Quicksorts of the 21st century

Sebastian Wild

based on joint work with Stefan Edelkamp, Conrado Martínez, Markus Nebel, Ralph Neininger, and Armin Weiß

UNIVERSITY OF LIVERPOOL

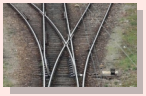
Isaac Newton Institute – Verified Software Workshop



Outline



1 Context



2 Dual-Pivot Quicksort



3 Average-Case Analysis



4 Quicksort + X

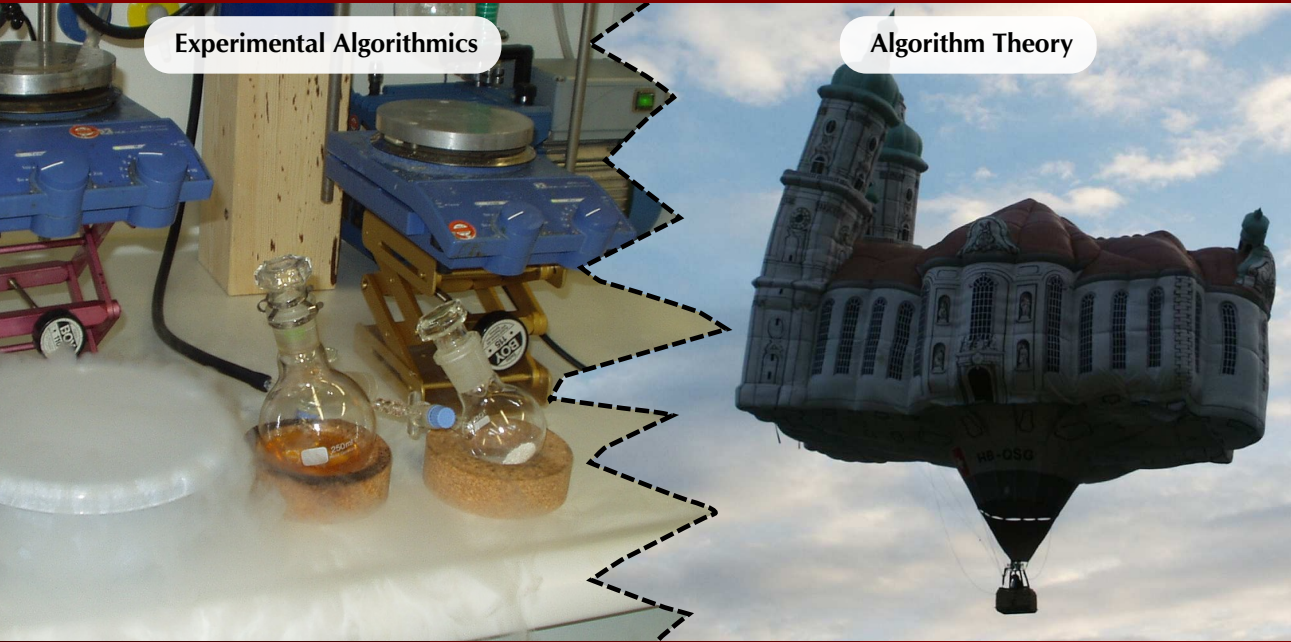


5 Conclusion



1 Context

Algorithm Science



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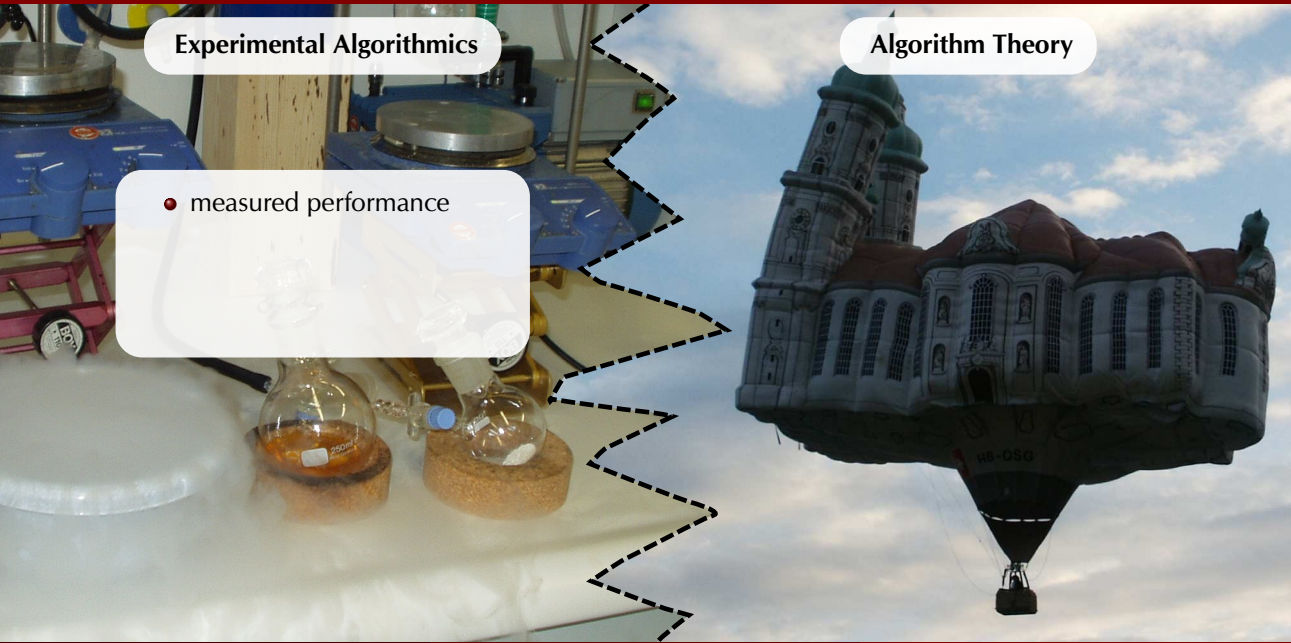

