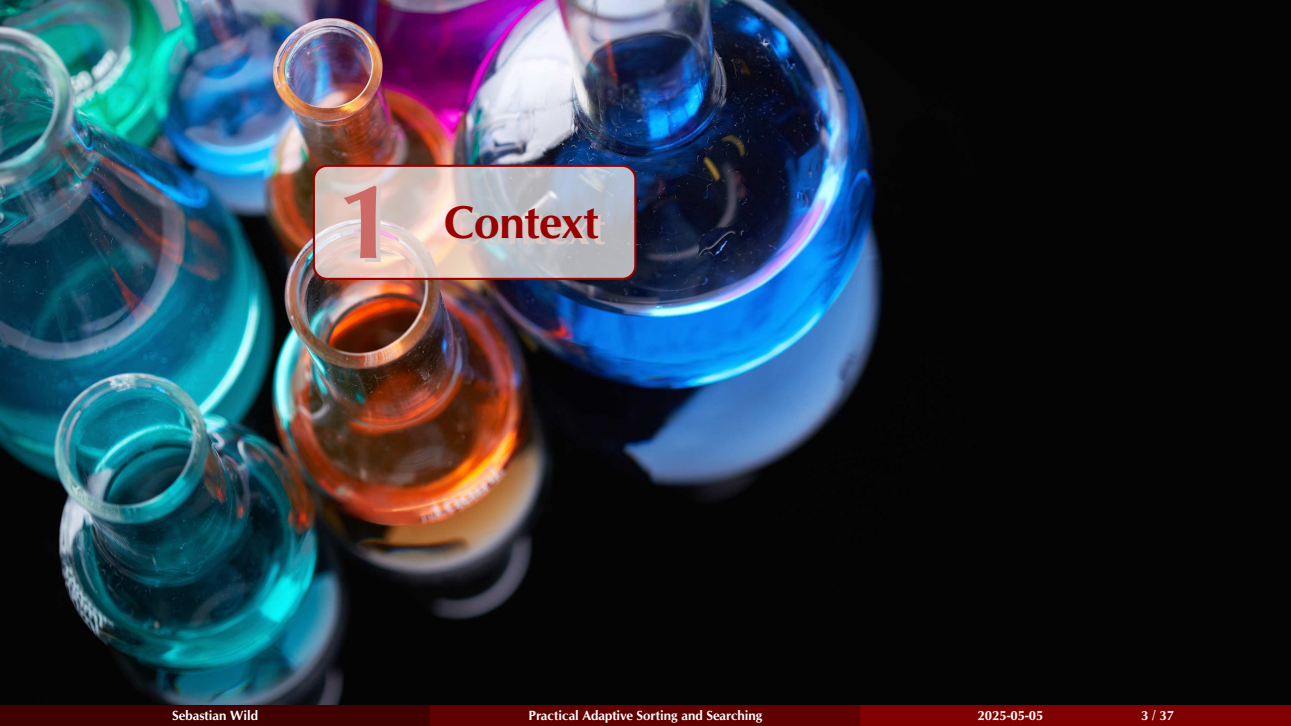
2 Dual-Pivot Quicksort



3 Quicksort with Equal Keys



4 Powersort



1 Context

Adaptive Sorting

Adaptive algorithm: exploit “structure” of input

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$\rightsquigarrow$ **adaptive sorting**: exploit “presortedness”

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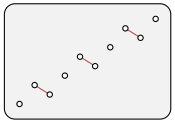

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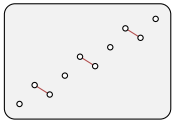

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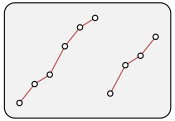

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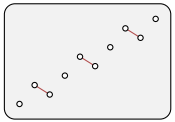

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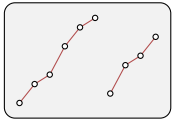

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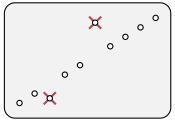

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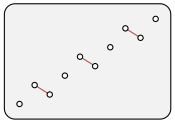

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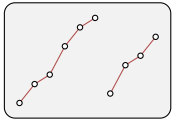

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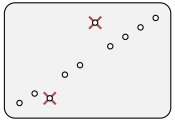

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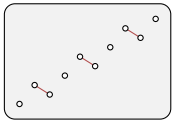

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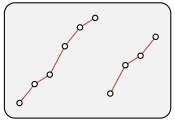

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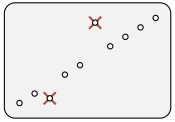

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optimal algorithms known for many measures of presortedness

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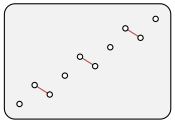

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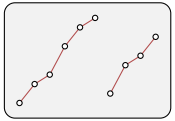

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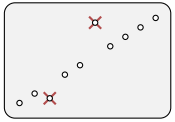

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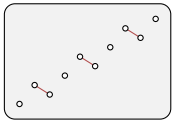

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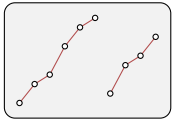

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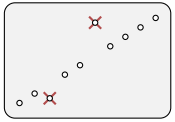

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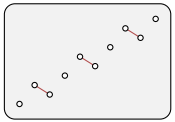

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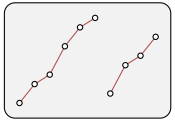

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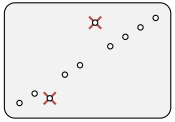

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optimal algorithms known for many measures of presortedness

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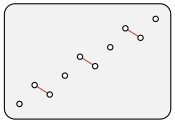

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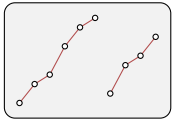

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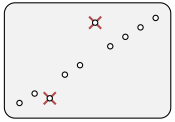

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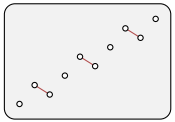

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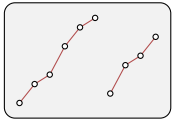

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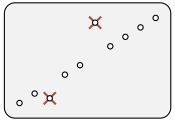

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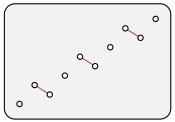

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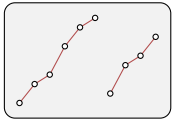

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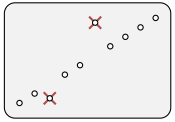

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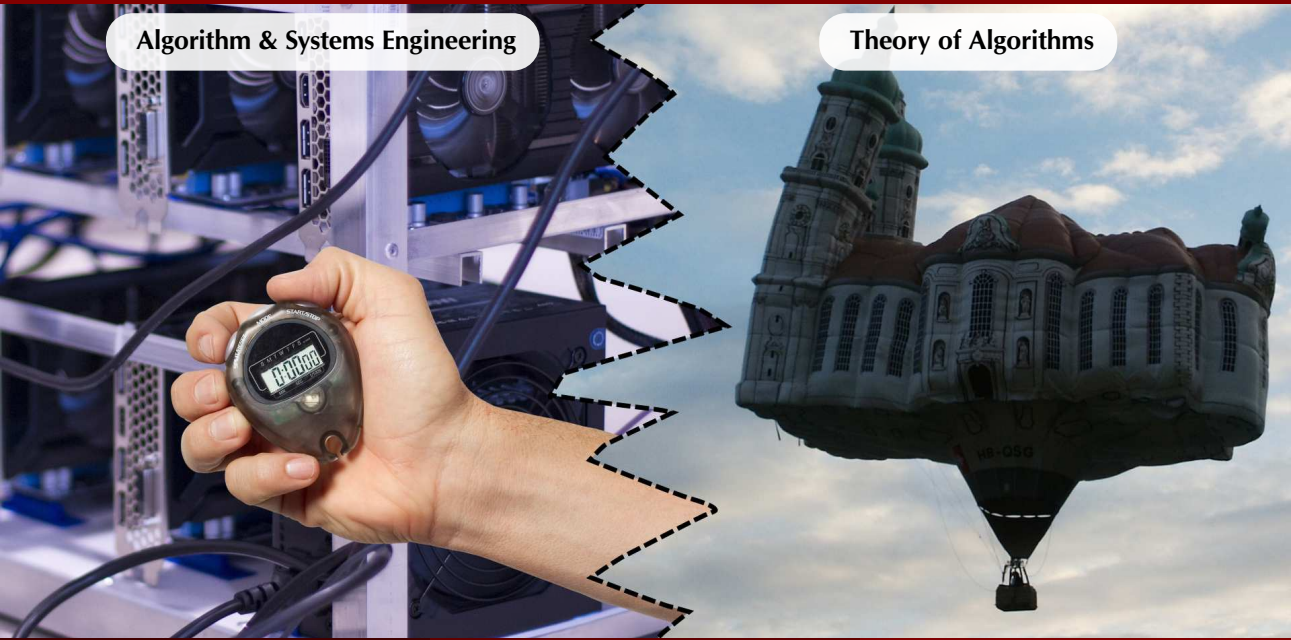
- competitive on inputs **without** presortedness

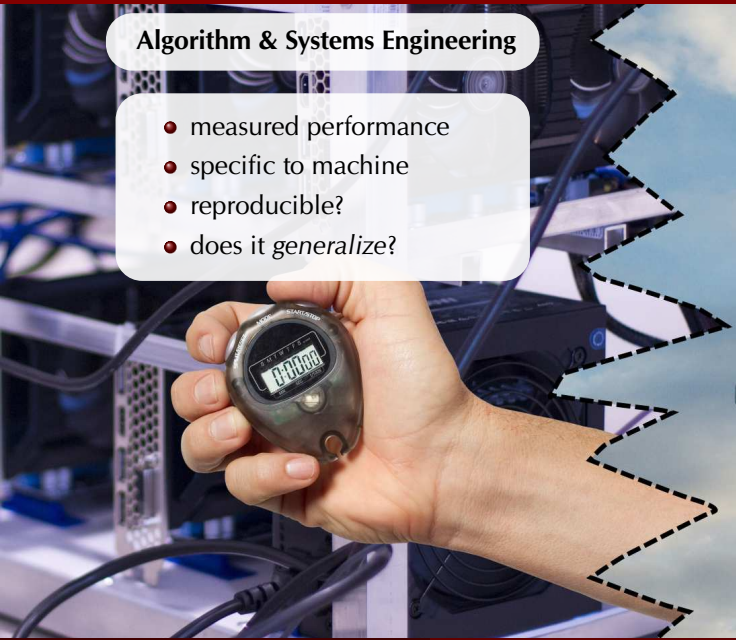
Algorithm Science



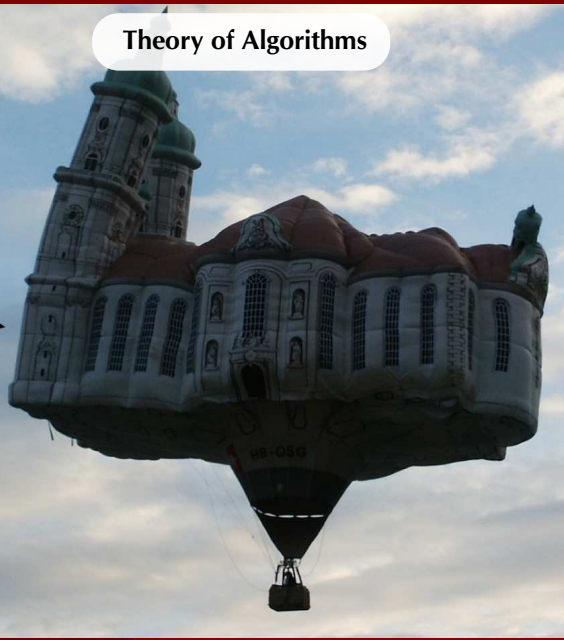
Algorithm Science

Algorithm & Systems Engineering

- measured performance
- specific to machine
- reproducible?
- does it *generalize*?



Theory of Algorithms



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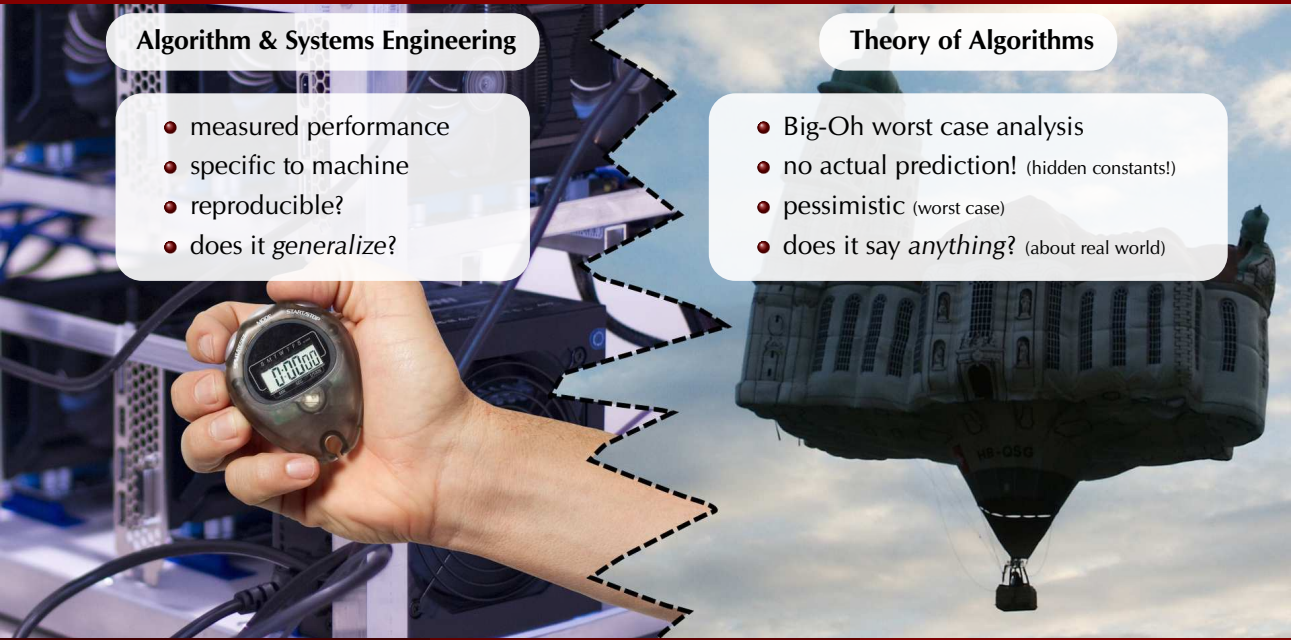

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- no actual prediction! (hidden constants!)
- pessimistic (worst case)
- does it say *anything*? (about real world)

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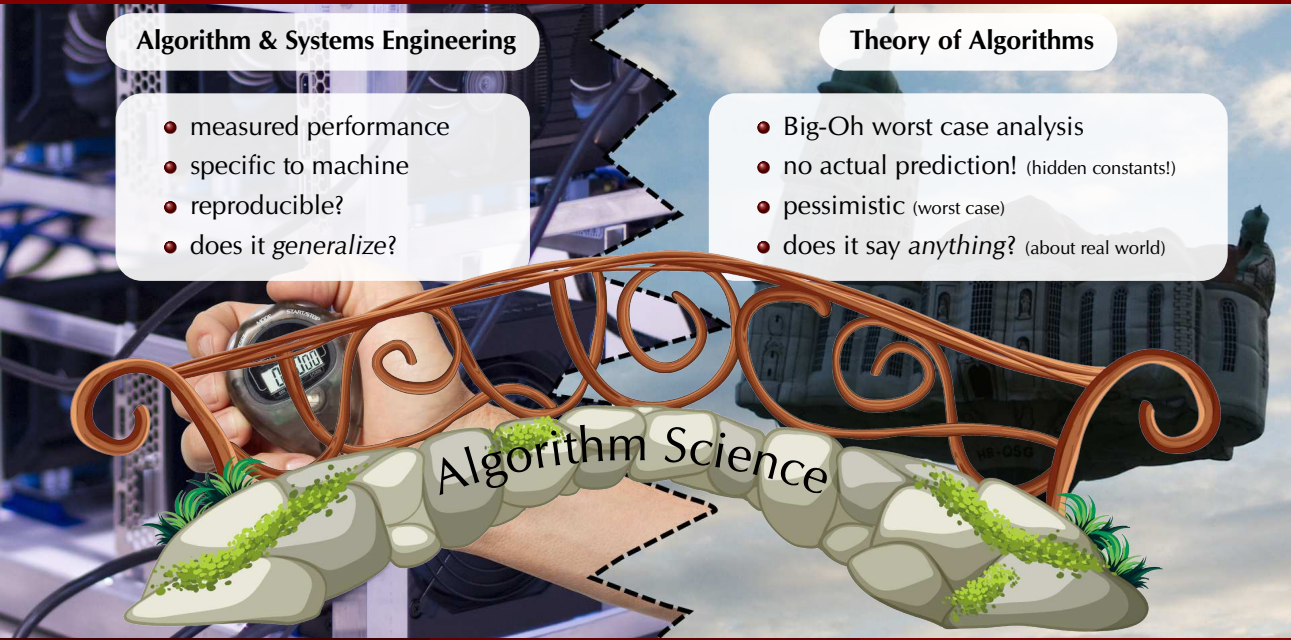

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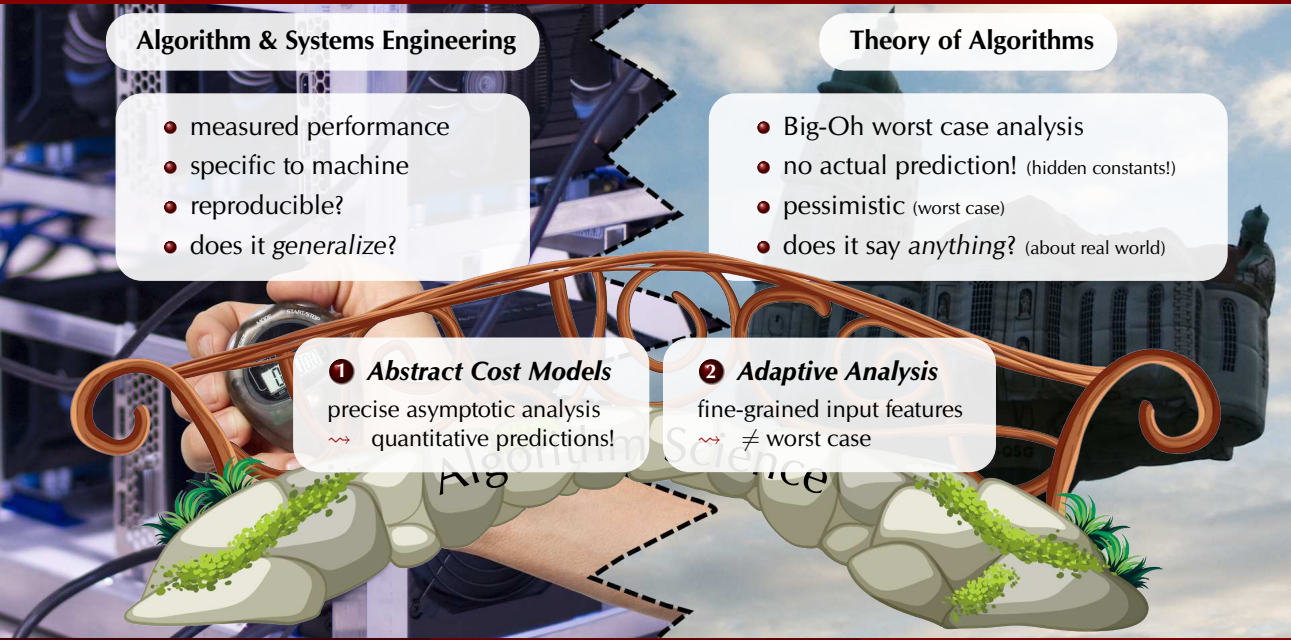

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precise asymptotic analysis
⇝ quantitative predictions!

2 ***Adaptive Analysis***

fine-grained input features
⇝ $\neq$ worst case

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Platform-independent performance guarantees

- concrete quantitative model
- validated by experiments

Case Study: Sorting in Java Library

Why Java?



Java™

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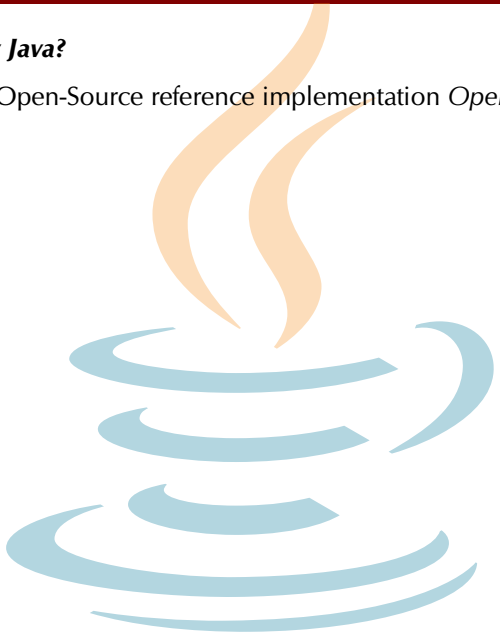
- Open-Source reference implementation *OpenJDK*



Java™

Case Study: Sorting in Java Library

Why Java?