Experimental Algorithmics

- measured performance

Algorithm Theory

Algorithm Science

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- specific to machine

Algorithm Theory



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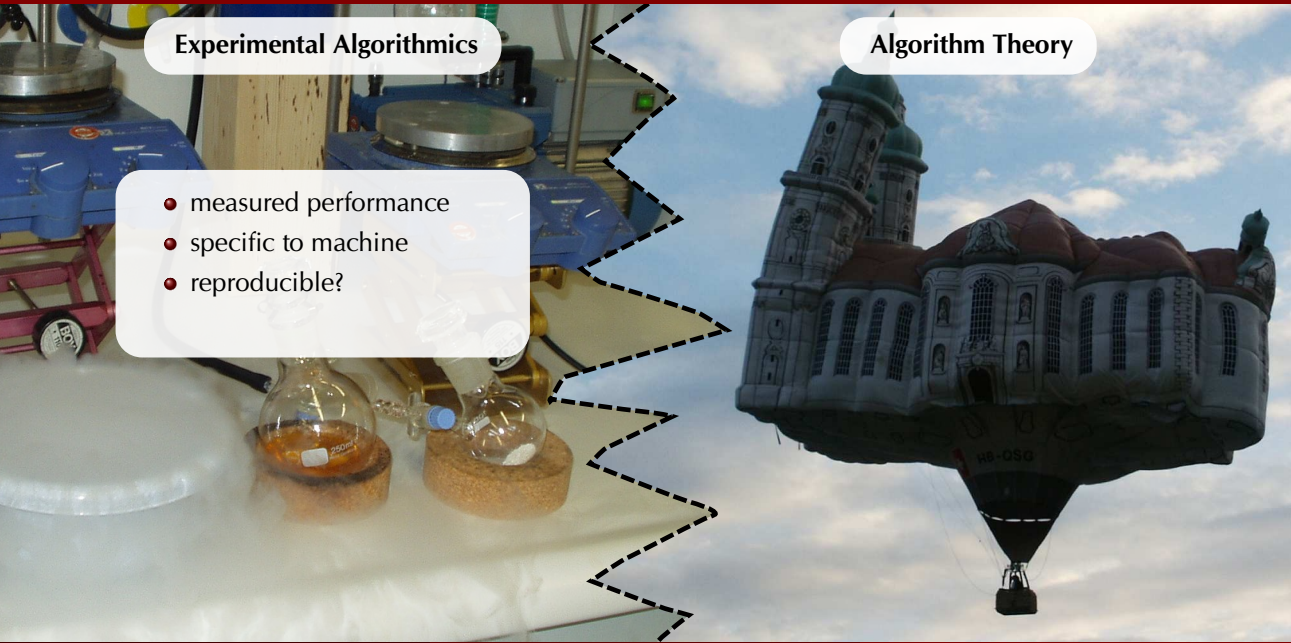

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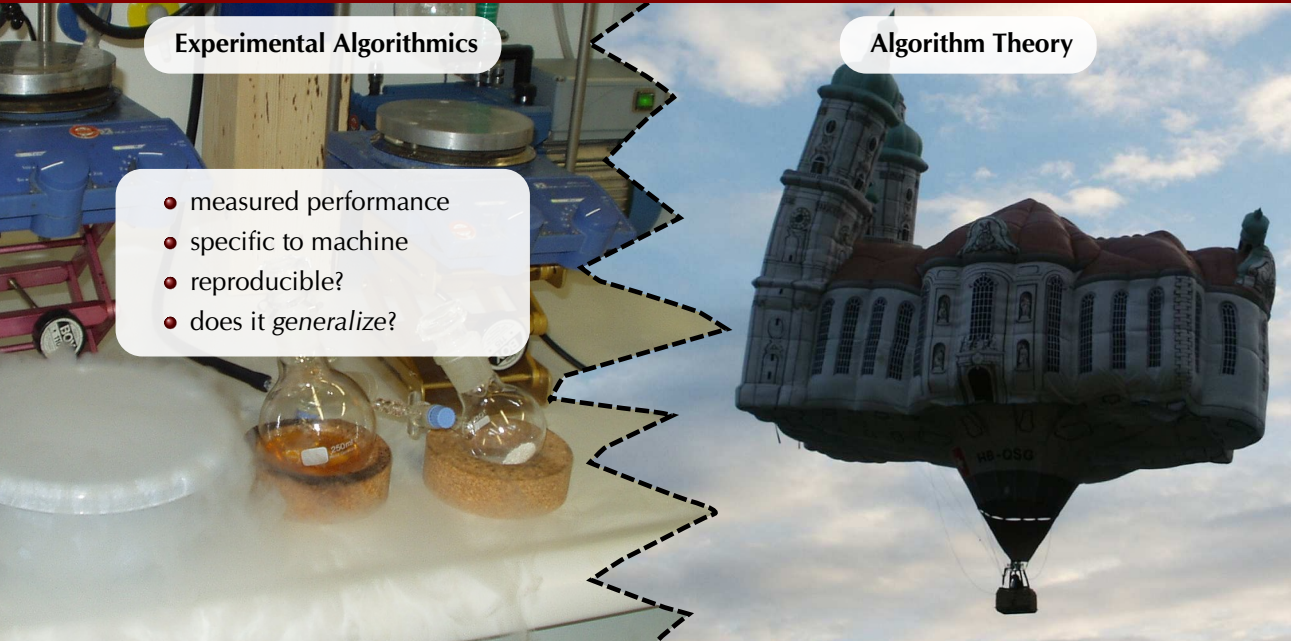

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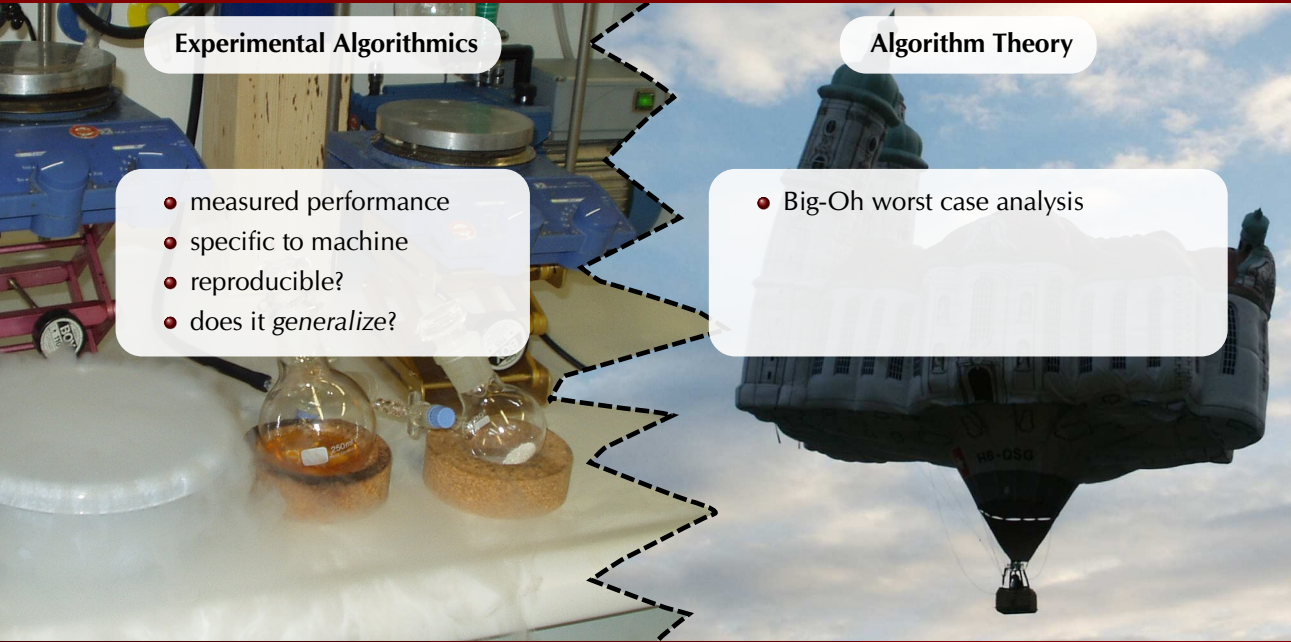