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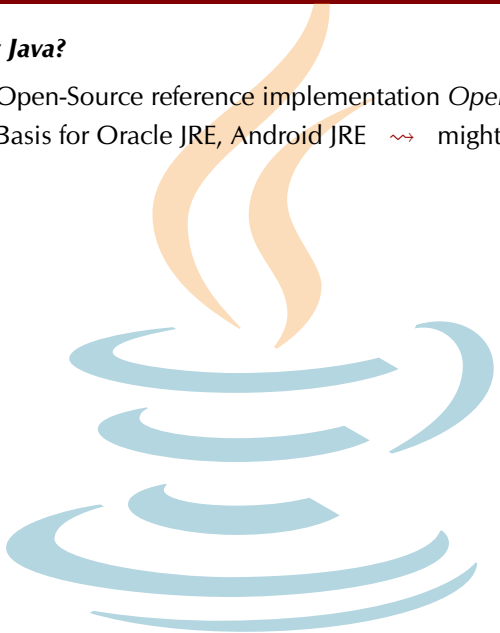
- Basis for Oracle JRE, Android JRE $\rightsquigarrow$ might contain executed algorithms in existence!



Java™

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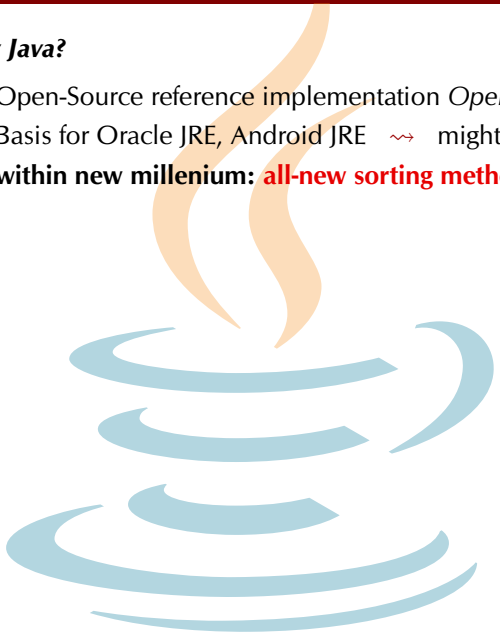

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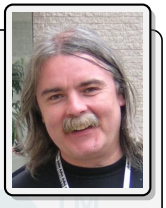

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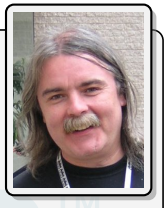

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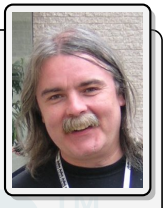

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⇝ **Focus:** Benchmarks & Unit-Tests not on proofs or analysis

Outline



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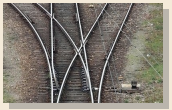


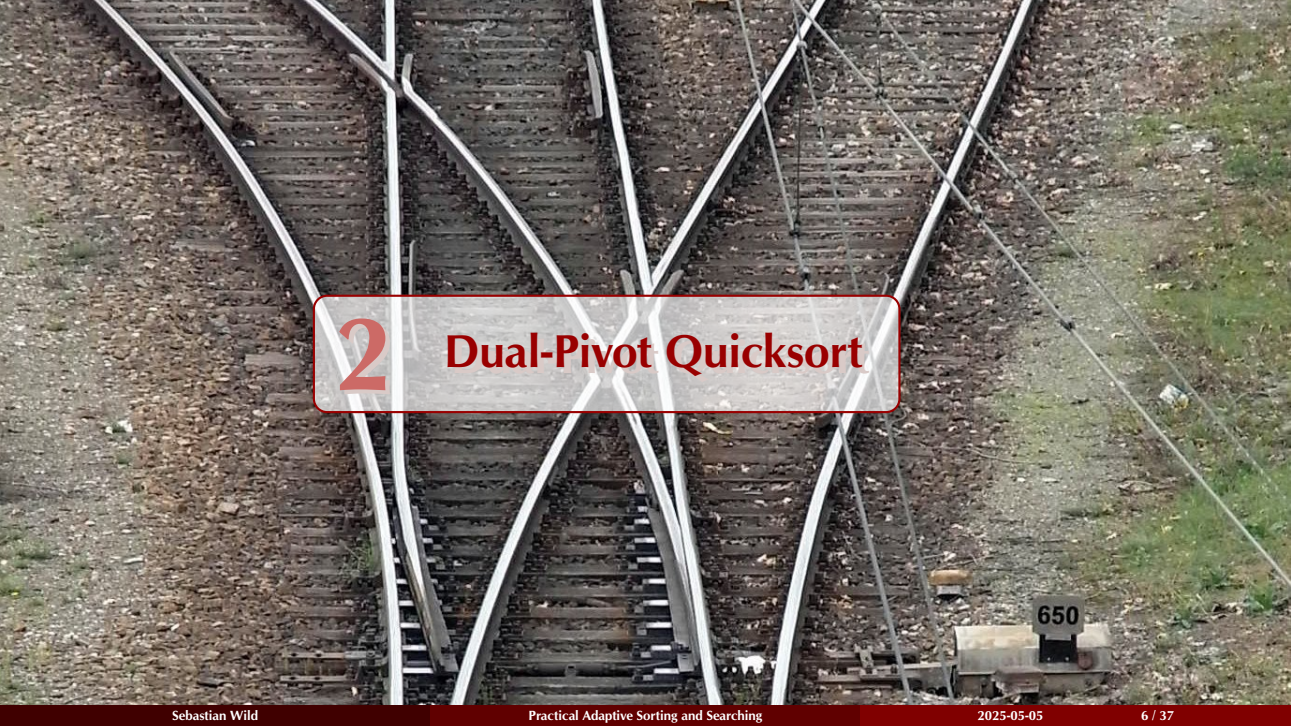
2 Dual-Pivot Quicksort



3 Quicksort with Equal Keys



4 Powersort



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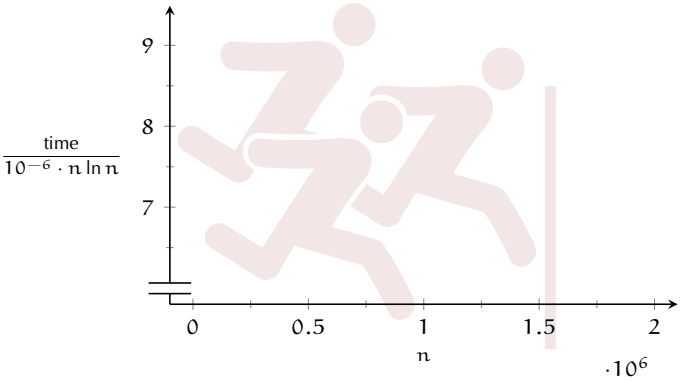

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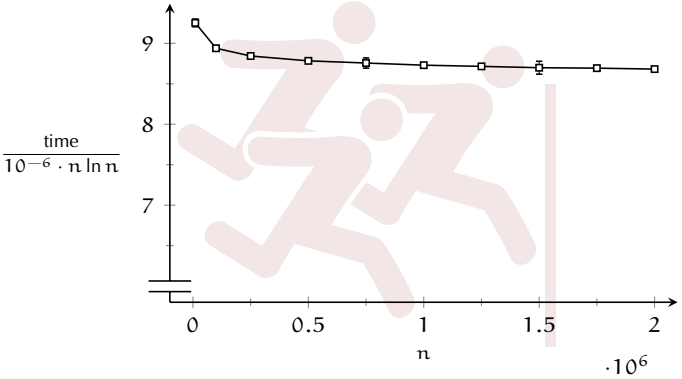

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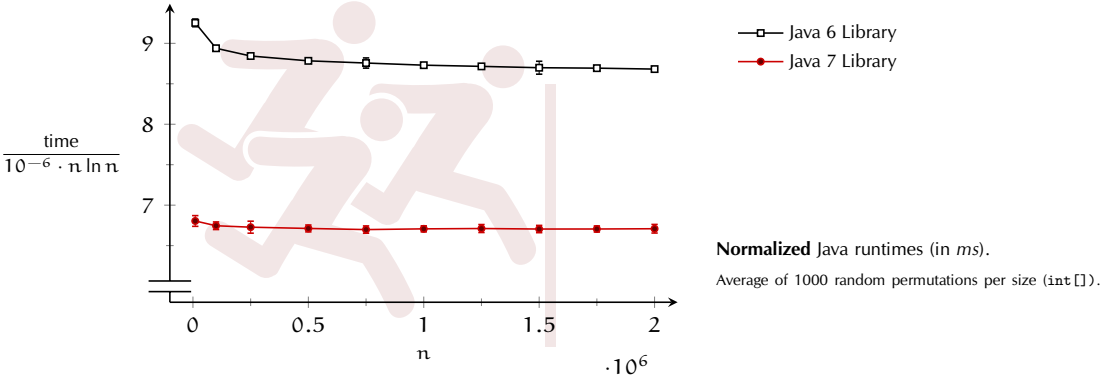

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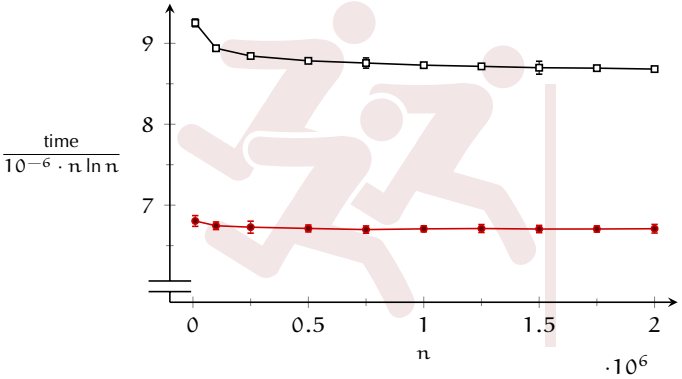

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⇝ Another Algorithm-Science case study!

What changes with 2 pivots?

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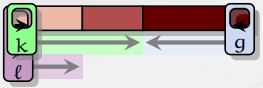
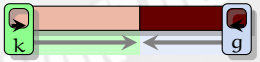
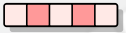
		 (corrected for pivot Sampling)	
Pivots	 (Tertiles of 5)		

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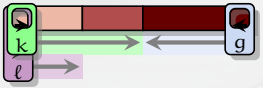
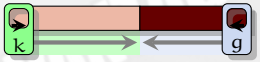
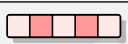
		 (corrected for pivot Sampling)	
Pivots	 (Tertiles of 5)		
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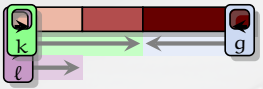
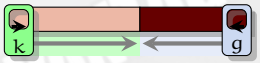
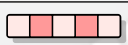
		 (corrected for pivot Sampling)	
Pivots	 (Tertiles of 5)		
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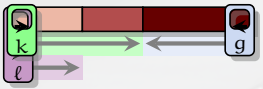
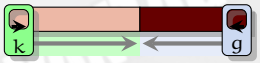
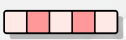
		 (corrected for pivot Sampling)	
Pivots	 (Tertiles of 5)		
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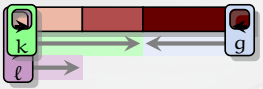
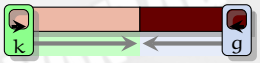
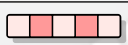
		 (corrected for pivot Sampling)	
Pivots	 (Tertiles of 5)		
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	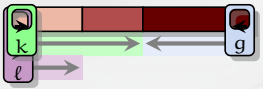	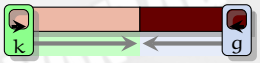 (corrected for pivot Sampling)	
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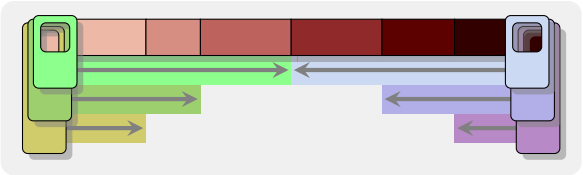

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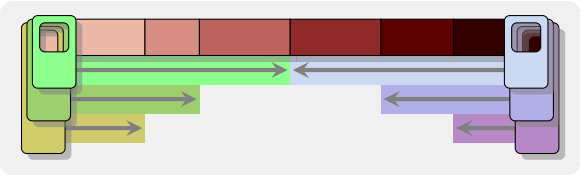

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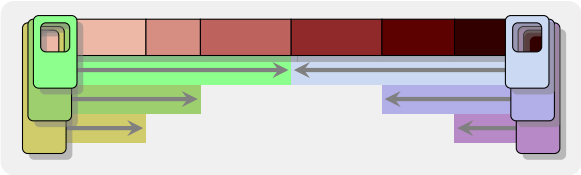

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Known Quicksort variants

Method	s	$\mathfrak{m}$	λ_k	λ_g
Classic	2	1		
Lomuto	2	2		
YBB	3	2		
Waterloo	4	2		

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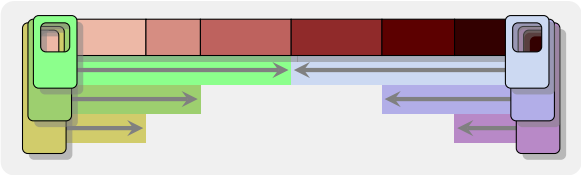
Can we do even better than YBB Quicksort?

Setup: *Generic s-way one-pass partitioning*

- **s-way partitioning:** split input into $s \geqslant 2$ parts
- **one pass** over input
- m segments grow from left, $s - m$ from right



- classify elements using BSTs: λ_k and λ_g
- pivots $P_1, \ldots, P_{s-1}$ chosen from sample

Known Quicksort variants

Method	s	m	λ_k	λ_g
Classic	2	1		
Lomuto	2	2		
YBB	3	2		
Waterloo	4	2		

Beyond Dual-Pivot: Multiway Partitioning

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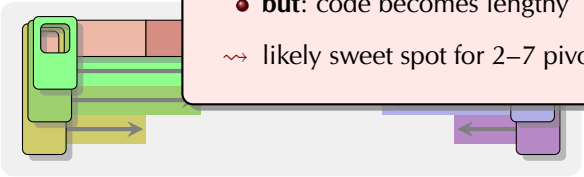
- m segments gro



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Findings for dual-pivot Quicksort are typical:

- **comparisons**: negligible improvements
- **scanned elements**: **significant** improvements
- **but**: code becomes lengthy

⇝ likely sweet spot for 2–7 pivots

…n Quicksort variants

	s	m	λ_k	λ_g
	2	1		
	2	2		
	3	2		
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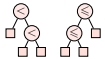

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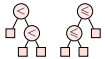

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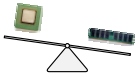

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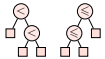

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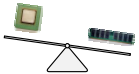

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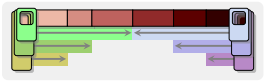

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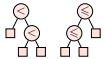

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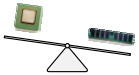

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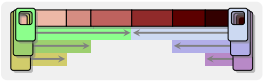

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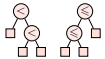

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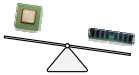

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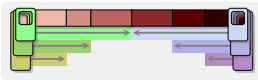

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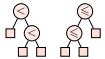

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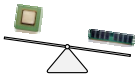

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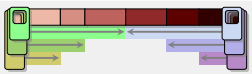

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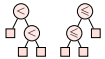

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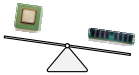

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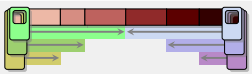

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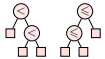

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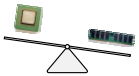

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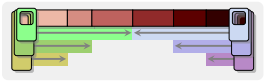

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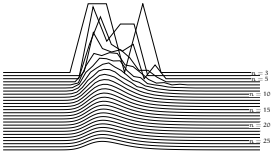

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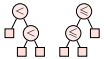

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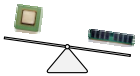

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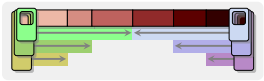

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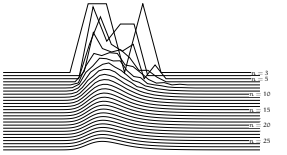

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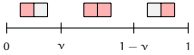

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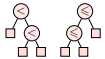

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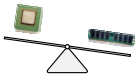

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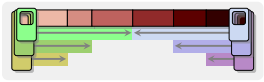

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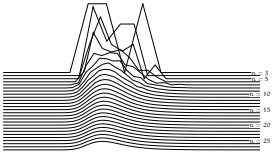

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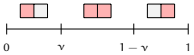

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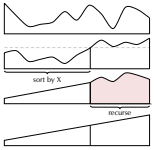

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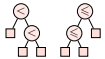

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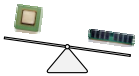

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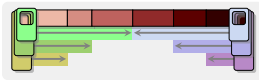

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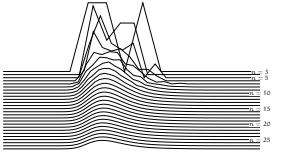

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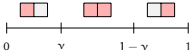

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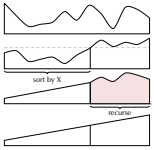

Open Problem

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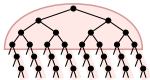
- Intriguing interplay of parameters



Outline



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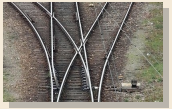

2 Dual-Pivot Quicksort



3 Quicksort with Equal Keys



4 Powersort



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*Here: Quicksort can indeed **exploit** equal keys!*

- a likely key reason its popularity in libraries
- specifically true for dual-pivot quicksort (where it doesn't need special handling)

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on analysis of Quicksort with equal keys:

- Sedgewick 1977: *Quicksort on Equal Keys*
- Sedgewick & Bentley 2002: *Quicksort is Optimal* (Talk at Knuthfest)
- Wild 2018: ANALCO

A bit more on BSTs:

- Burge 1976: *An Analysis of BSTs Formed from Sequences of Nondistinct Keys*
- Kemp 1996: *BSTs constructed from nondistinct keys with/without specified probabilities*
- Archibald & Clément 2006: *Average depth in a BST with repeated keys*

Equal Keys Model

Assumptions:

1. **Input:**

Equal Keys Model

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1. **Input:** (A)



Multiset Model:
Random permutation $U_1, \ldots, U_n$ of **fixed multiset**
$x_1, \ldots, x_u$ number of occurrences of values $1, \ldots, u$

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(B) 

Discrete i.i.d. Model:

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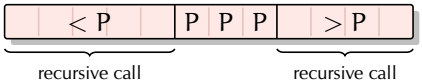

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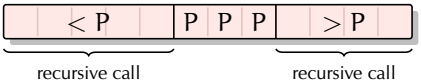

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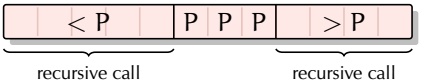

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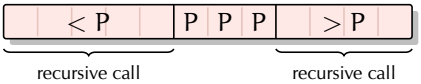

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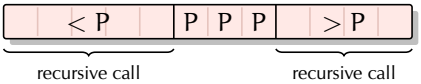

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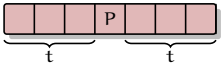
Median-of-(2t+1) Quicksort:

Example:
$t = 3$



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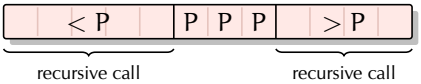

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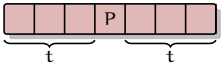

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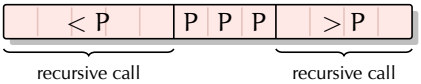

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- (extension to asymmetric sampling possible) ← not today

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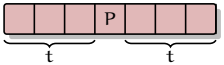

Analysis for classic Quicksort

Classic Quicksort (without pivot sampling):

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Analysis of Quicksort with equal keys

1. Define $C(x_1,\dots,x_n) = C(1,n)$ to be the mean # compares to sort the file

$$C(1,n) = N - 1 + \frac{1}{N}\sum_{1\le j\le n} x_j(C(1,j-1) + C(j+1,n))$$

2. Multiply both sides by $N = x_1 + \dots + x_n$

$$NC(1,n) = N(N-1) + \sum_{1\le j\le n} x_j C(1,j-1) + \sum_{1\le j\le n} x_j C(j+1,n)$$

3. Subtract same equation for $x_2,\dots,x_n$ and let $D(1,n) = C(1,n) - C(2,n)$

$$(x_1 + \dots + x_n)D(1,n) = x_1^2 - x_1 + 2x_1(x_2 + \dots + x_n) + \sum_{2\le j\le n} x_j D(1,j-1)$$

4. Subtract same equation for $x_1,\dots,x_{n-1}$

$$(x_1 + \dots + x_n)D(1,n) - (x_1 + \dots + x_{n-1})D(1,n-1) = 2x_1x_n + x_nD(1,n-1)$$

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$$D(1,n) = D(1,n-1) + \frac{2x_1x_n}{x_1 + \dots + x_n}$$

6. Telescope (twice)

$$C(1,n) = N - n + \sum_{1\le k<j\le n} \frac{2x_kx_j}{x_k + \dots + x_j}$$

THEOREM. Quicksort (with 3-way partitioning, randomized) uses

$N - n + 2QN$ compares (where $Q = \sum_{1\le k<j\le n} \frac{p_kp_j}{p_k + \dots + p_j}$, with $p_i = x_i/N$)

to sort an $(x_1,\dots,x_n)$-file, on the average .