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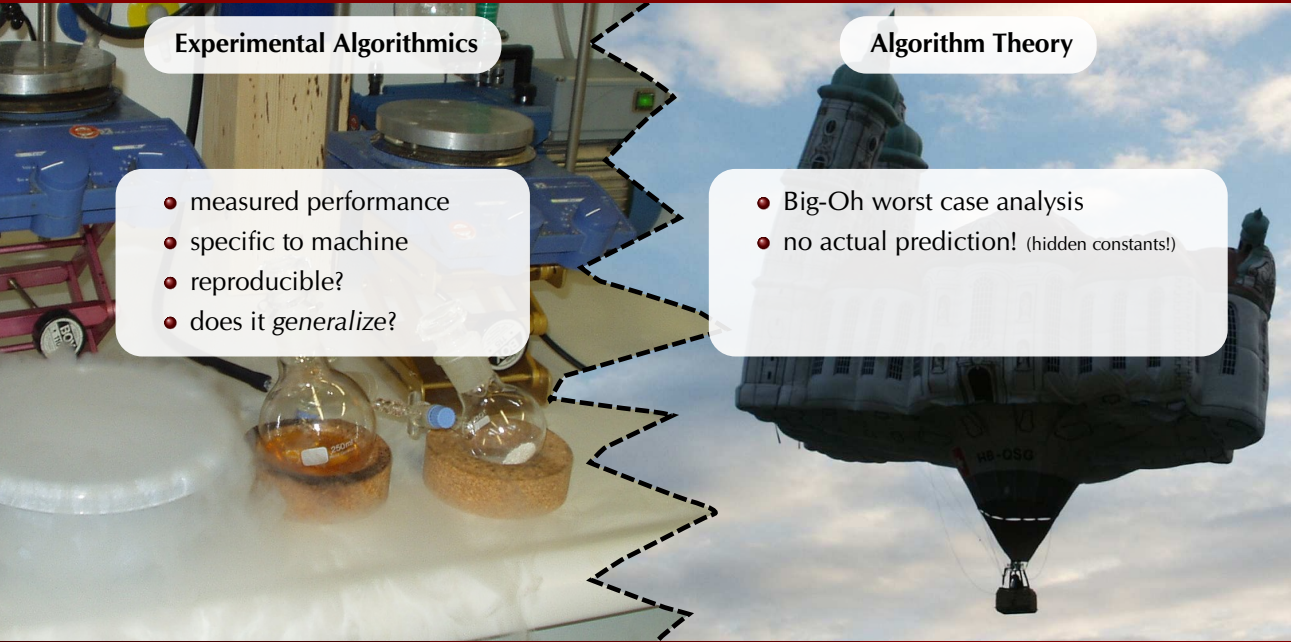

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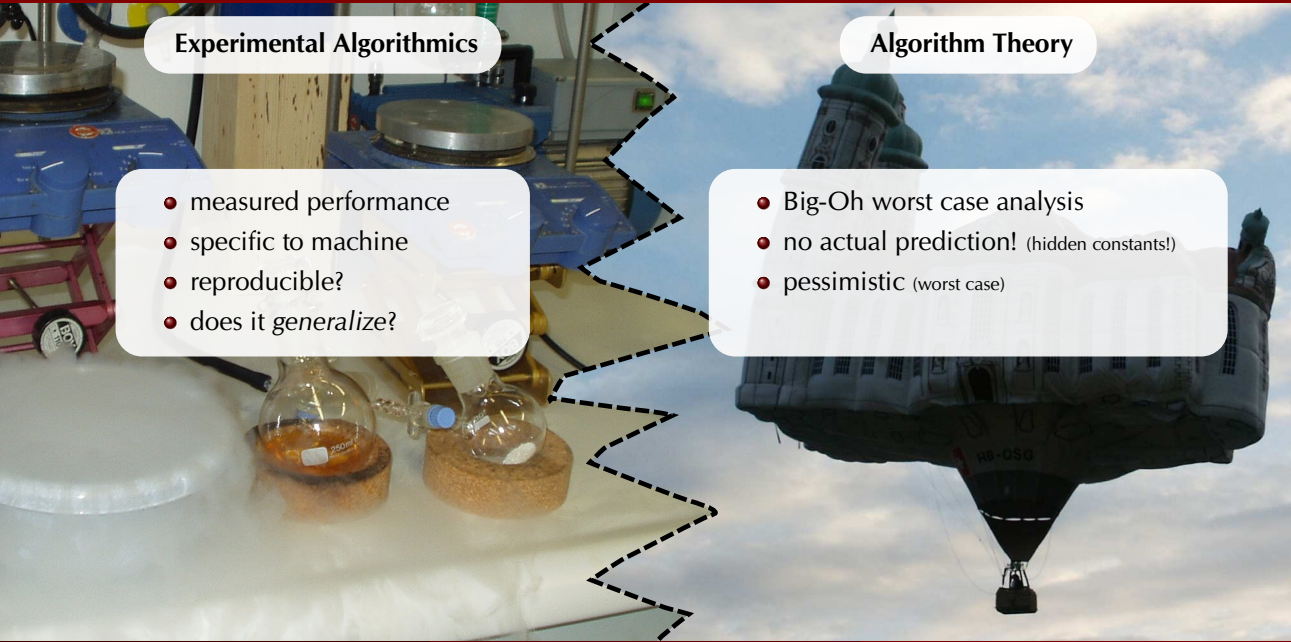