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Classic Quicksort (without pivot sampling): Expected comparisons expressible **exactly**.

Analysis of Quicksort with equal keys

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The conjecture of Sedgewick and Bentley

Quicksort is optimal

The average number of compares per element C/N is always within a constant factor of the entropy H

lower bound: $C > NH - N$ (information theory)
upper bound: $C < 2\ln 2NH + N$ (Burge analysis, Melhorn bound)

No comparison-based algorithm can do better.

Conjecture: With sampling, $C / N \to H$ as sample size increases.

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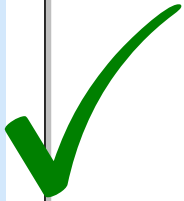

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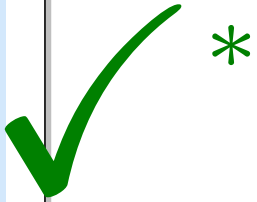

* subject to some restrictions

Quicksort & search trees

Classic Fact:

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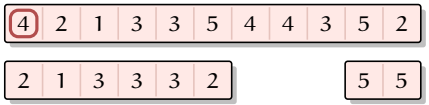

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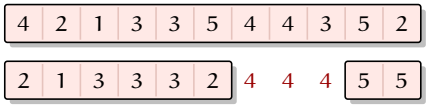

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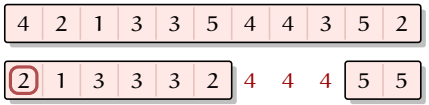

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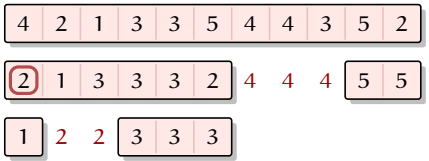

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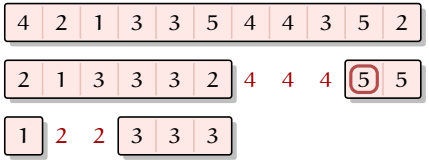

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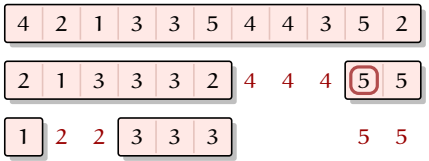

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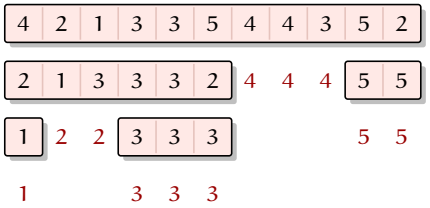

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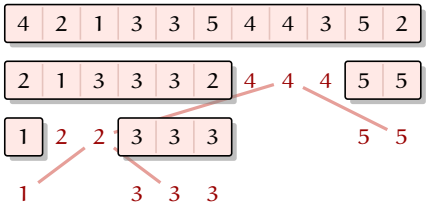

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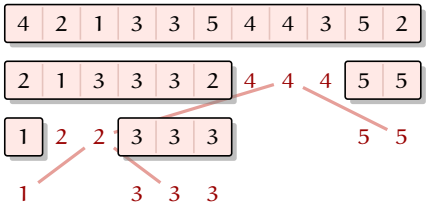

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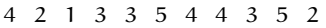

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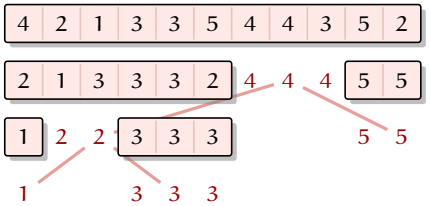

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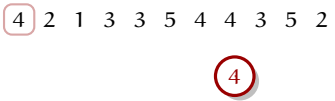

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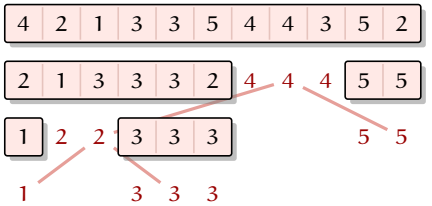

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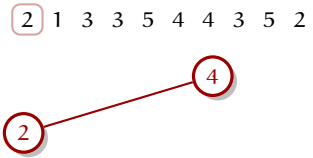

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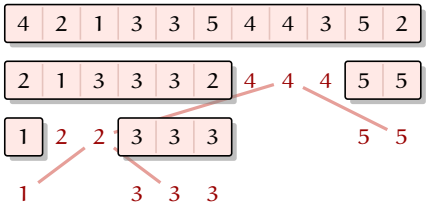

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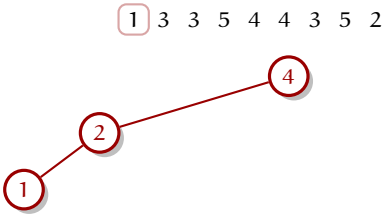

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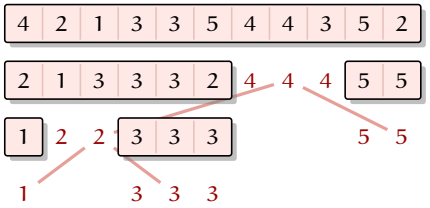

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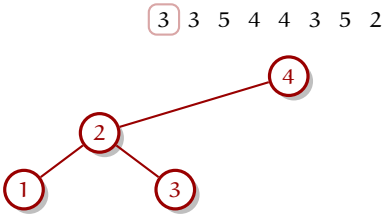

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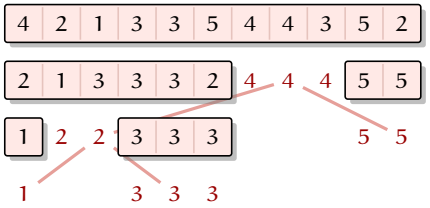

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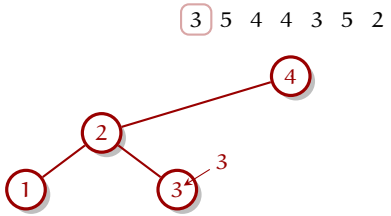

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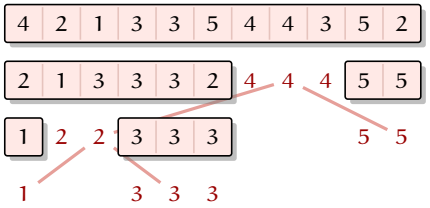

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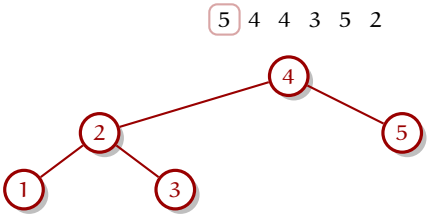

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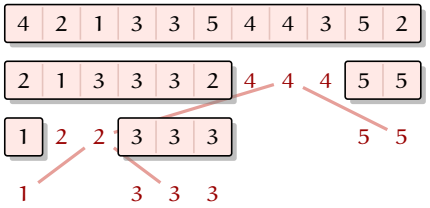

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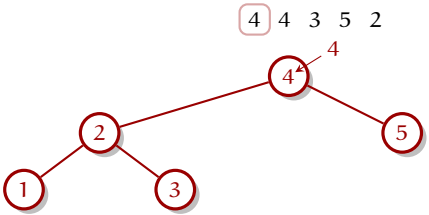

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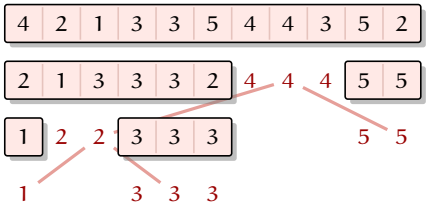

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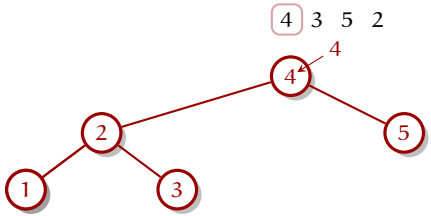

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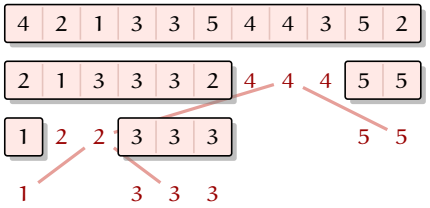

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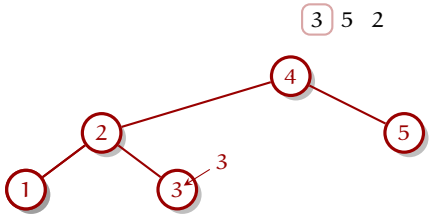

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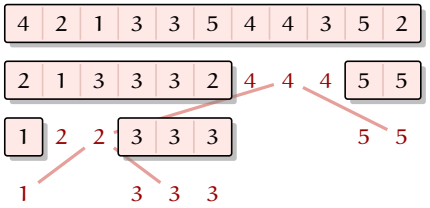

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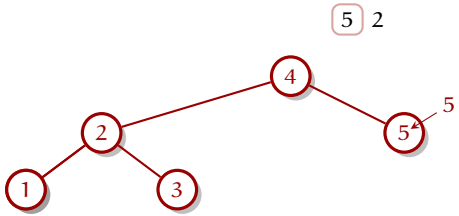

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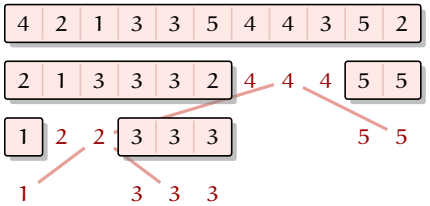

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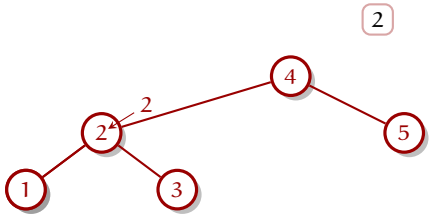

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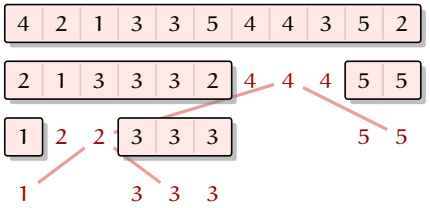

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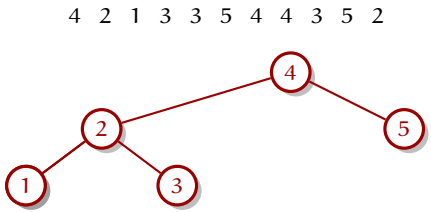

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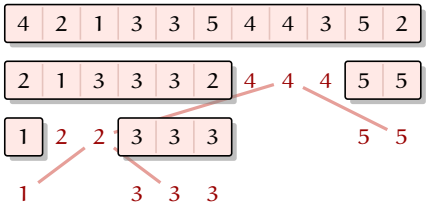

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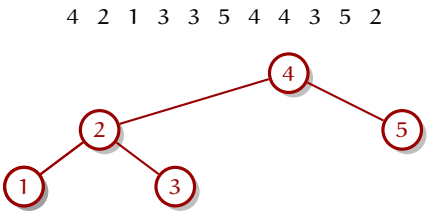

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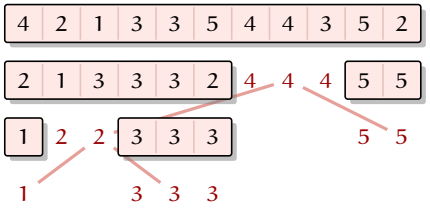

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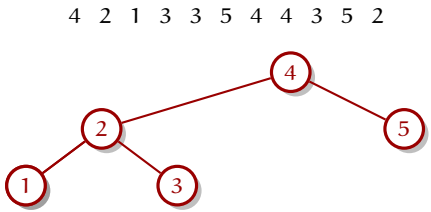

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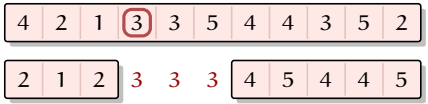

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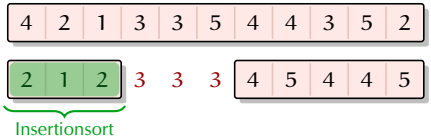

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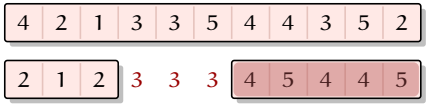

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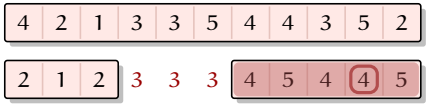

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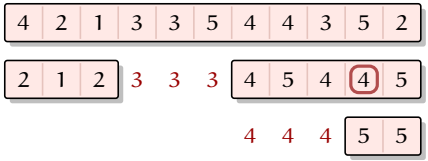

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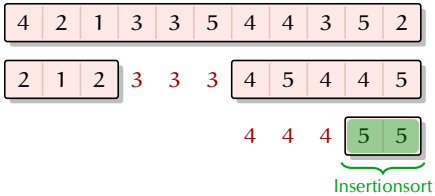

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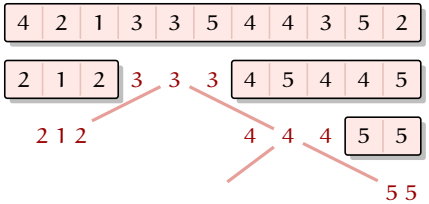

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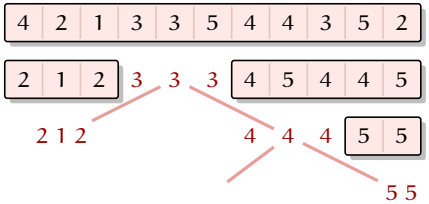

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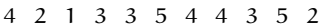

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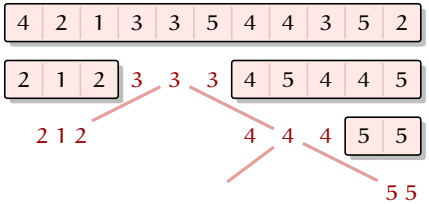

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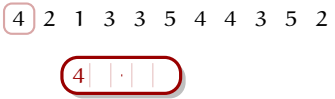

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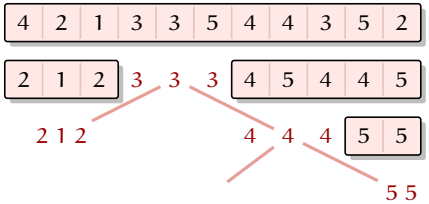

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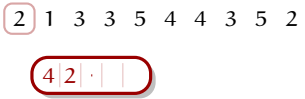

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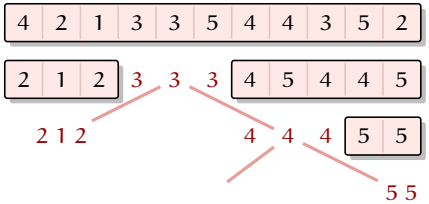

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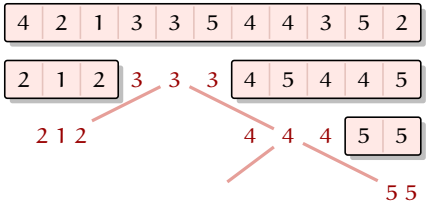

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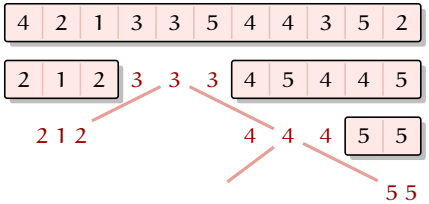

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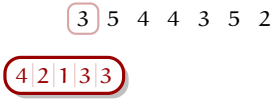

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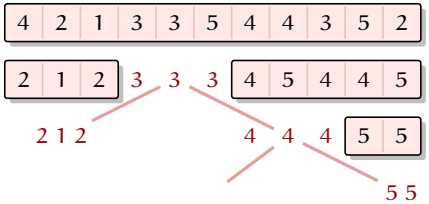

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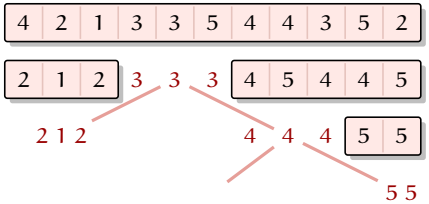

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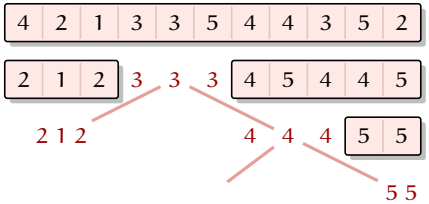

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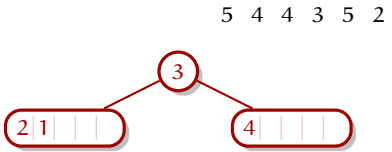

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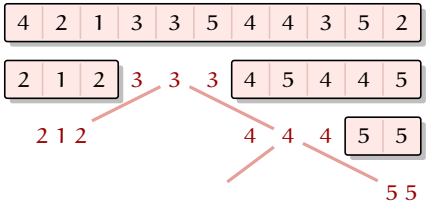

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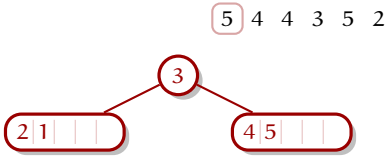

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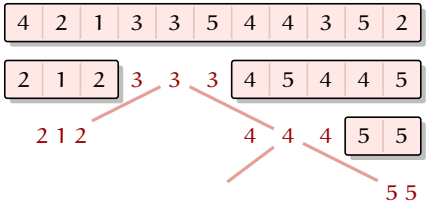

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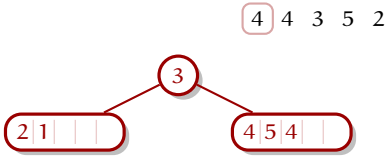

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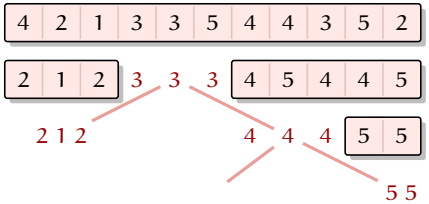

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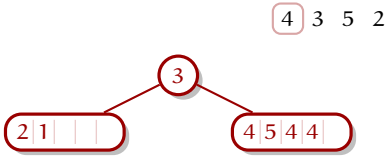

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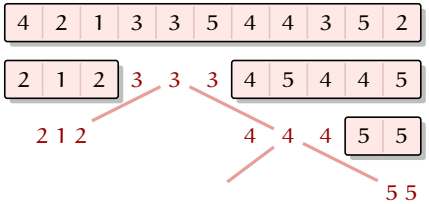

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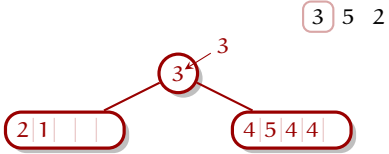

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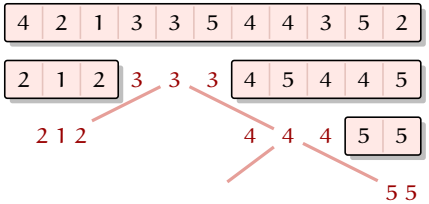

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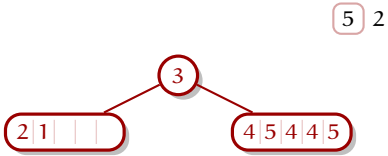

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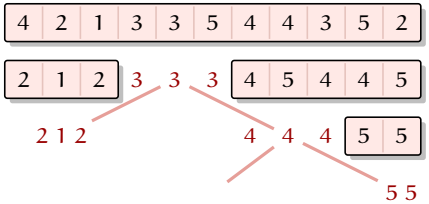

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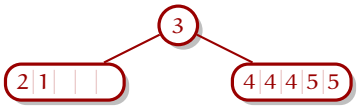

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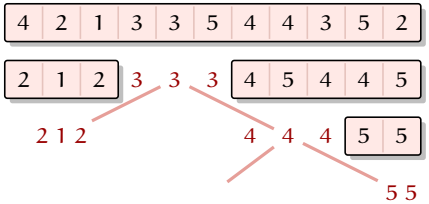

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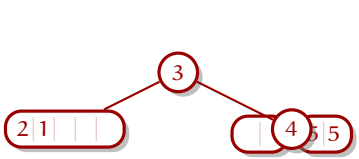

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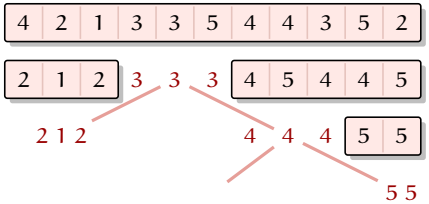

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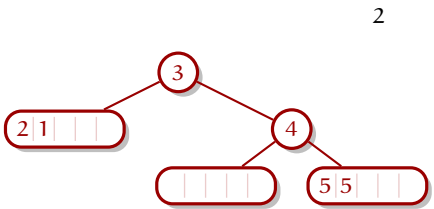

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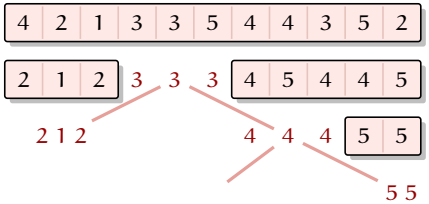

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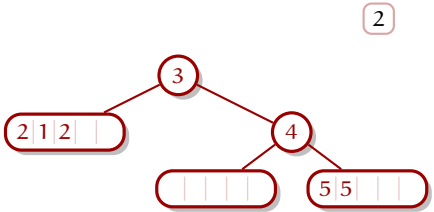

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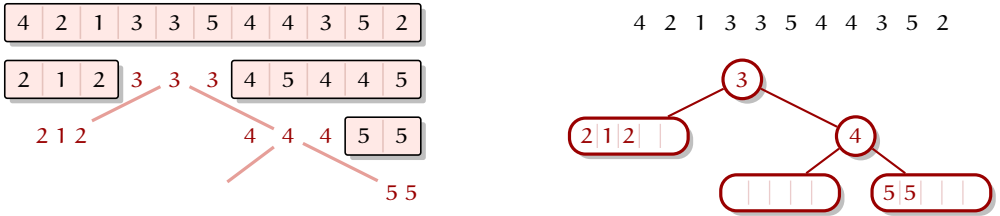

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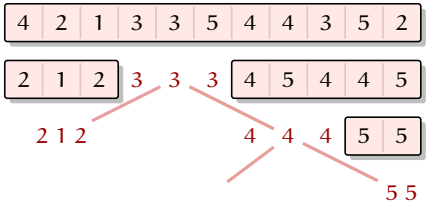

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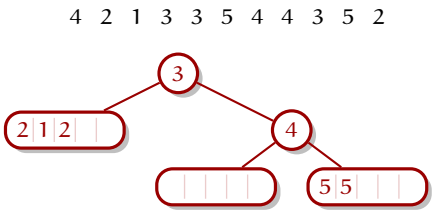

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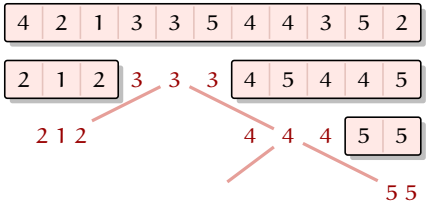

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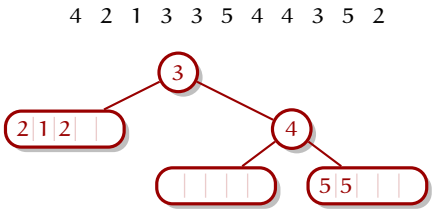

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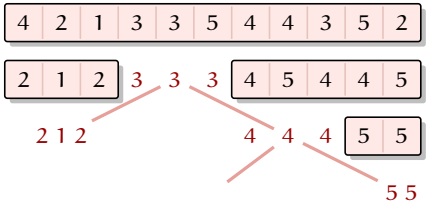

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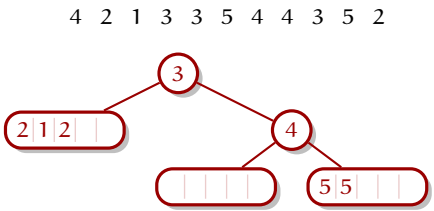

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$\rightsquigarrow$ Analyze search trees instead of Quicksort.

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4	5	5	5	2	3	5	4	2	2	2	5	5	5	4	2	5	1	5	4	2	3	4	2	2	1	2	4	2	2	5	5	2	3	4	2	3	4	1	3	2	2	1	4	4	4	1	2	3	3

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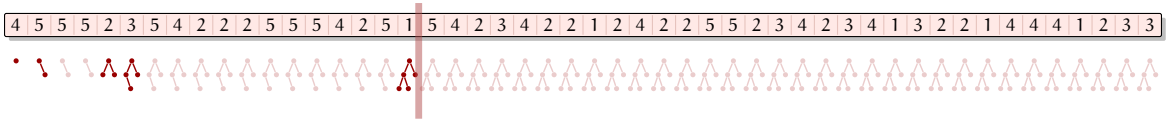

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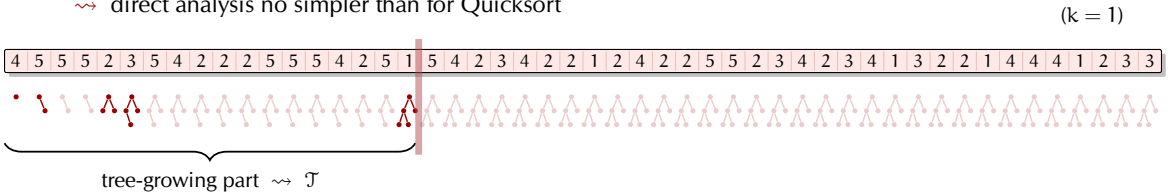

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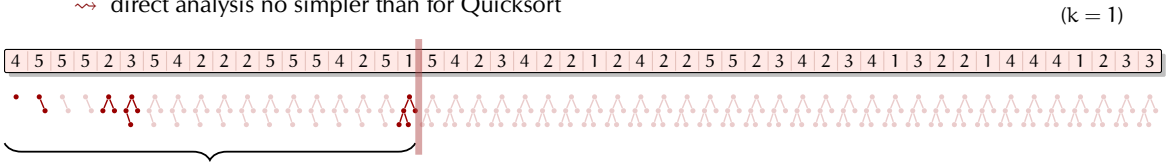

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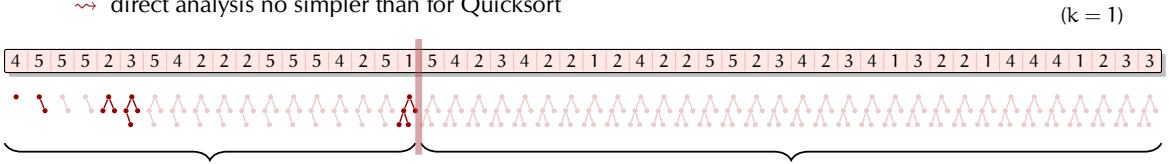

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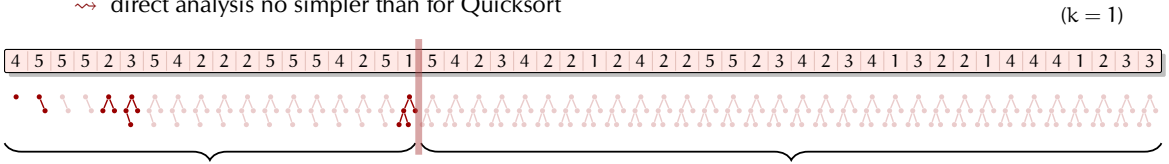

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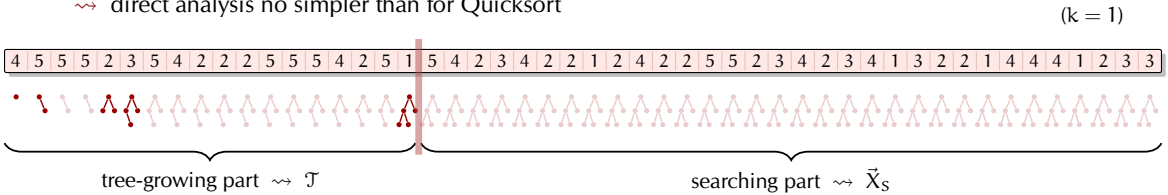

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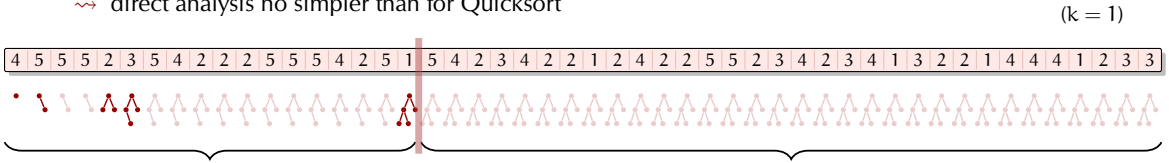

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Results in the i.i.d. Model



Two pieces to the puzzle

Results in the i.i.d. Model



Two pieces to the puzzle

- **Separation Theorem**:
  Quicksort costs
  - in the **i.i.d. model** 
  - with **"many duplicates"**
    ($\Omega(n^{\varepsilon})$ duplicates of each value in expectation)

  are given by (as $n \to \infty$, for any $\delta \in (0, \varepsilon)$)

$$\mathbb{E}[C_{n,\vec{q}}] \;=\; \alpha(\vec{q}) \cdot n \;\pm\; O(n^{1-\delta})$$

  (show: tree is "saturated" after $n^{1-\varepsilon}$ elements whp.)

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- **Average search costs**
  in saturated k-fringe-balanced trees:

(as $\mathcal{H} \to \infty$)

$$\alpha(\vec{q}) \;=\; \alpha_k \cdot \mathcal{H} \;\pm\; O\big(\mathcal{H}^{\frac{t+2}{t+3}} \log \mathcal{H}\big)$$

  - $\mathcal{H} = \mathcal{H}_{\mathsf{Id}}(\vec{q})$
  - $\alpha_k = \dfrac{\ln 2}{H_{k+1} - H_{t+1}}$

  (Compare recurrence of entropy to search cost, tour de force to bound α in terms of $\mathcal{H}$)

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$\rightsquigarrow$

Quicksort Costs (i.i.d. model)

Under the assumptions above, we have for any $\delta \in \big(0, \frac{1}{t+3}\big)$
$\mathbb{E}[C_{n,\vec{q}}] \;=\; \alpha_k \mathcal{H}_{\mathsf{Id}}(\vec{q}) \cdot n \;\pm\; O\big((\mathcal{H}(\vec{q})^{1-\delta} + 1)n\big)$.

Quicksort with Equal Keys Summary

Findings

- Partial Answer to **conjecture of Sedgewick & Bentley** (for "many duplicates"):
  - $\rightsquigarrow$ Same relative speedup as for random permutations.

  Median-of-k *Quicksort approaches lower bound for* $k \to \infty$.

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Open Problem

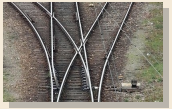
- Get rid of "many duplicates" restriction

  … by going back to fixed profile model?

Outline



1 Context



2 Dual-Pivot Quicksort



3 Quicksort with Equal Keys

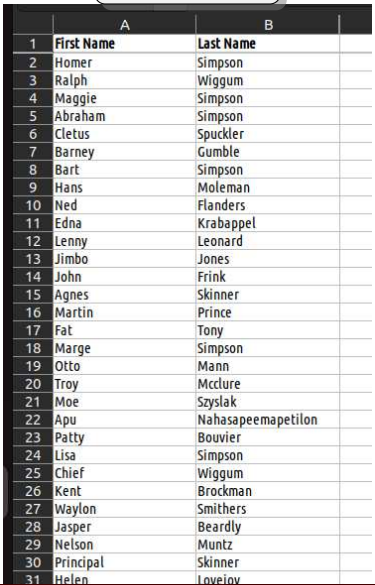


4 Powersort


4 Powersort

Stable Sorting

unsorted input



	A	B
1	First Name	Last Name
2	Homer	Simpson
3	Ralph	Wiggum
4	Maggie	Simpson
5	Abraham	Simpson
6	Cletus	Spuckler
7	Barney	Gumble
8	Bart	Simpson
9	Hans	Moleman
10	Ned	Flanders
11	Edna	Krabappel
12	Lenny	Leonard
13	Jimbo	Jones
14	John	Frink
15	Agnes	Skinner
16	Martin	Prince
17	Fat	Tony
18	Marge	Simpson
19	Otto	Mann
20	Troy	Mcclure
21	Moe	Szyslak
22	Apu	Nahasapeemapetilon
23	Patty	Bouvier
24	Lisa	Simpson
25	Chief	Wiggum
26	Kent	Brockman
27	Waylon	Smithers
28	Jasper	Beardly
29	Nelson	Muntz
30	Principal	Skinner
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Stable Sorting

unsorted input

	A	B
1	First Name	Last Name
2	Homer	Simpson
3	Ralph	Wiggum
4	Maggie	Simpson
5	Abraham	Simpson
6	Cletus	Spuckler
7	Barney	Gumble
8	Bart	Simpson
9	Hans	Moleman
10	Ned	Flanders
11	Edna	Krabappel
12	Lenny	Leonard
13	Jimbo	Jones
14	John	Frink
15	Agnes	Skinner
16	Martin	Prince
17	Fat	Tony
18	Marge	Simpson
19	Otto	Mann
20	Troy	Mcclure
21	Moe	Szyslak
22	Apu	Nahasapeemapetilon
23	Patty	Bouvier
24	Lisa	Simpson
25	Chief	Wiggum
26	Kent	Brockman
27	Waylon	Smithers
28	Jasper	Beardly
29	Nelson	Muntz
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sorted by First Name

	A	B
1	First Name	Last Name
2	Abraham	Simpson
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4	Apu	Nahasapeemapetilon
5	Barney	Gumble
6	Bart	Simpson
7	Carl	Carlson
8	Charles Montgomery	Burns
9	Chief	Wiggum
10	Cletus	Spuckler
11	Edna	Krabappel
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13	Hans	Moleman
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15	Homer	Simpson
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sorted by Last Name

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3	Patty	Bouvier
4	Selma	Bouvier
5	Kent	Brockman
6	Charles Montgomery	Burns
7	Carl	Carlson
8	Ned	Flanders
9	John	Frink
10	Barney	Gumble
11	Lionel	Hutz
12	Snake	Jailbird
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CPython Sorting History

Python Version (Year)	Sorting method	Remarks	Stable?
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3.11.1 – ∞?	2022	***Powersort*** NEW	Timsort with Powersort merge policy [MW18]	✓

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[MW18] **Munro & Wild**: *Nearly-Optimal Mergesorts: Fast, Practical Sorting Methods That Optimally Adapt to Existing Runs*, ESA **2018**

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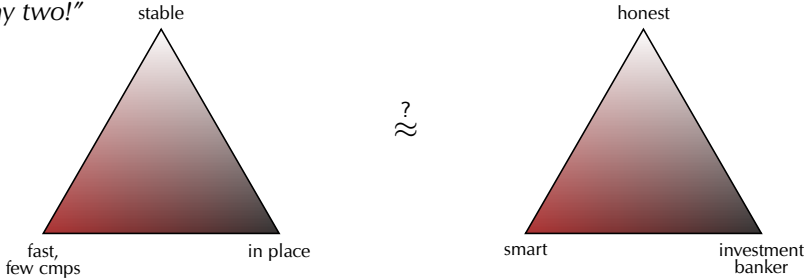

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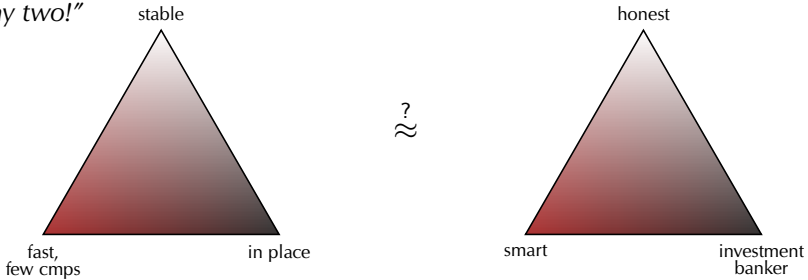
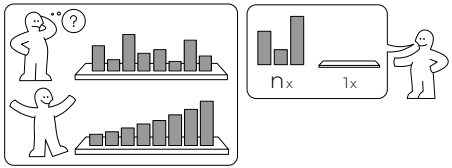


- Python has lots of objects and pointers anyways … "in-place" property least painful to sacrifice

⇝ Mergesort!

MERGE SÖRT

idea-instructions.com/merge-sort/
v1.2, CC by-nc-sa 4.0



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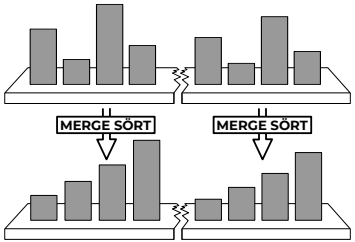

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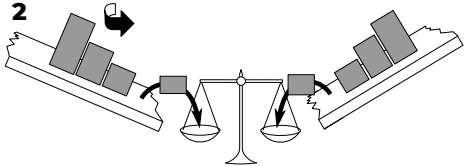

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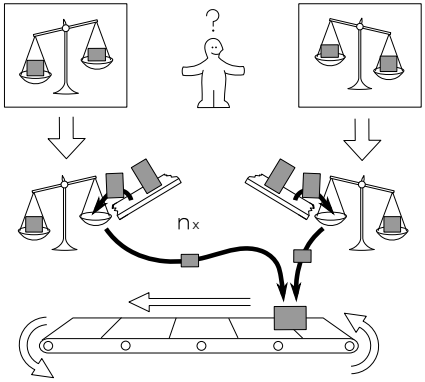

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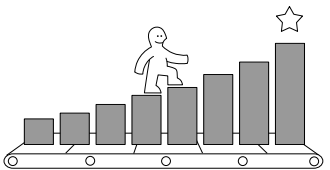

Timsort = Mergesort++

Various modifications

1. detect existing **runs**
   run = any (weakly) increasing or
   (strictly) decreasing range
2. **merge policy** decides when to merge which runs
3. keeping runs on a **fixed-size** runstack
4. **galloping** merges
   use exponential searches
   to find position in other run



Much more on galloping ⇝ ask Vincent

Ghasemi, Jugé, Khalighinejad, Yazdanyar: *Galloping in Fast-Growth Natural Merge Sorts*, Algorithmica **2025**

5. **minimum run lengths** with binary insertion sort
   choose $32 \leqslant \texttt{minrun} \leqslant 64$, so that $\lceil \frac{n}{\texttt{minrun}} \rceil$ is 2^k or slightly smaller
   fill runs to `minrun` elements