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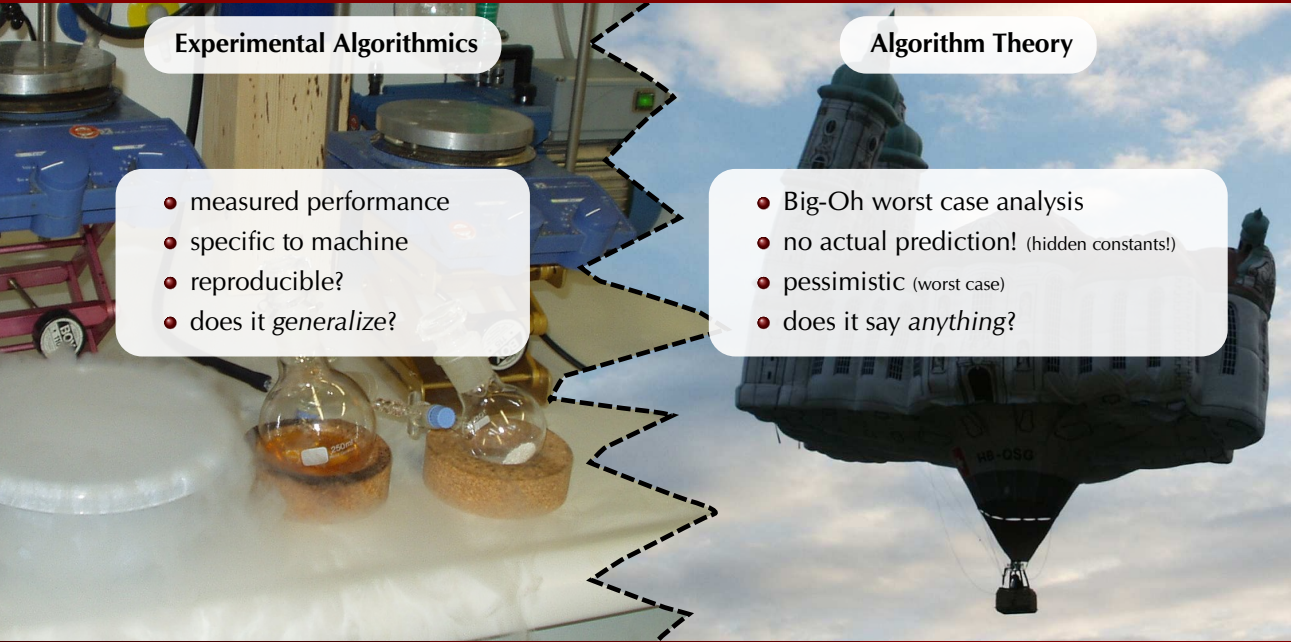

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Platform-independent performance guarantees

- concrete, useful predictions
- concrete, analyzable model
- validated by experiments

Case Study: Sorting in Java Library

Why Java?

- Open-Source reference implementation *OpenJDK*
- Basis for Oracle JRE, Android JRE $\rightsquigarrow$ might contain executed algorithms in existence!
- **within new millenium: all-new sorting methods!**



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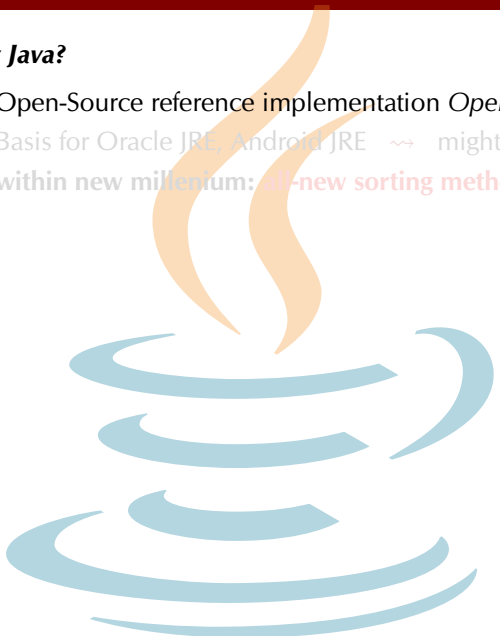

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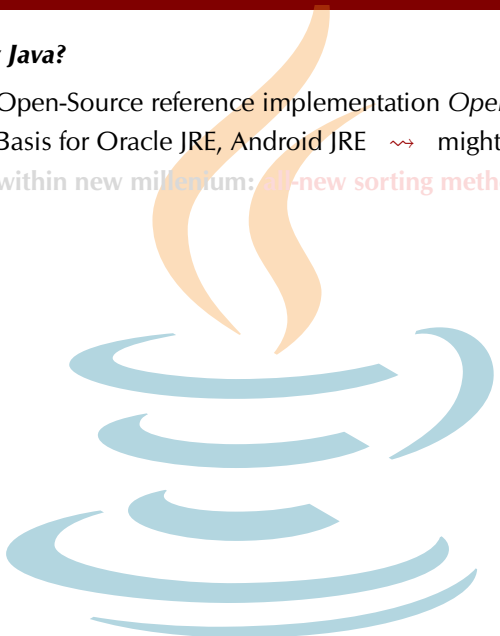

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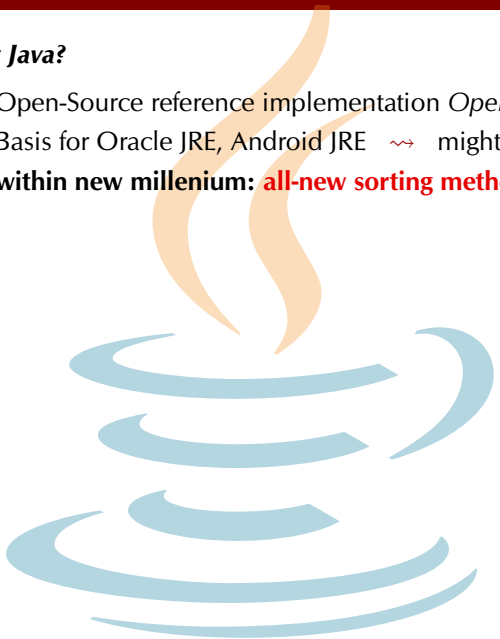

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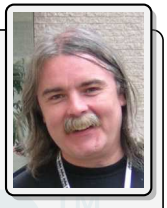

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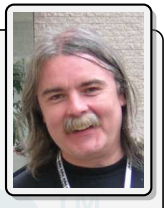

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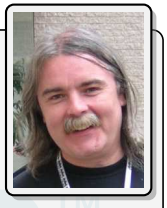

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$\rightsquigarrow$ **Focus:** Benchmarks & Unit-Tests not on proofs or analysis