Tim Peters et al.: *listsort.txt*, CPython sources

Timsort merge policy (original)

```
def timsort(lst):
    i = 0; runs = []
    while i < len(lst):
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full code:
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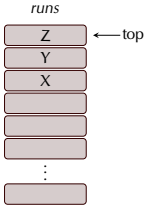

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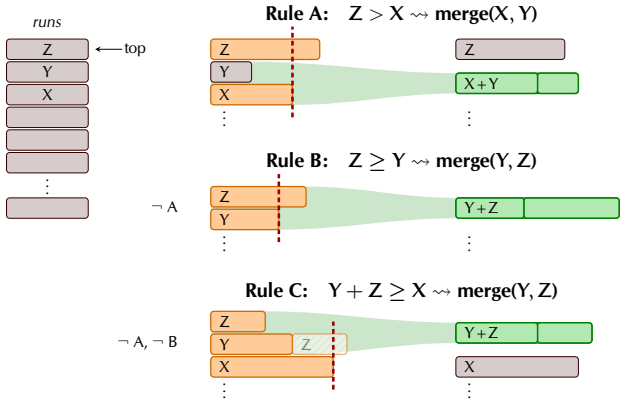

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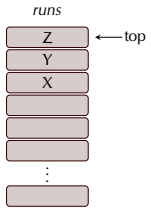

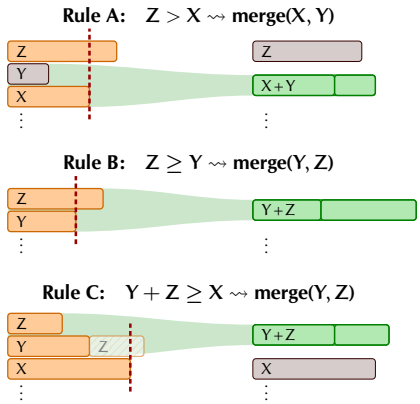

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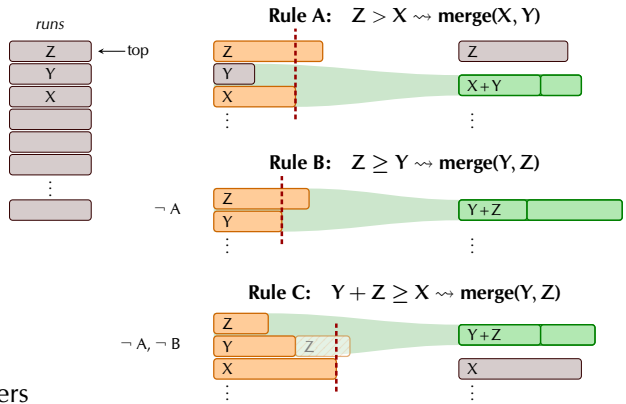

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  $\rightsquigarrow$ max stack height $\approx \log_\phi(n)$
- Why exactly these rules?
  - Tim Peters: *"first thing I tried that 'worked well'"*

(a) small runs stack,
(b) balanced on equal runs,
(c) $O(n \log n)$ worst case*

Invariant trouble

Recall this?

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> Proving that Android's, Java's and Python's sorting algorithm is broken (and showing how to fix it)
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> February 24, 2015 Envisage Written by Stijn de Gouw. $s



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- In Java: `Arrays.sort(Object[])`
  could throw `ArrayIndexOutOfBoundException`
  for specific input of size 67,108,864
  - Timsort in Java since 2009, but issue never reported?
  - first (incorrectly!) patched in 2013, then a second time in 2015 ...
- for CPython: patched in 2015
  - mostly a theoretical issue (needs $\geqslant 2^{49}$ elements)

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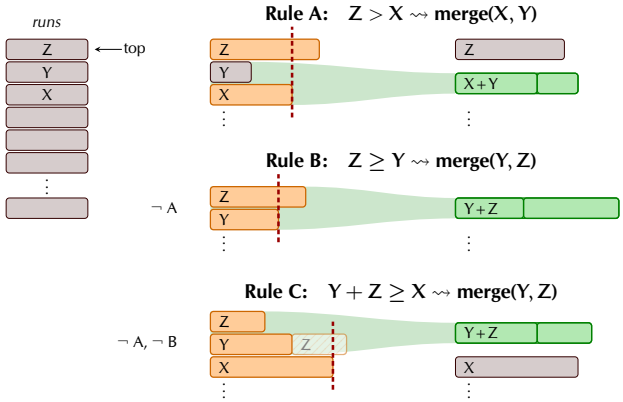

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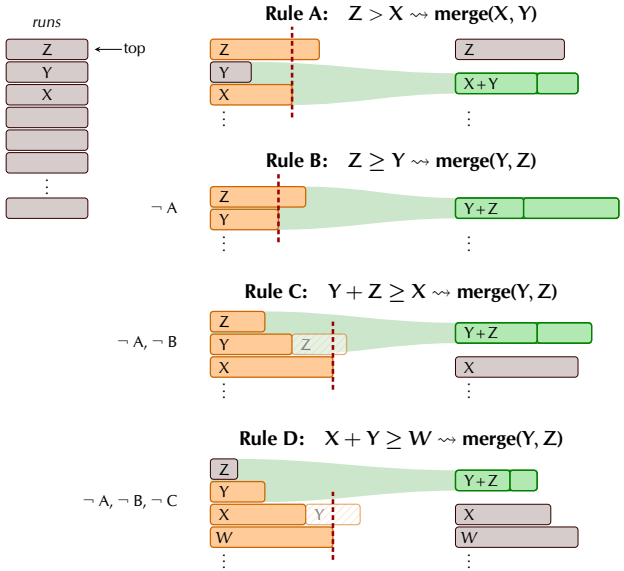

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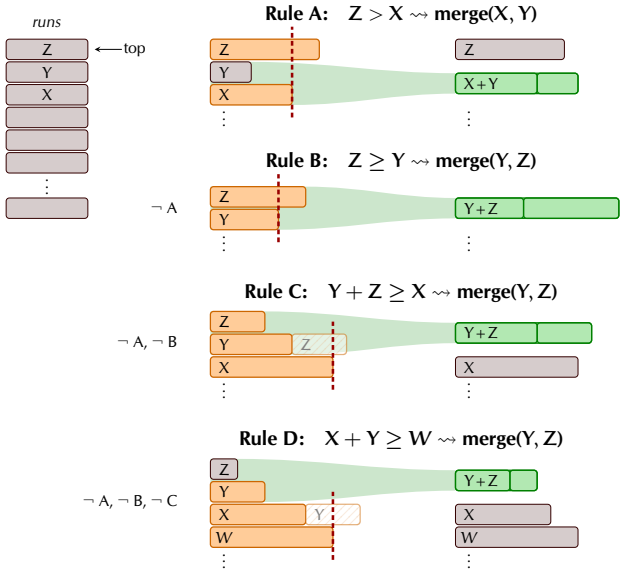

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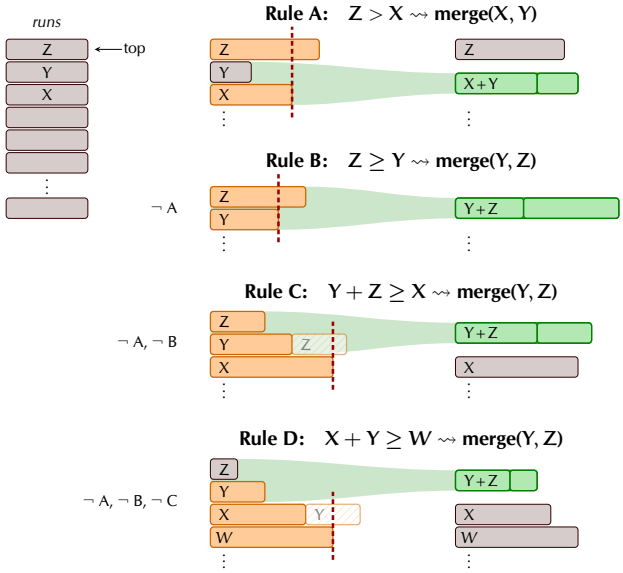
- running time indeed $O(n \log n)$
  …*very* complicated to prove



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Merge policies from first principles

Run-adaptive mergesort

interleaved in code

Conceptually two steps:

1. *Find runs in input.*
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determined by **merge policy**

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⇝ 2 becomes:
merge 2 runs,
repeat until single run

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⇝ *merge 2 **adjacent** runs*

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Merge policies from first principles

Run-adaptive mergesort

Conceptually two steps: (interleaved in code)

1. *Find runs in input.*
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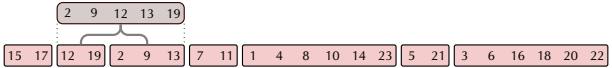

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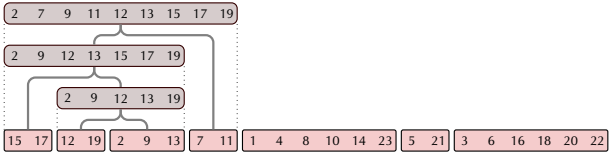

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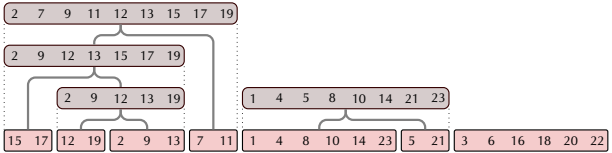

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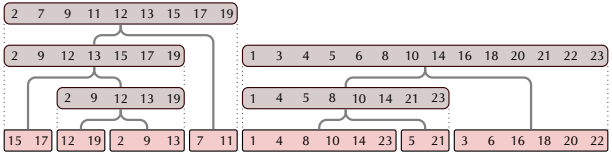

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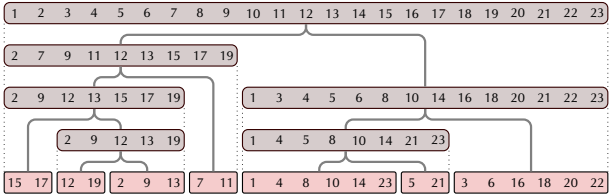

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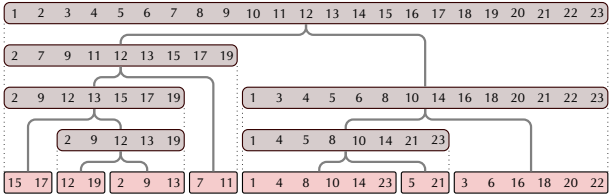

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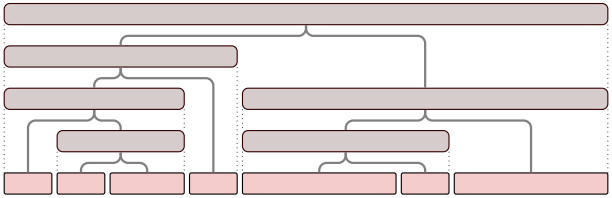

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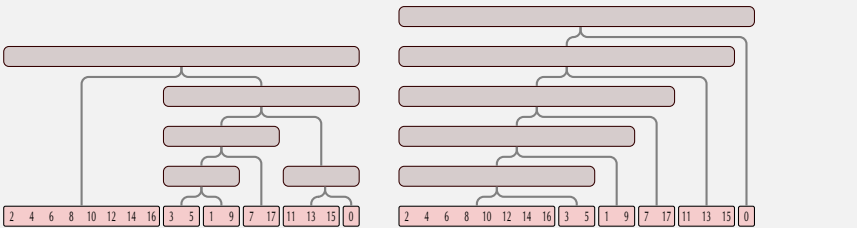
Different merge policies yield different merge costs!



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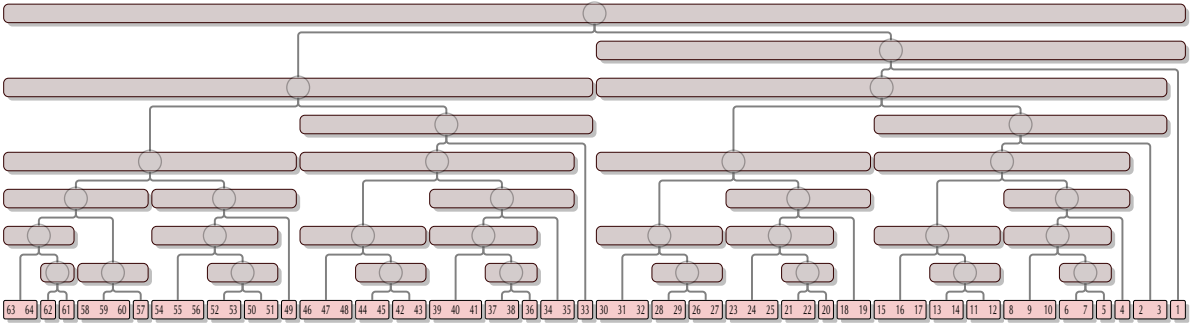

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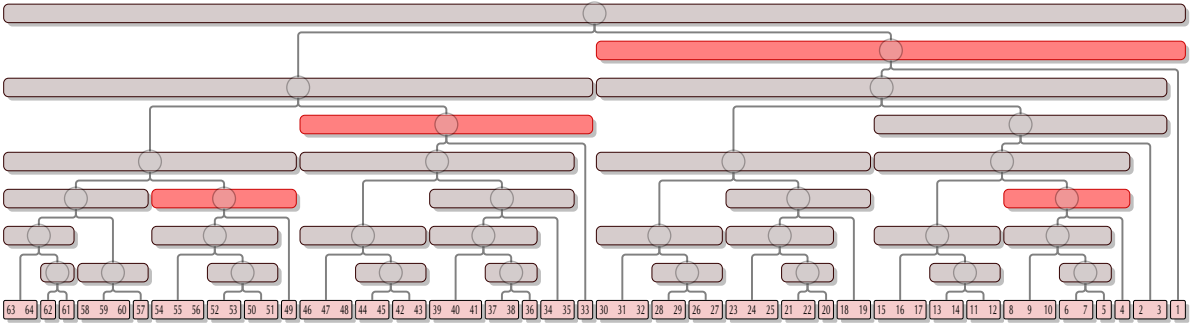

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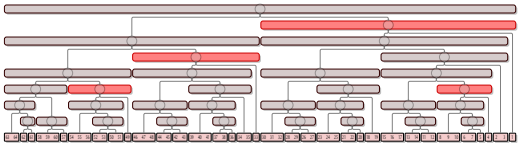

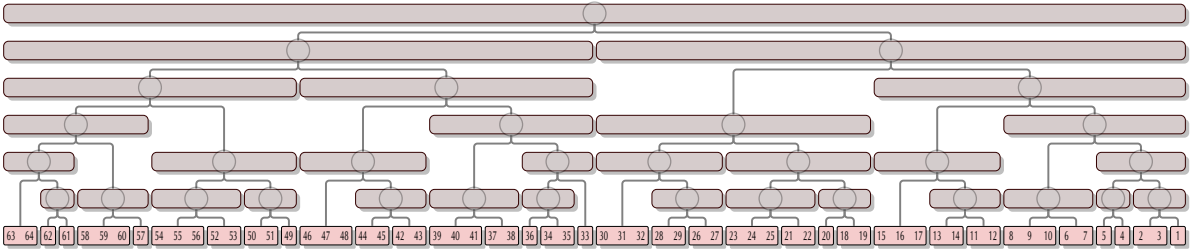

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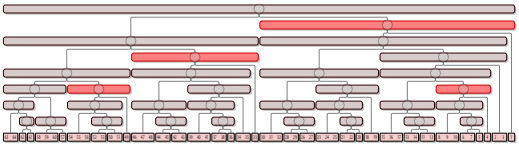

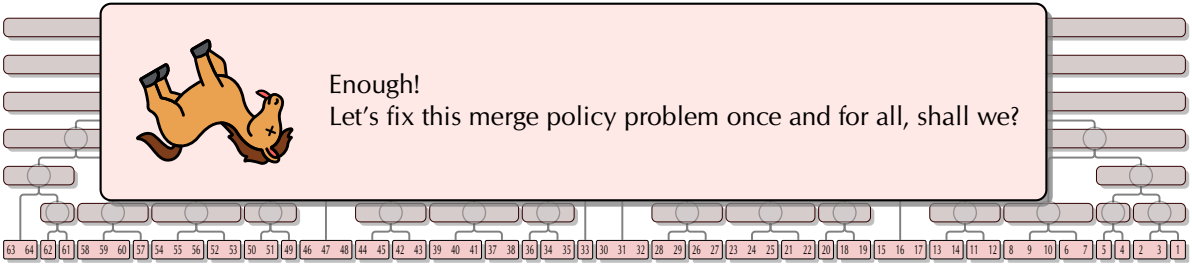

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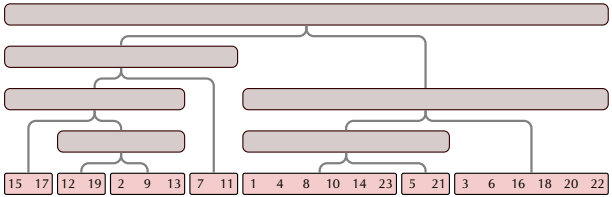

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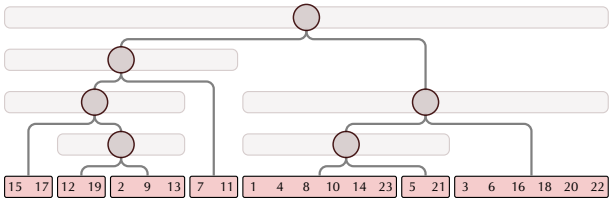

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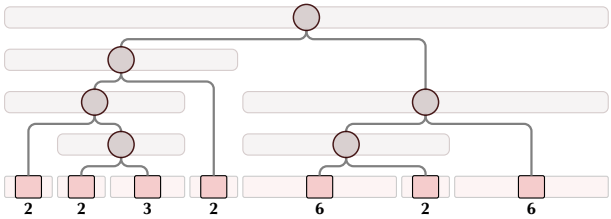

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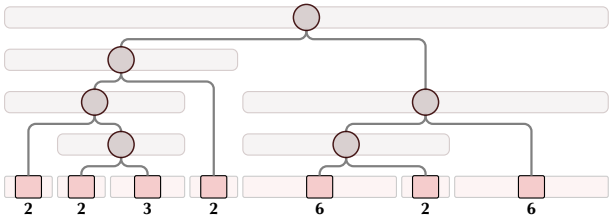

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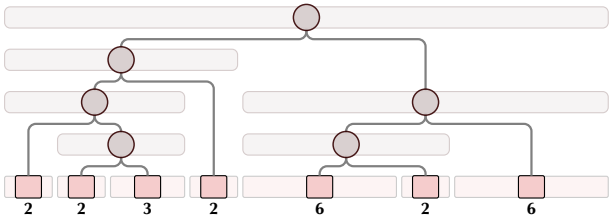