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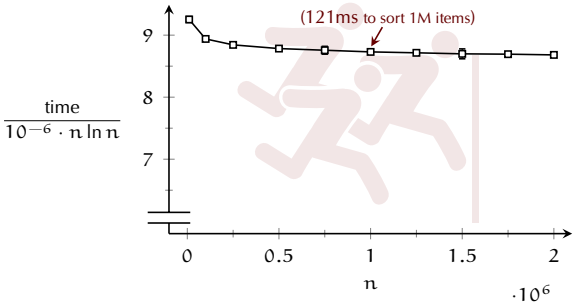

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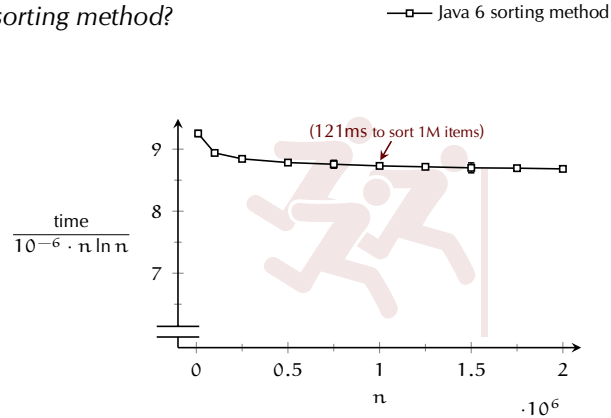

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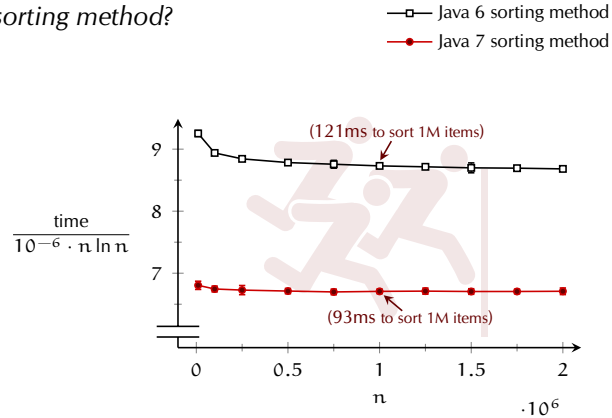

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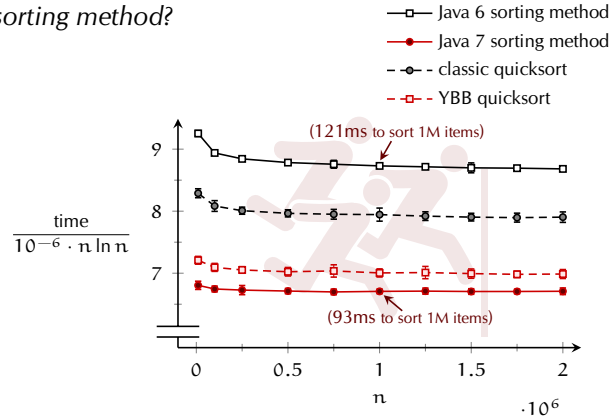

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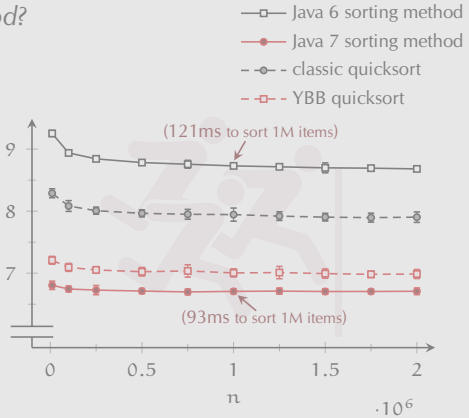

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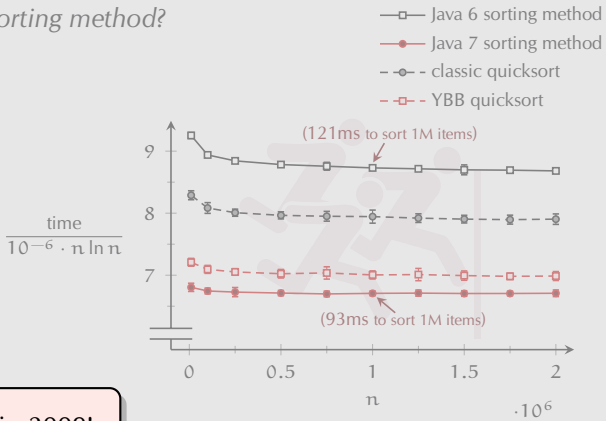

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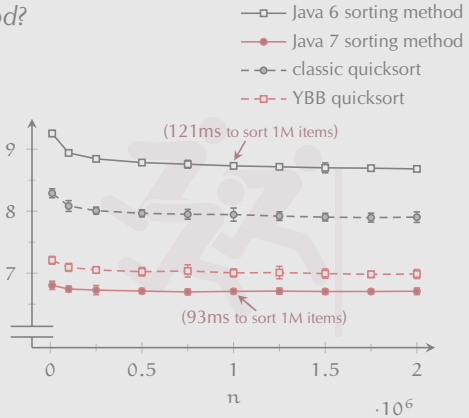

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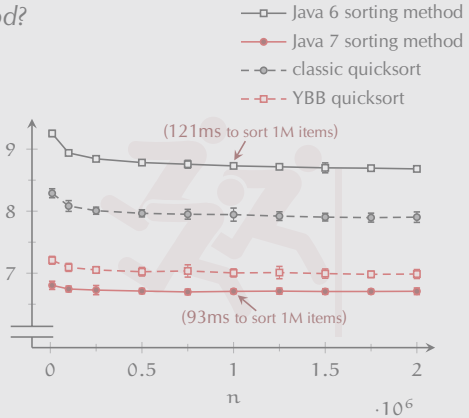

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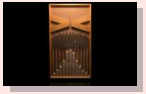