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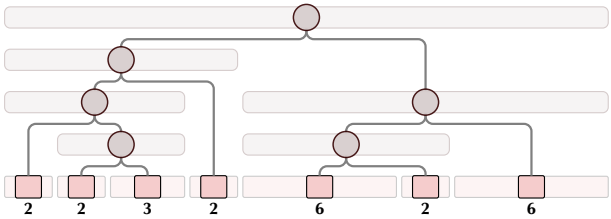

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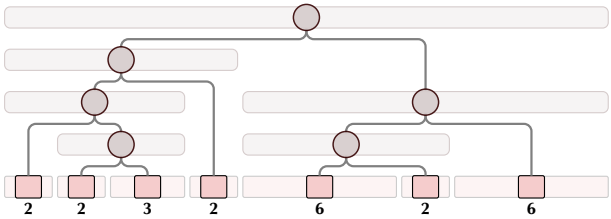

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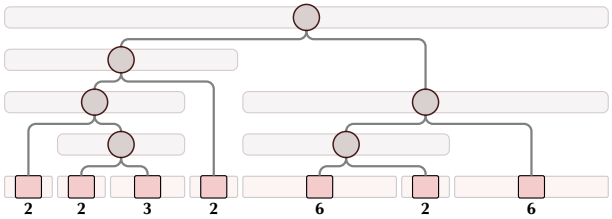

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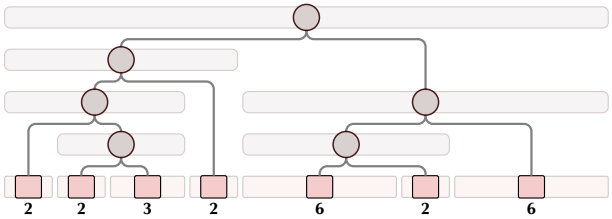

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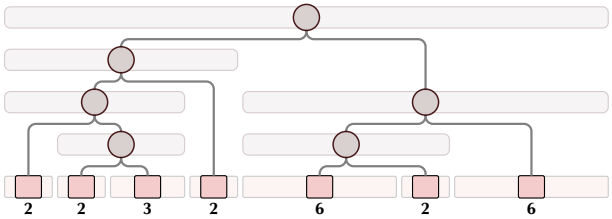

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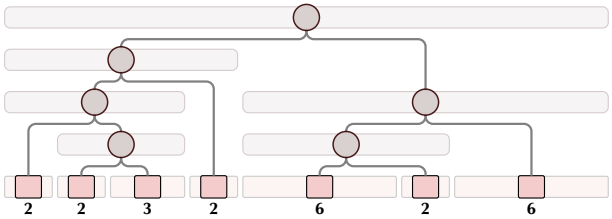

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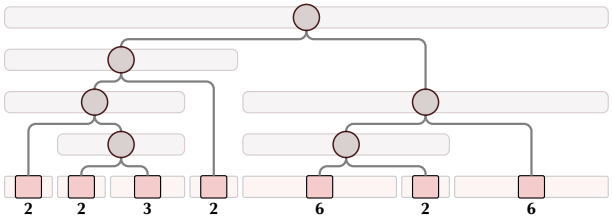

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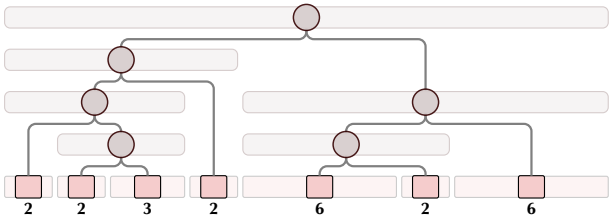

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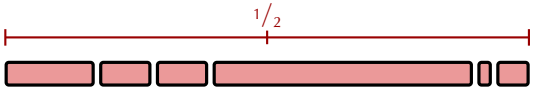

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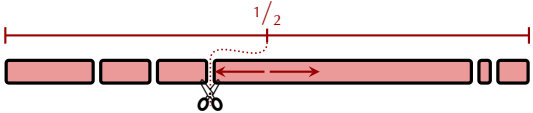
The Bisection Method

- Powersort is based on the **bisection method**

  Mehlhorn: *A best possible bound for the weighted path length of binary search trees,* SIAM J Comp **1977**



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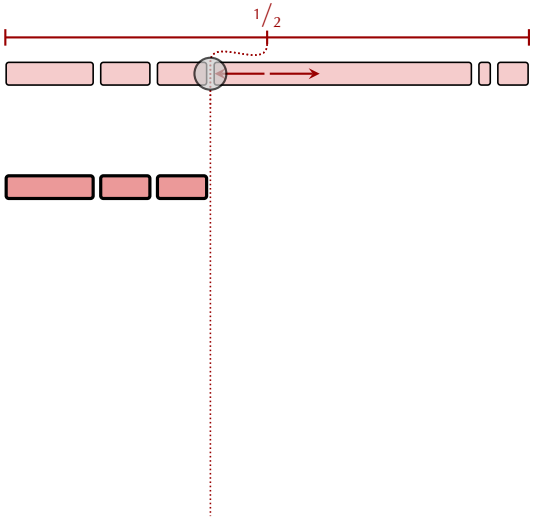

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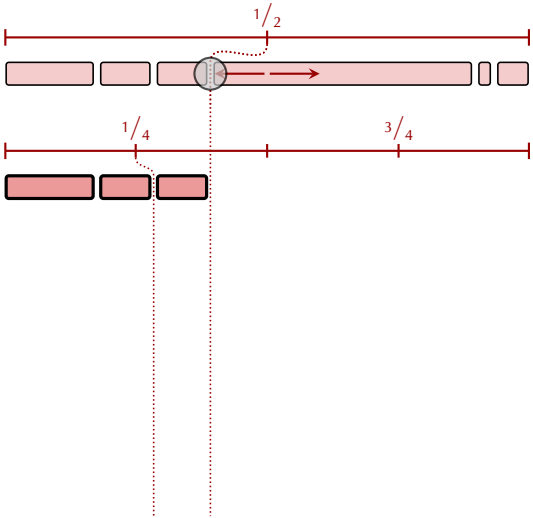

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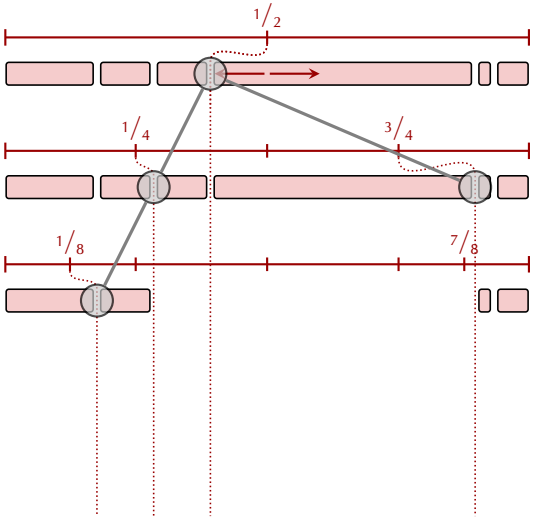

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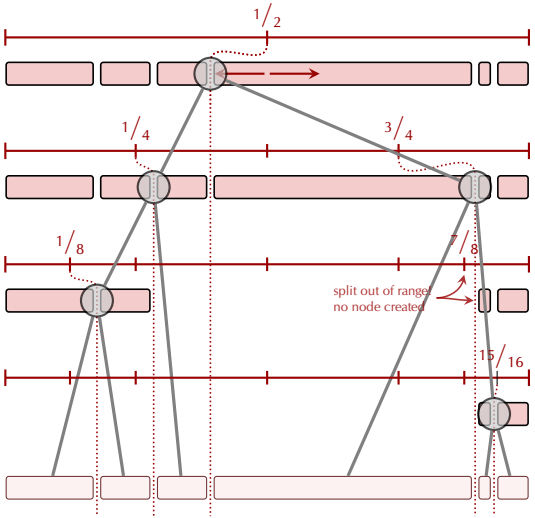

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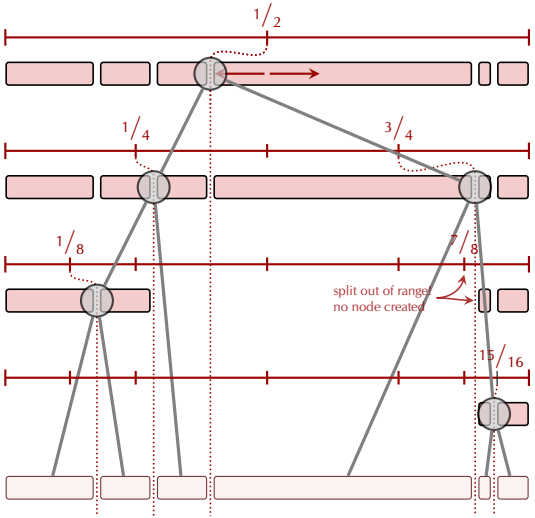
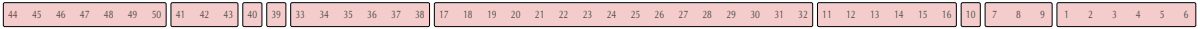
- *Up next:* Don't use recursion!



Run-Boundary Powers



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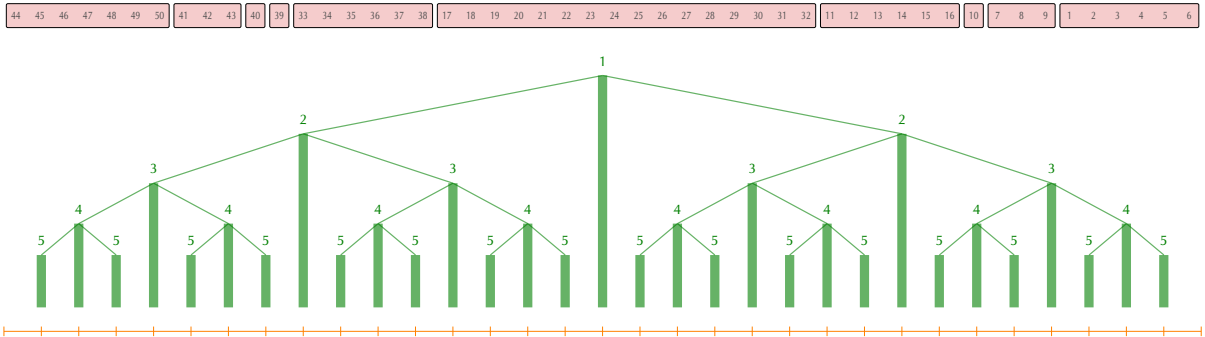
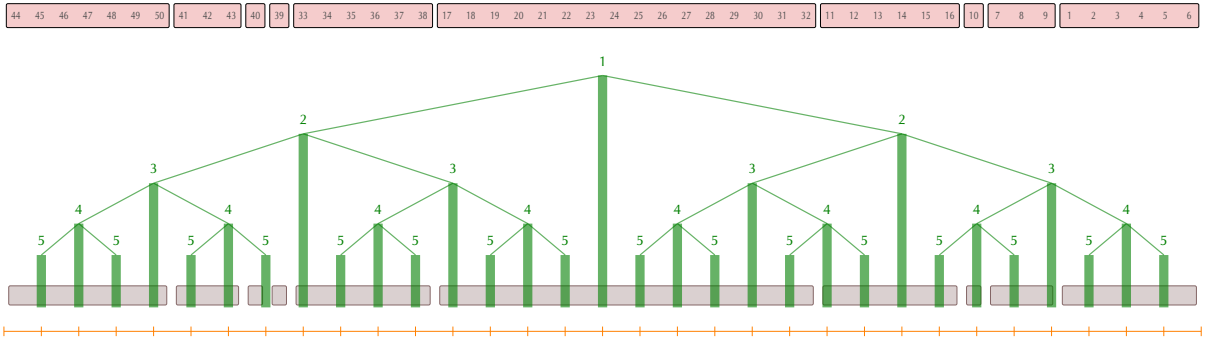


- (virtual) perfect balanced binary tree

Run-Boundary Powers



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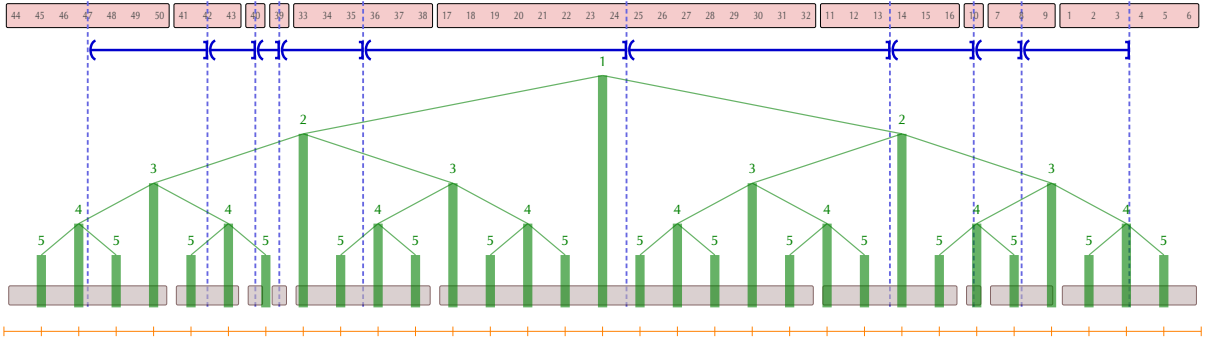

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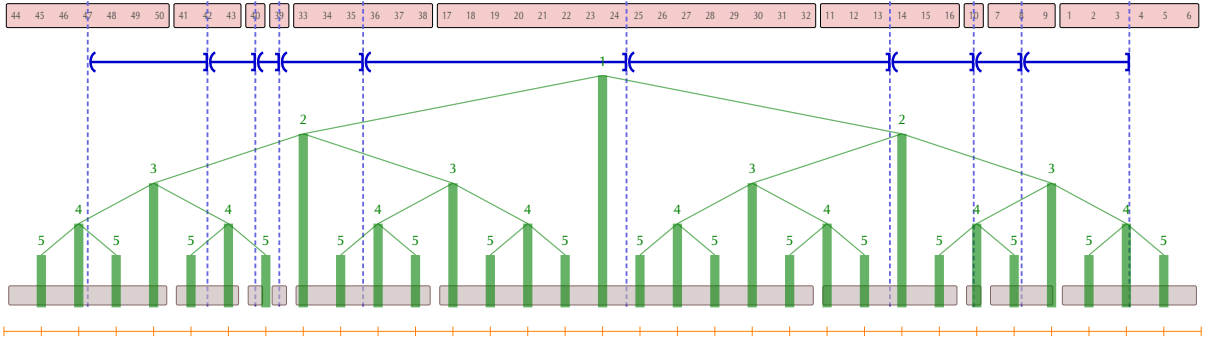

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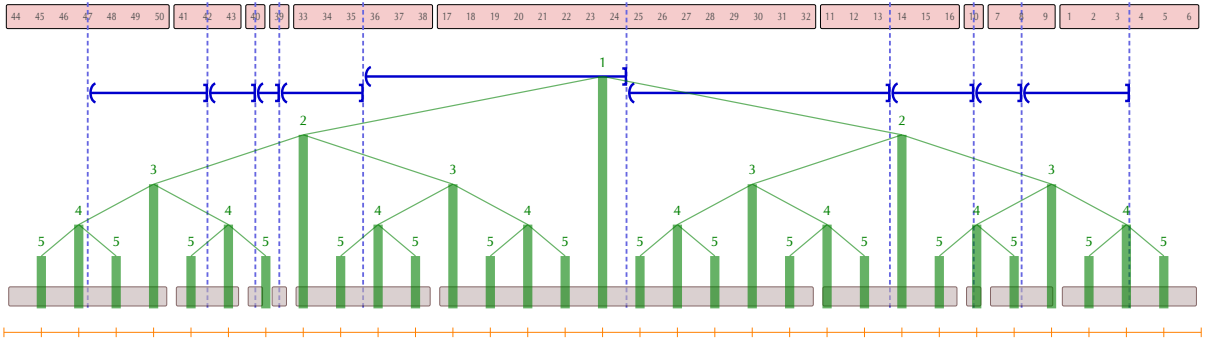

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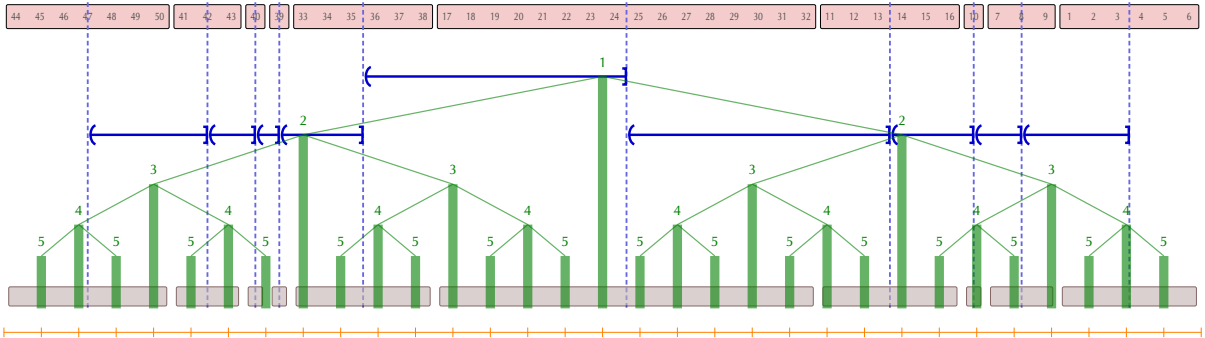

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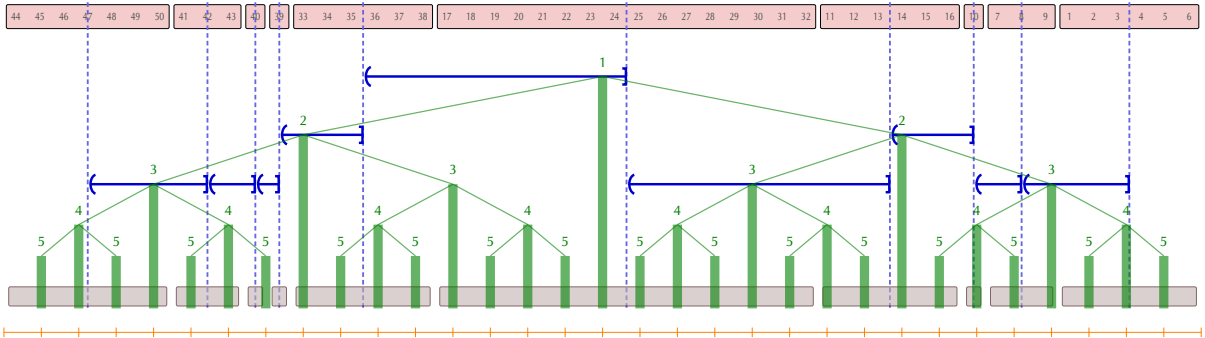

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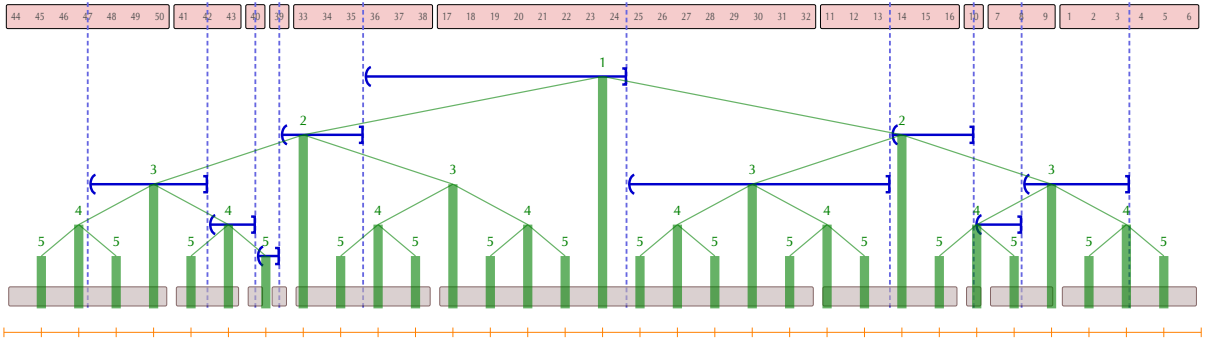

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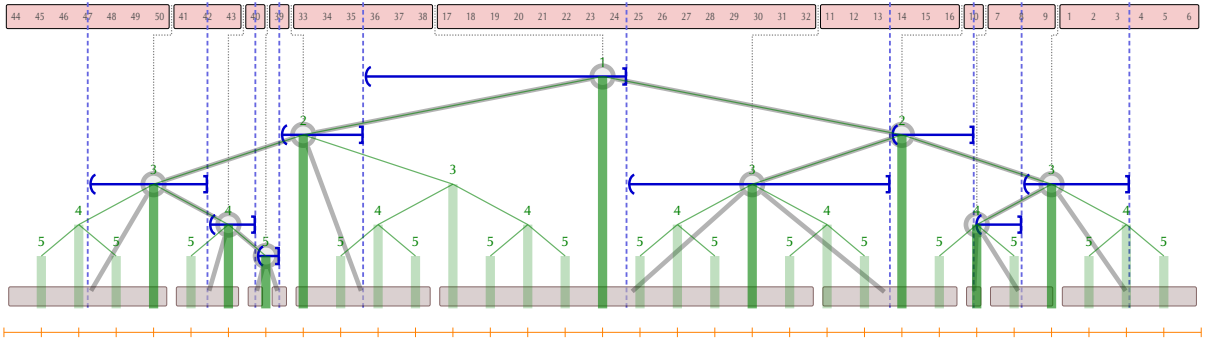

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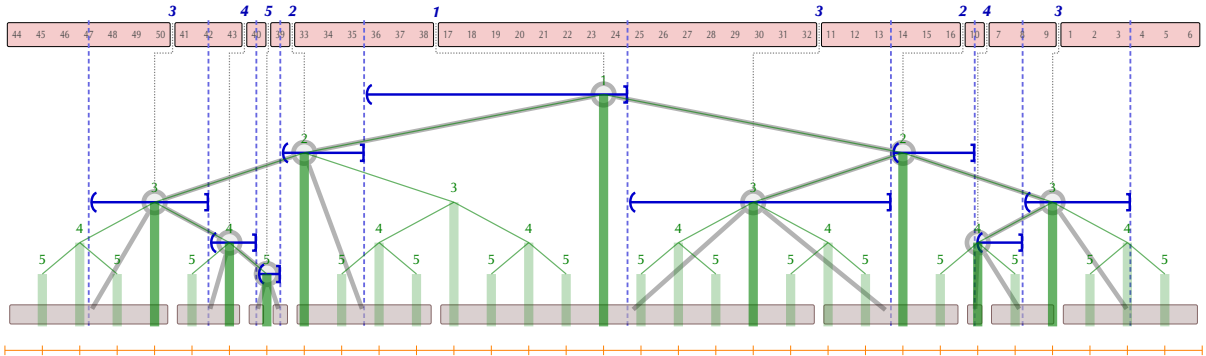

- (virtual) perfect balanced binary tree
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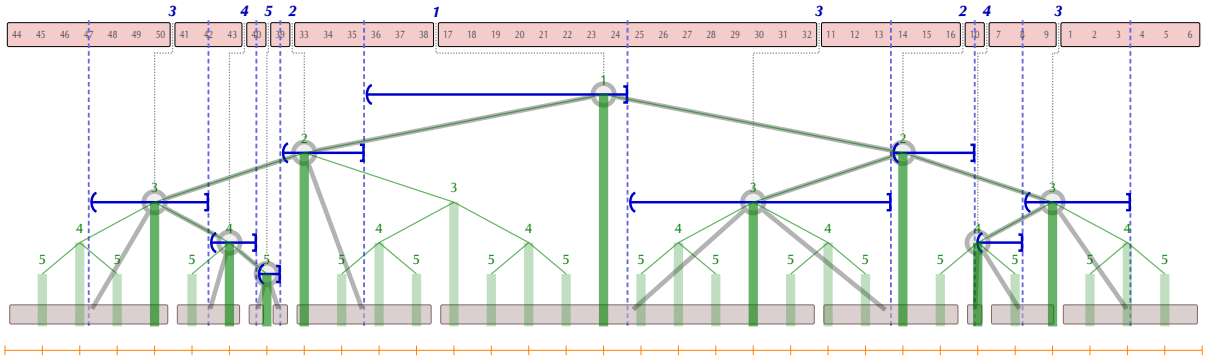
  - ⇝ assigns each run boundary a depth

Run-Boundary Powers



- (virtual) perfect balanced binary tree
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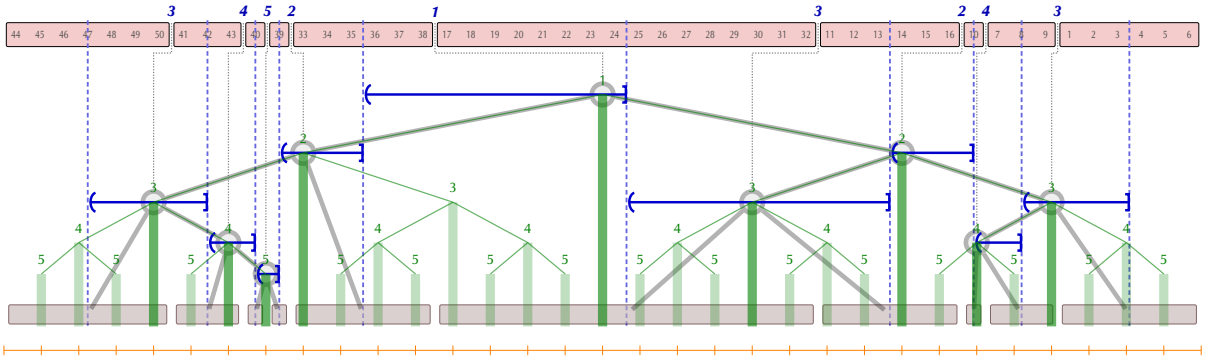
Run-Boundary Powers



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⇝ merge tree follows virtual tree



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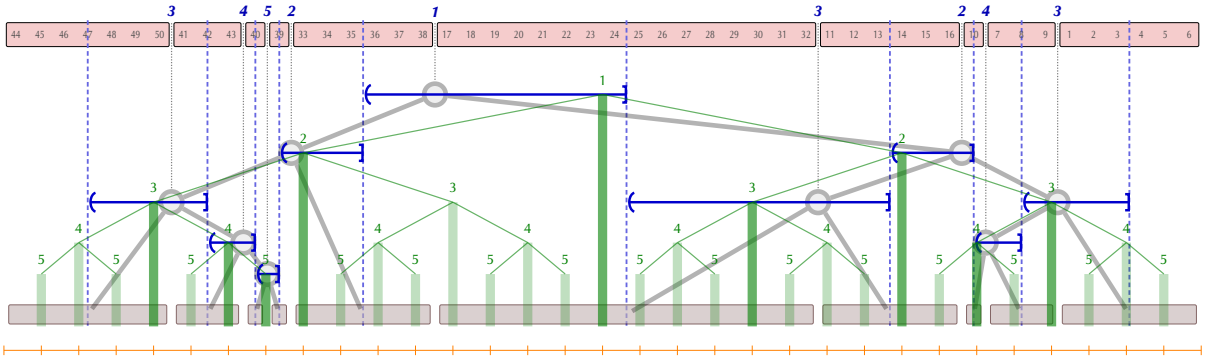

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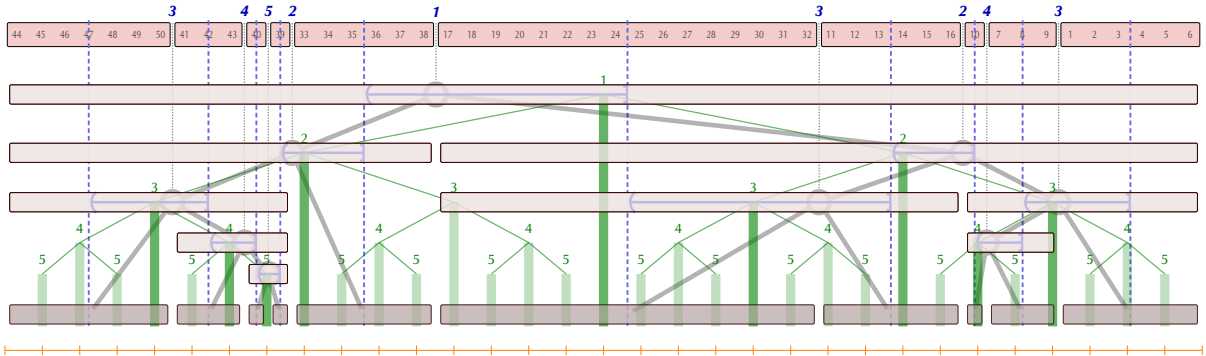

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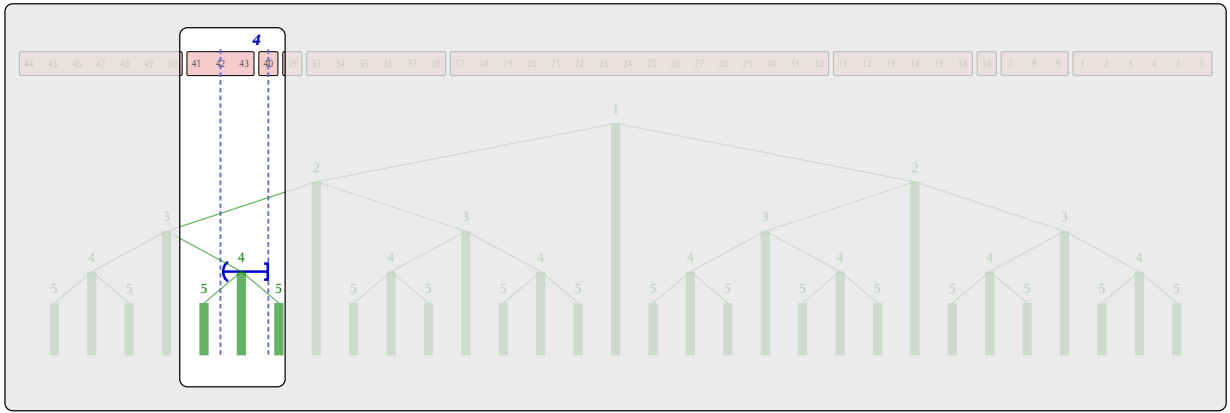
Run-Boundary Powers are Local



Computation of powers only depends on two adjacent runs.

Powersort

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1  def powersort(lst)
2      i = 0; n = len(lst)
3      runs = [];
4      j = extend_run(lst, i)
5      runs.append((i,j-i,0)); i = j
6      while i < n:
7          j = extend_run(lst, i)
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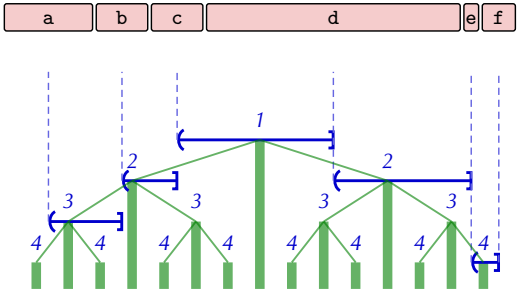
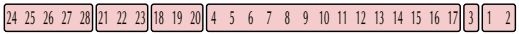
full code:
`tiny.cc/timsort`





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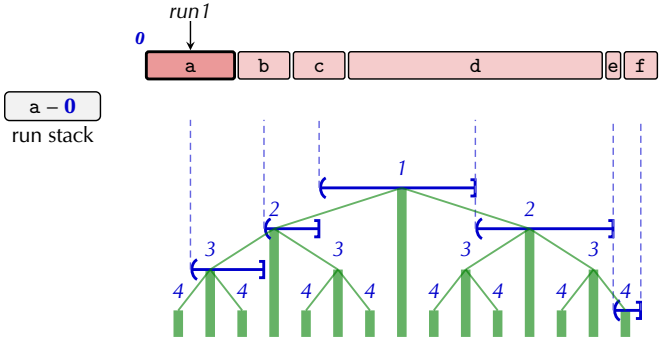

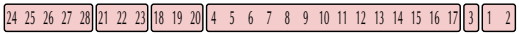

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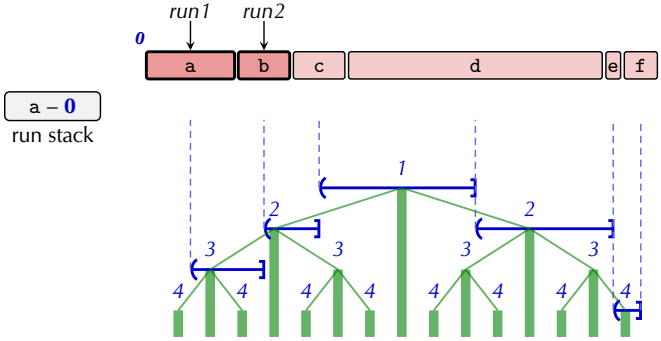

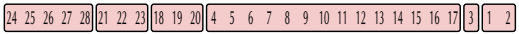

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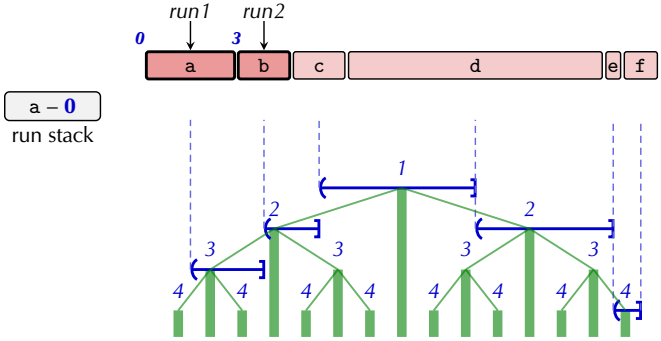

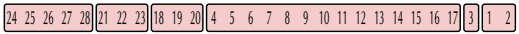

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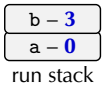

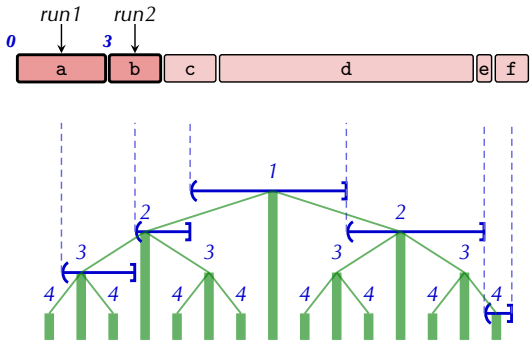

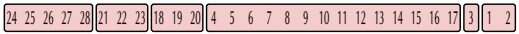