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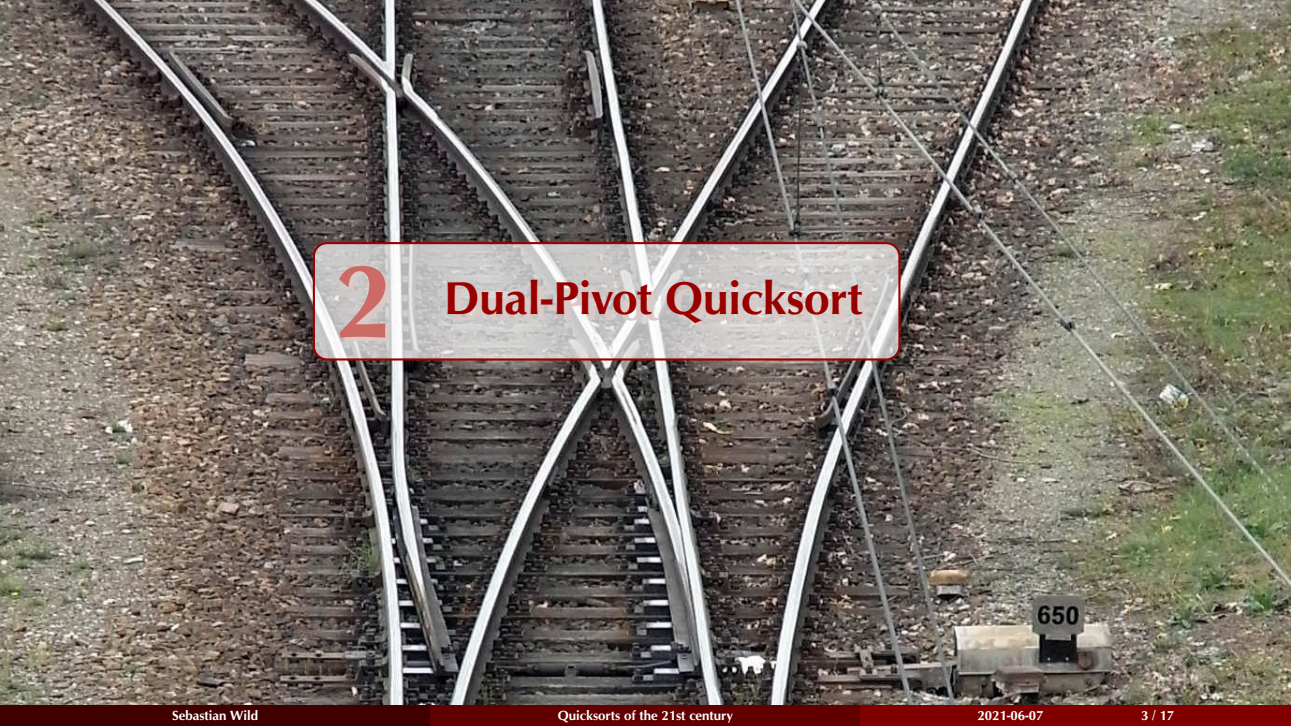

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Dual-Pivot Quicksort: Conceptual View

Standard Quicksort

1. Choose a pivot element P.
2. For each element x, determine its **class** by comparing x to P.
   - **small** if $x < P$
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3. Rearrange elements according to classes:

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Standard Quicksort

1. Choose a pivot
2. For each element x, determine its **class** by comparing x to P.
   - **small** if $x < P$
   - **large** if $P < x$

assuming distinct elements in this talk

3. Rearrange elements according to classes:



4. Recurse on left & right subarray.

How to do ❸ efficiently and in place? ⇝ *YBB Partitioning*

Dual-Pivot Quicksort

2. For each element x, determine its **class** by comparing x to **pivots** P_1 and P_2
   - **small** for $x < P_1$
   - **medium** for $P_1 < x < P_2$
   - **large** for $P_2 < x$

Dijkstra's Dutch National Flag Problem

3. Arrange elements according to classes:



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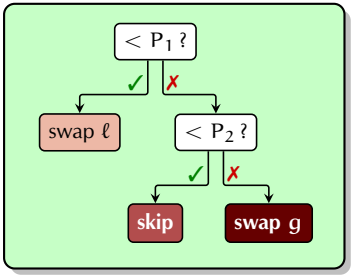
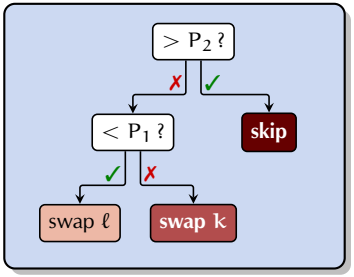
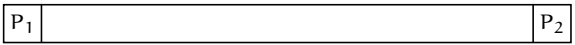
YBB Partitioning Example







Invariant: 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YBBQUICKSORT(A, left, right)
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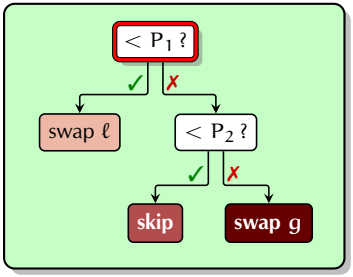
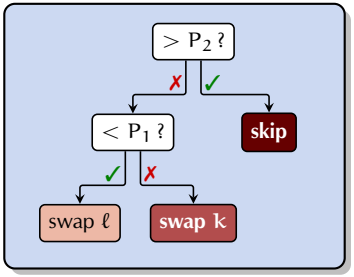
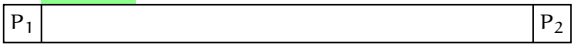






Invariant: 

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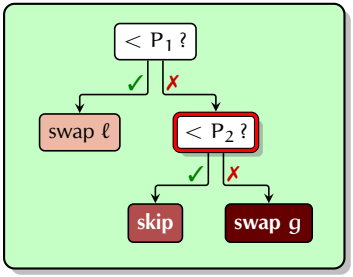
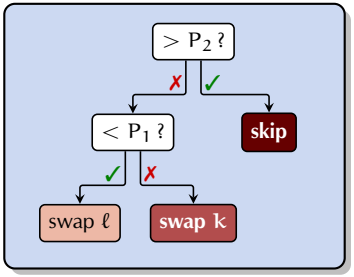
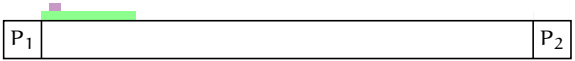






Invariant: 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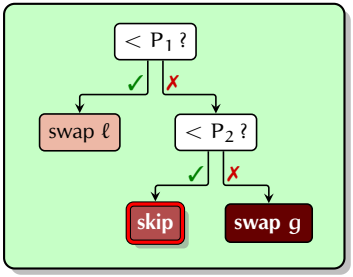
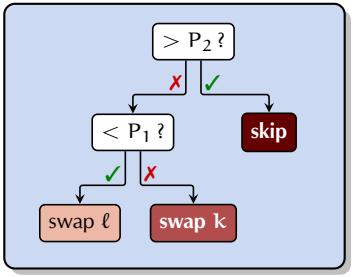
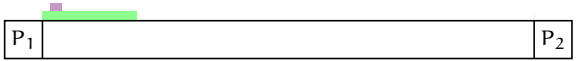