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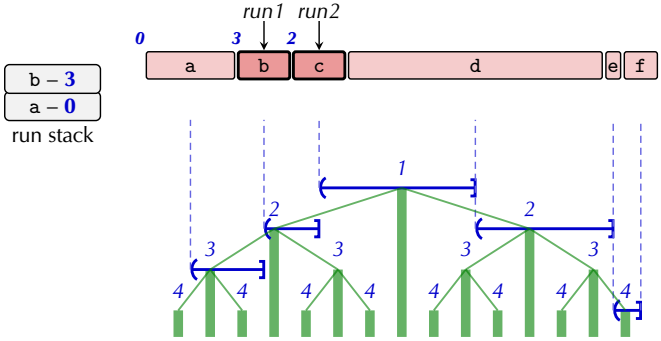

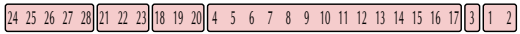

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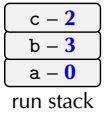

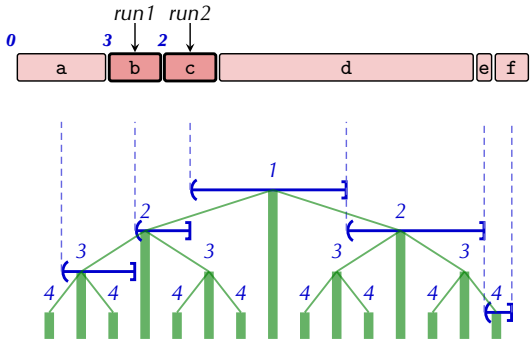

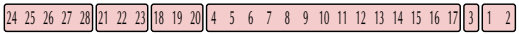

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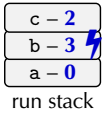

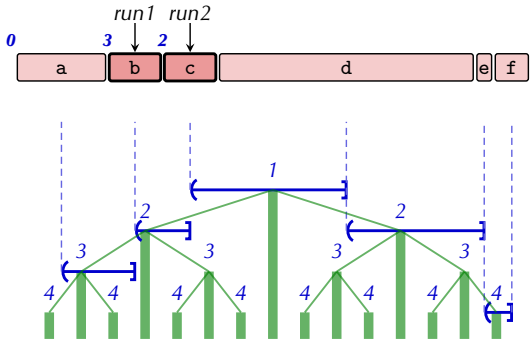

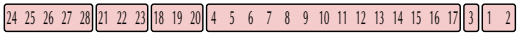

Powersort

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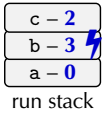

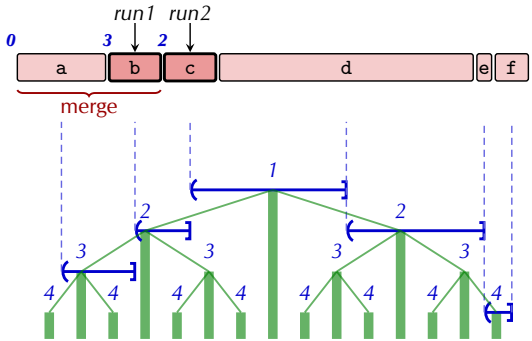

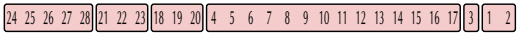

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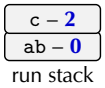

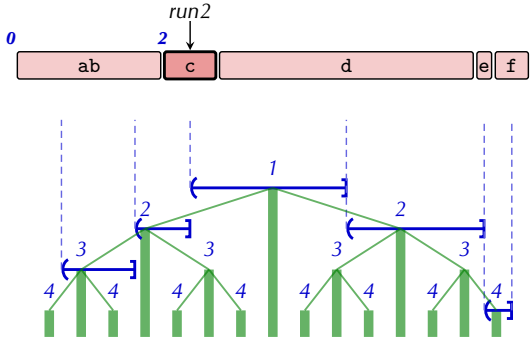

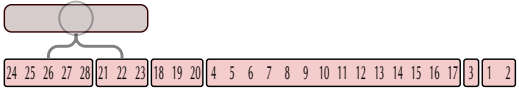

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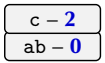

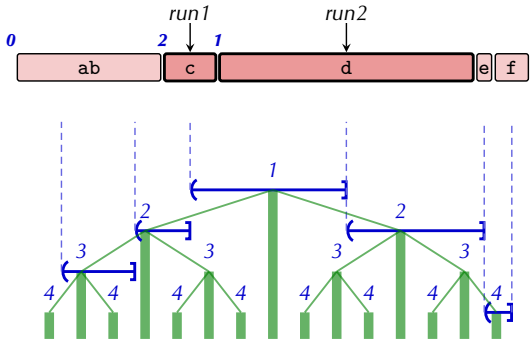

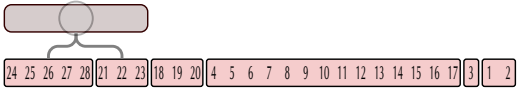

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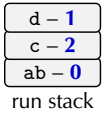

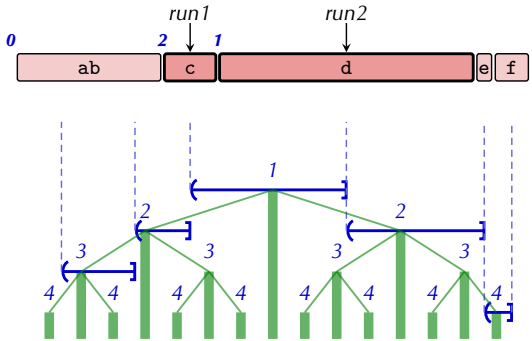

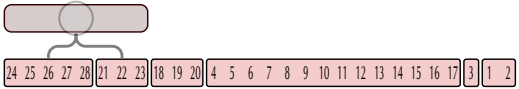

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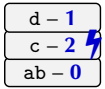
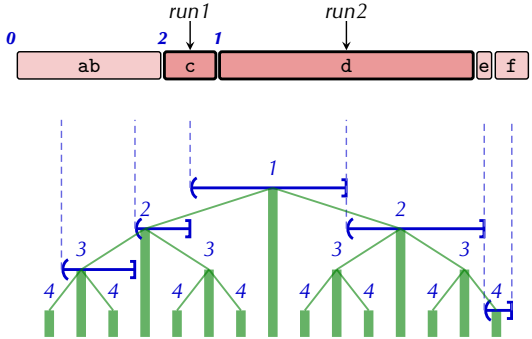
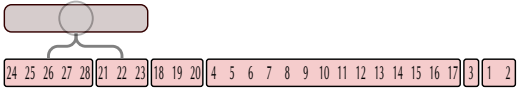


run stack





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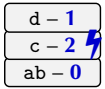

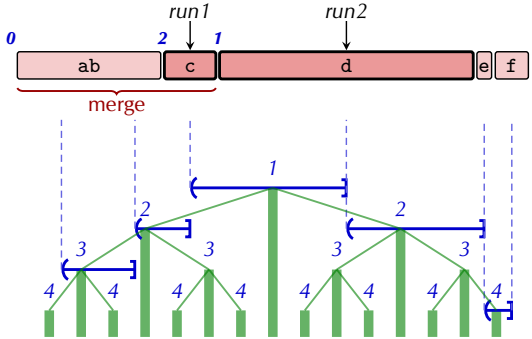

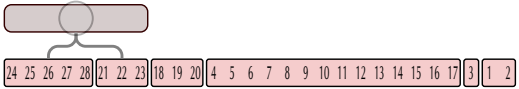

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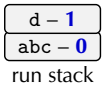

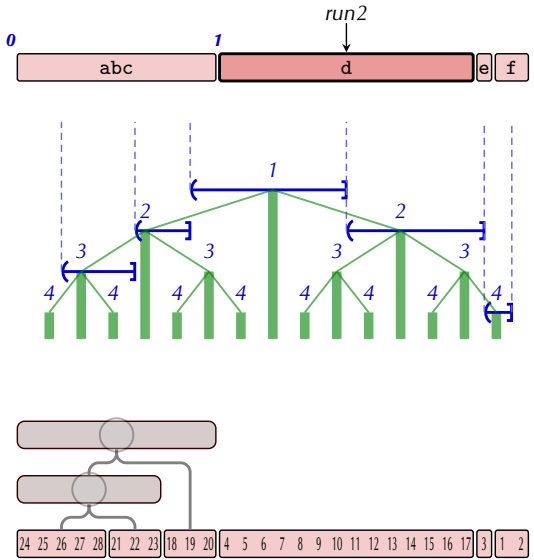

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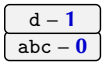
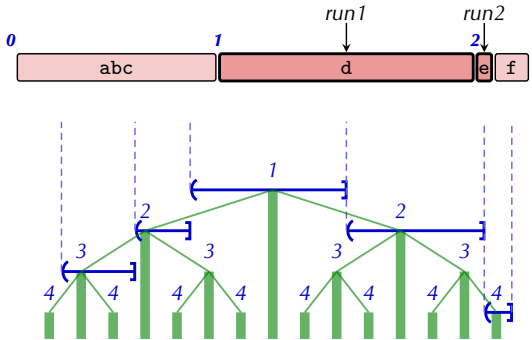
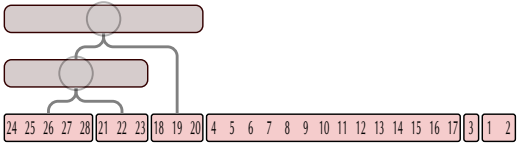


run stack





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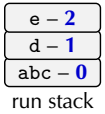

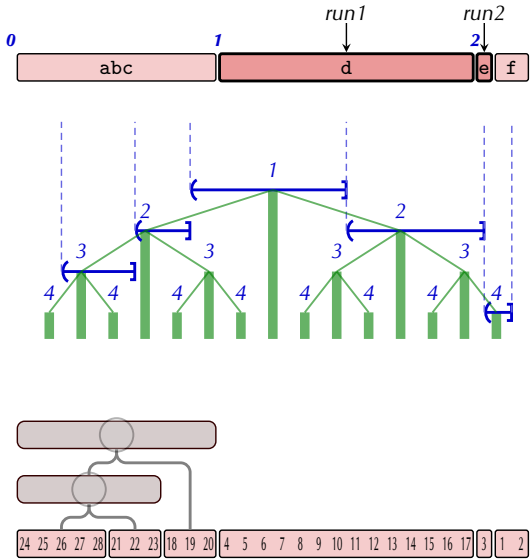

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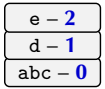

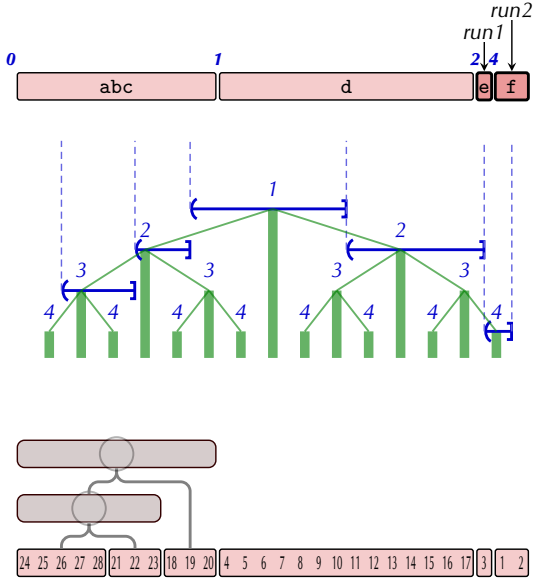

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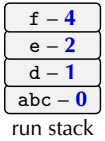

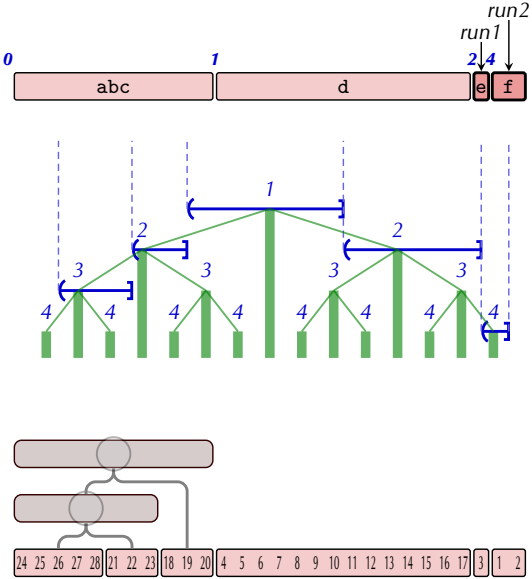

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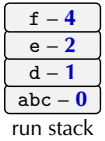

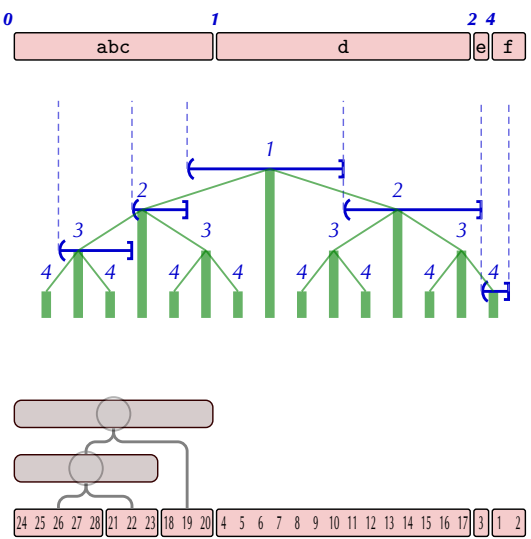

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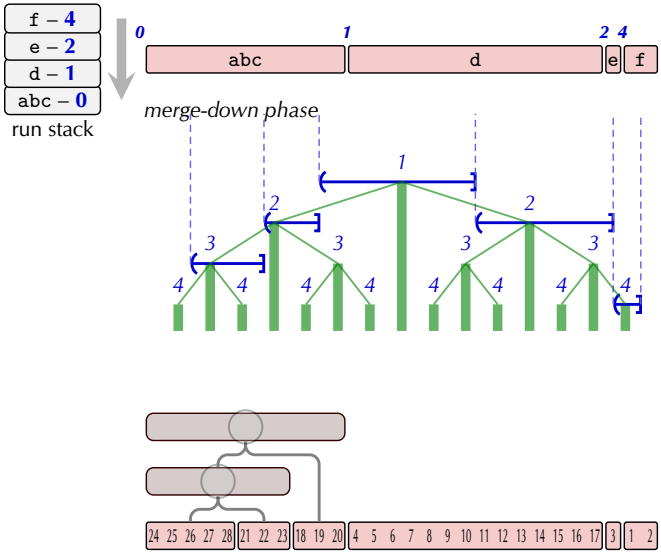

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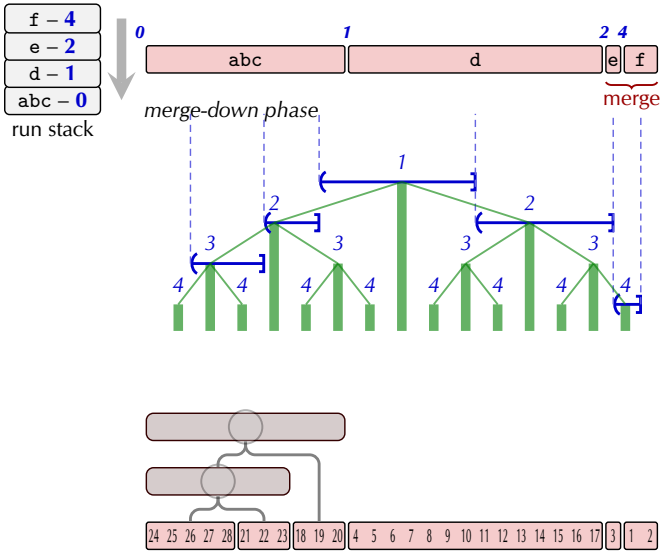

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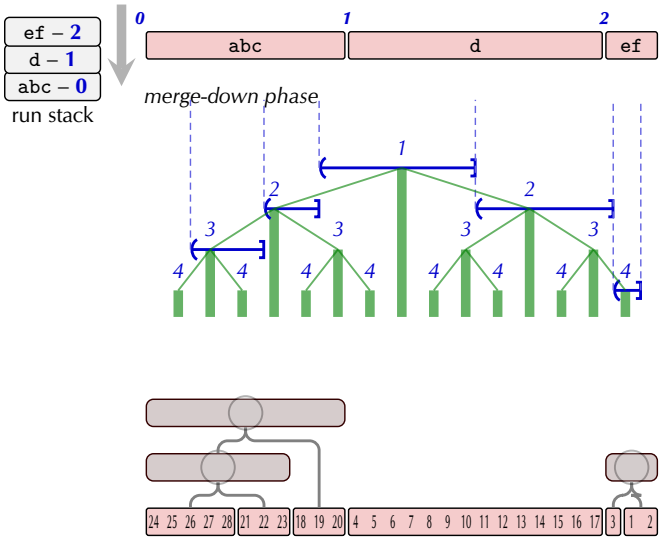

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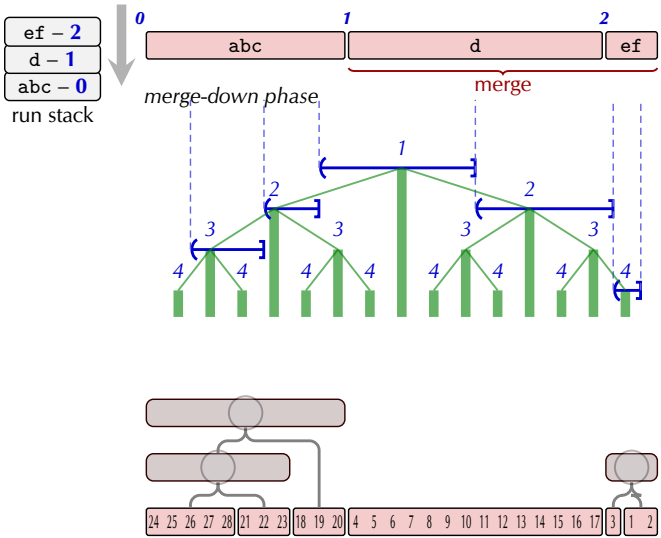
`tiny.cc/timsort`



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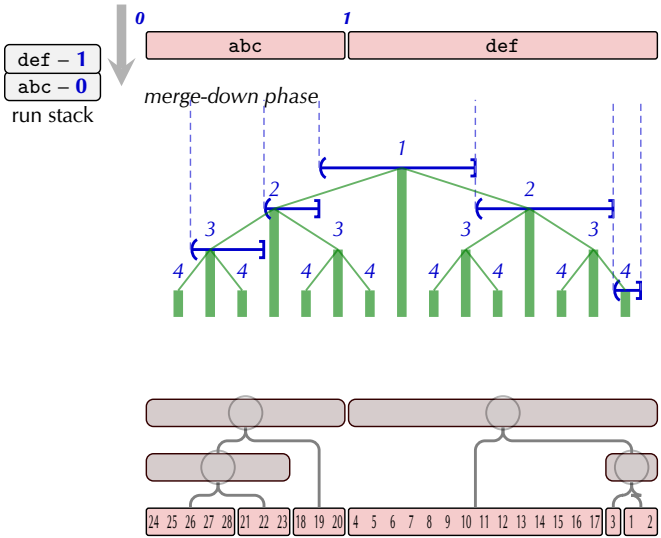

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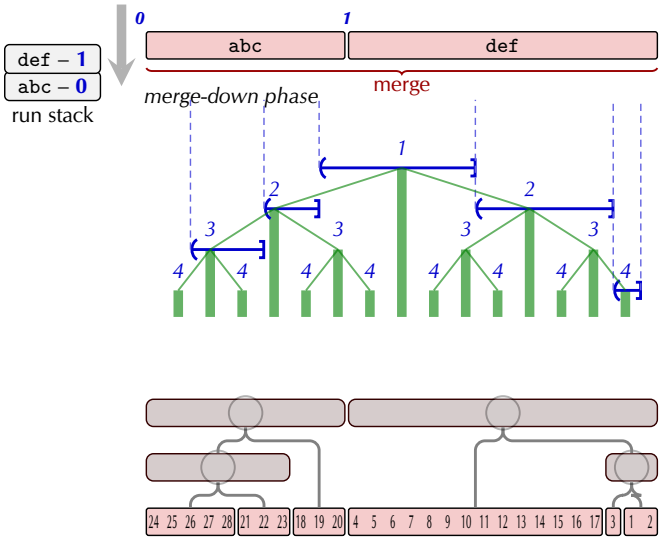

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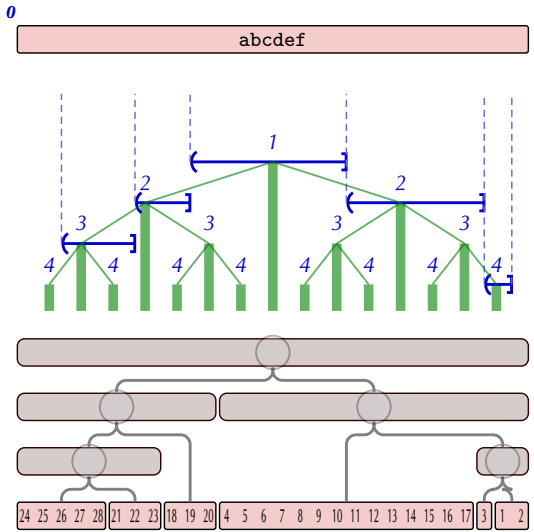
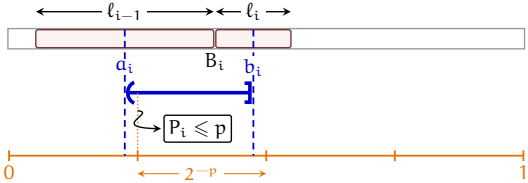
abcdef – **0**
run stack



Computing Powers

- Computing the power of (run boundary between) two runs
  - $\left[\!\left[\,\right]\!\right]$ = normalized midpoint interval
  - power = min ℓ s.t. $\left[\!\left[\,\right]\!\right]$ contains $c \cdot 2^{-\ell}$





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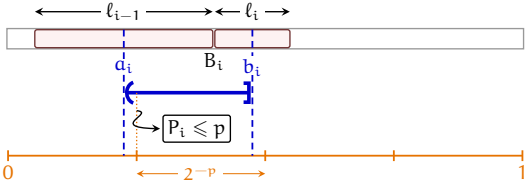


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3      i2, n2 = run2
4      A = 2 * i1 + n1    # A = 2na_i
5      B = A + n1 + n2    # B = 2nb_i
6      p = 0
7      while True:
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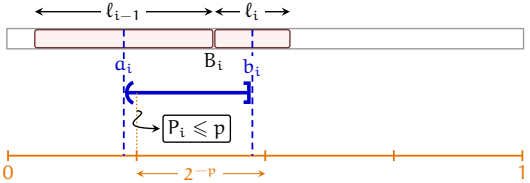

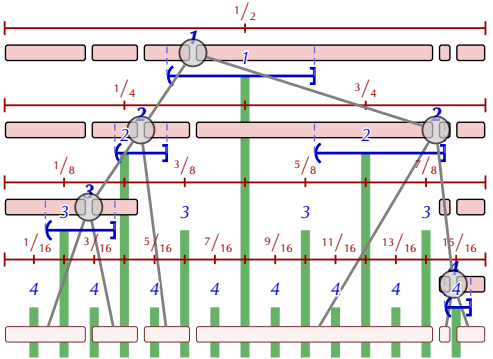

Mergecost Analysis

- **Given:** input with runs $R_1, \ldots, R_r$ of **lengths** $L_1, \ldots, L_r$; $n = L_1 + \cdots + L_r$
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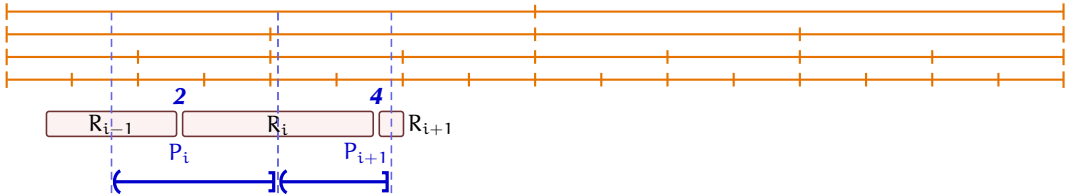

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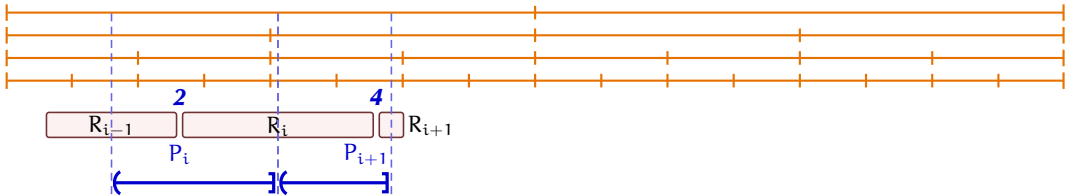

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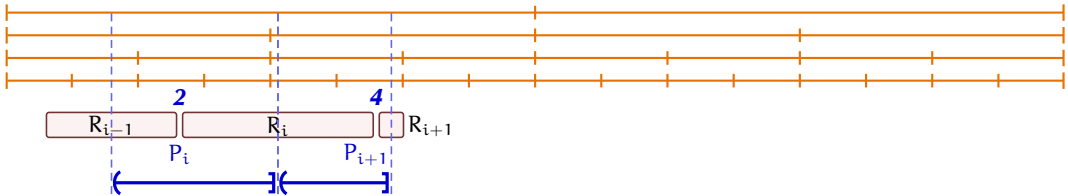

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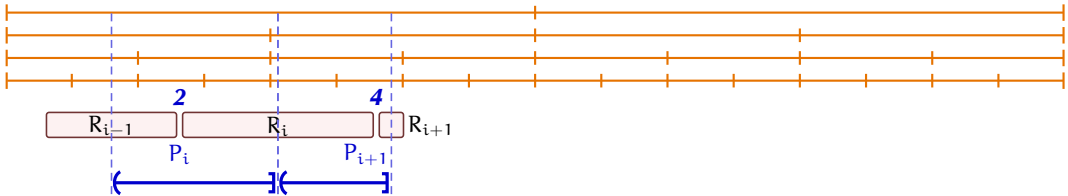

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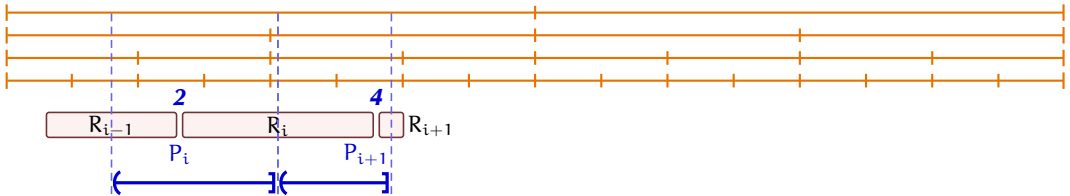

Conclusion



```
            a = 2 * s1 + n1          # 2 * a'
            b = a + n1 + n2          # 2 * b'
            result = 0
            while True:
                result += 1
                if a >= n:
                    assert b >= a
                    a -= n
                    b -= n
                elif b >= n:
                    break
                assert a < b < n
                a <<= 1
                b <<= 1
            return result

    def found_new_run(self, run):

        p = self.pending
        if p:
            s1 = p[-1].base
            n1 = p[-1].len
            power = powerloop(s1, n1, run.len, self.listlength)
            while len(p) > 1 and p[-2].power > power:
                self.merge_at(-2)
            assert len(p) < 2 or p[-2].power < power
            p[-1].power = power;

    def merge_force_collapse(self):
        p = self.pending
        while len(p) > 1:
```

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Summary



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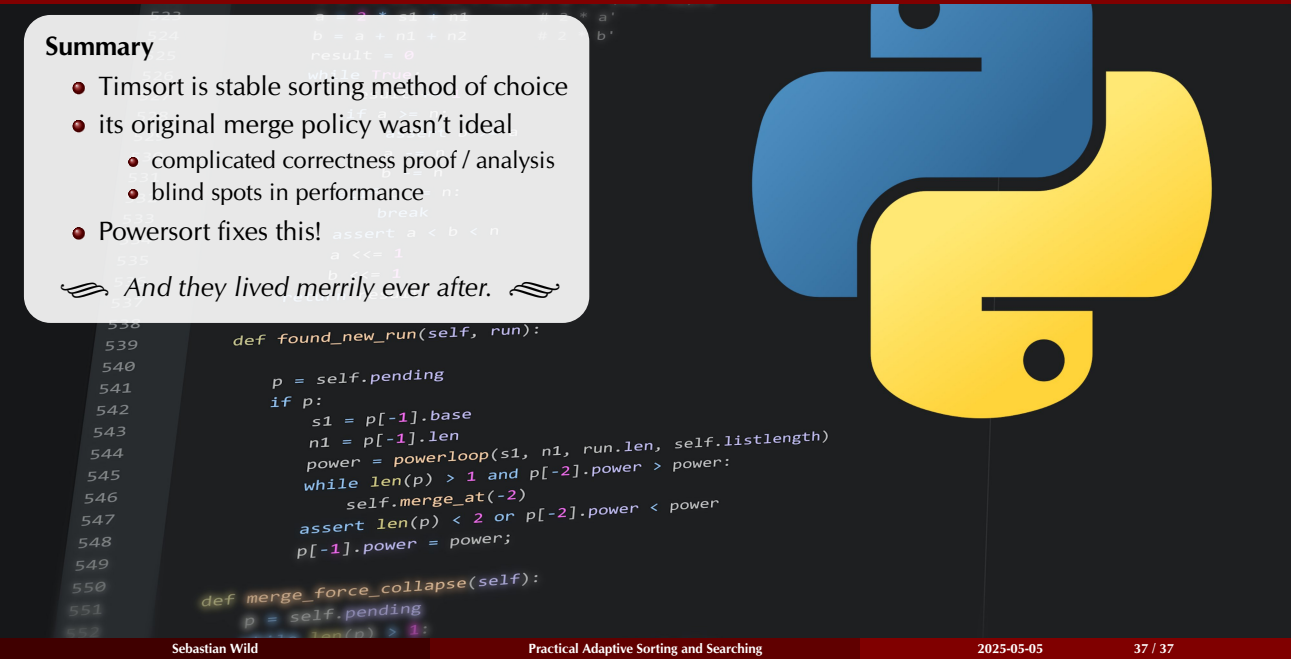

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Cawley Gelling, Nebel, Smith, Wild: *Multiway Powersort*, ALENEX 2023

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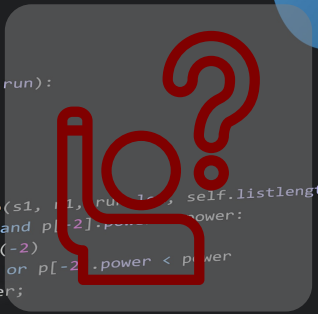
Open problems

1. Good **probabilistic models** for inputs with runs?
2. Analyze **minrunlen** heuristic
3. Powersort in **other libraries** using Timsort
   - numpy & pandas
   - OpenJDK
   - Android Java runtime
   - V8 JavaScript engine
   - GNU STL `std::stable_sort`

I'm hiring postdocs!

- **2 positions**, both can be AofA topics
  1. 2 year project *Computing over compressed graph-structured data*
  2. flexible post, with teaching (»Landesstelle«)
- Ample travel budget (for more AofAs ;))

Marburg sounds cool? → Talk to me!



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