Invariant:



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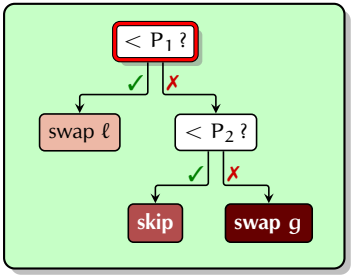

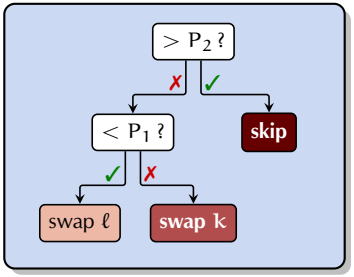



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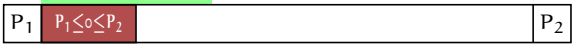

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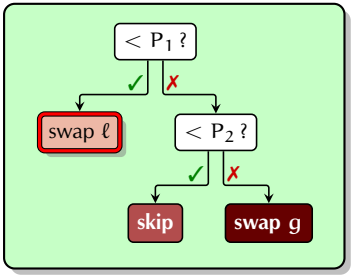

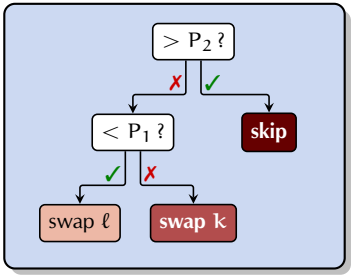



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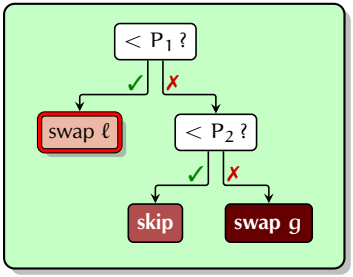

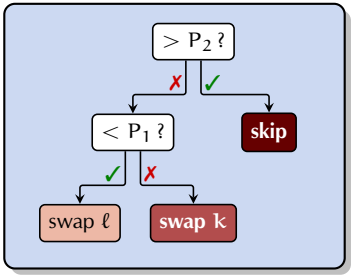

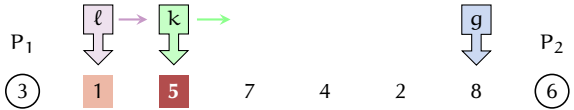

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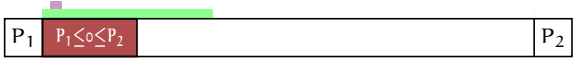

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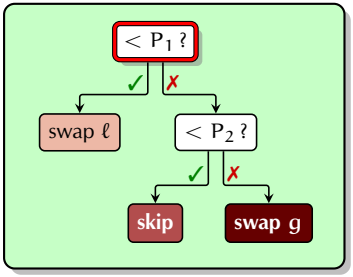

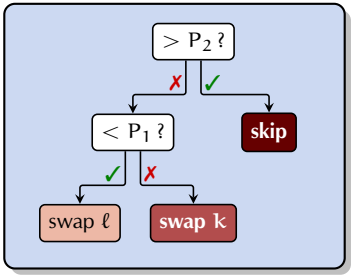

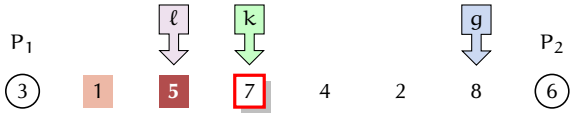

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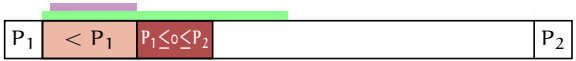


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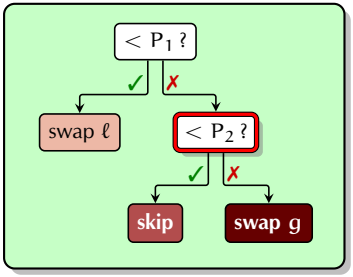

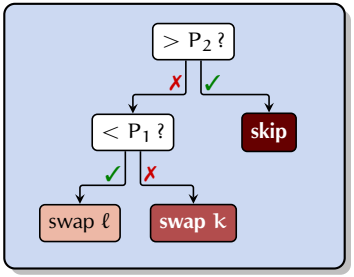

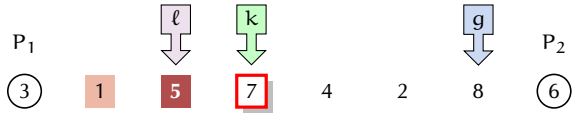

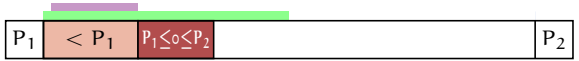

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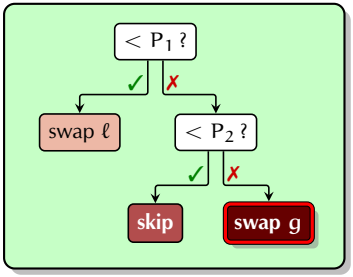
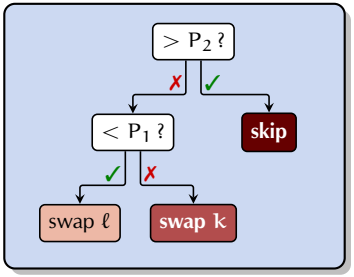
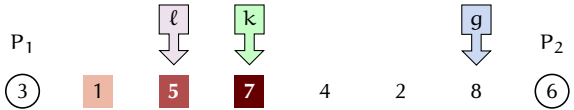
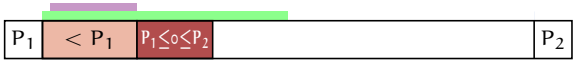




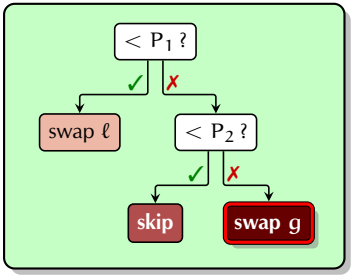


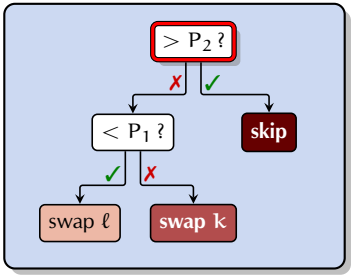
Invariant:

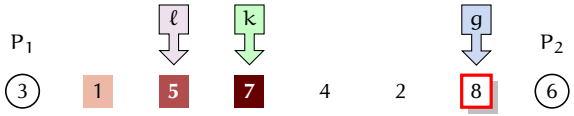



```
YBBQUICKSORT(A, left, right)
1  if right ⩽ left then return
2  P1 := A[left];    P2 := A[right]
3  if P1 > P2 then Swap P1 and P2 end if
4  ℓ := left + 1;    g := right − 1;    k := ℓ
5  while k ⩽ g
6      if A[k] < P1
7          Swap A[k] and A[ℓ] ;  ℓ := ℓ + 1
8      else if A[k] ⩾ P2
9          while A[g] > P2 and k < g
10             g := g − 1
11         end while
12         Swap A[k] and A[g] ;  g := g − 1
13         if A[k] < P1
14             Swap A[k] and A[ℓ] ;  ℓ := ℓ + 1
15         end if
16     end if
17     k := k + 1
18 end while
19 ℓ := ℓ − 1;    g := g + 1
20 Swap A[left] and A[ℓ] ; Swap A[right] and A[g]
21 YBBQUICKSORT(A, left , ℓ − 1)
22 YBBQUICKSORT(A, ℓ + 1, g − 1)
23 YBBQUICKSORT(A, g + 1, right )
```

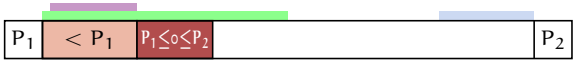
YBB Partitioning Example







Invariant:




```
YBBQUICKSORT(A, left, right)
1  if right ⩽ left then return
2  P1 := A[left];    P2 := A[right]
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7          Swap A[k] and A[ℓ] ;  ℓ := ℓ + 1
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11         end while
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14             Swap A[k] and A[ℓ] ;  ℓ := ℓ + 1
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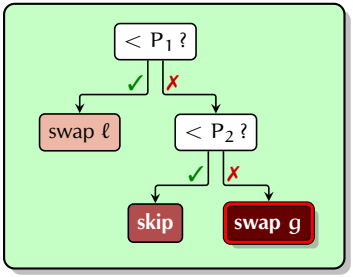
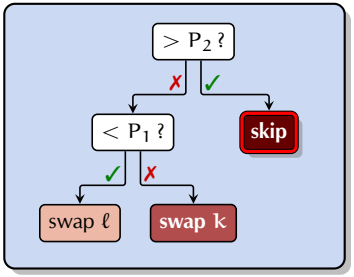
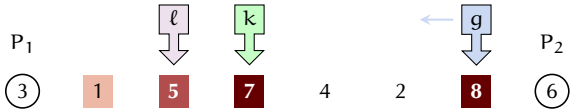
```

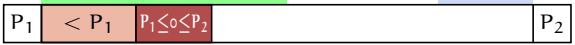
YBB Partitioning Example







Invariant:



YBBQUICKSORT(A, *left*, *right*)

```
1  if right ⩽ left then return
2  P1 := A[left];    P2 := A[right]
3  if P1 > P2 then Swap P1 and P2 end if
4  ℓ := left + 1;    g := right − 1;    k := ℓ
5  while k ⩽ g
6      if A[k] < P1
7          Swap A[k] and A[ℓ] ;  ℓ := ℓ + 1
8      else if A[k] ⩾ P2
9          while A[g] > P2 and k < g
10             g := g − 1
11         end while
12         Swap A[k] and A[g] ;  g := g − 1
13         if A[k] < P1
14             Swap A[k] and A[ℓ] ;  ℓ := ℓ + 1
15         end if
16     end if
17     k := k + 1
18 end while
19 ℓ := ℓ − 1;    g := g + 1
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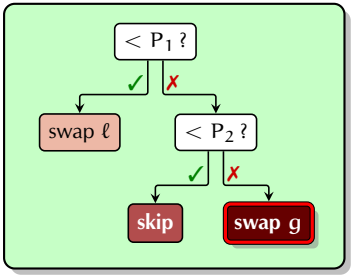
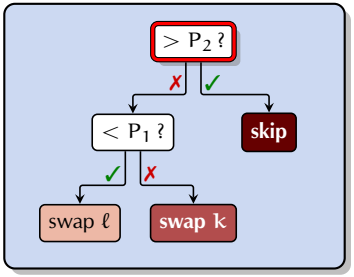
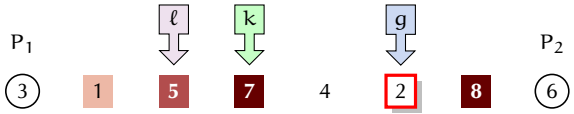
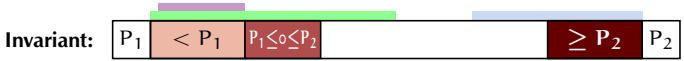
YBB Partitioning Example







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YBBQUICKSORT(A, left, right)
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6      if A[k] < P1
7          Swap A[k] and A[ℓ] ;  ℓ := ℓ + 1
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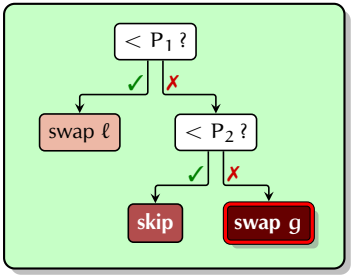
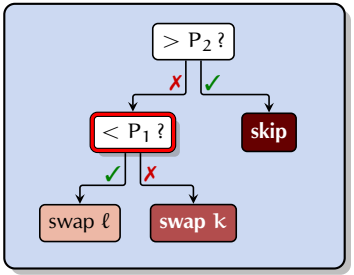
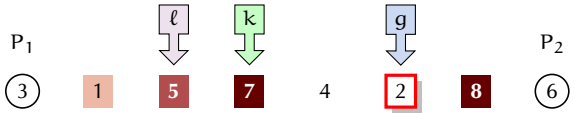
```

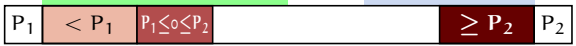
YBB Partitioning Example







Invariant:



YBBQUICKSORT(A, *left*, *right*)

```
1  if right ⩽ left then return
2  P1 := A[left];    P2 := A[right]
3  if P1 > P2 then Swap P1 and P2 end if
4  ℓ := left + 1;    g := right − 1;    k := ℓ
5  while k ⩽ g
6      if A[k] < P1
7          Swap A[k] and A[ℓ] ;  ℓ := ℓ + 1
8      else if A[k] ⩾ P2
9          while A[g] > P2 and k < g
10             g := g − 1
11         end while
12         Swap A[k] and A[g] ;  g := g − 1
13     → if A[k] < P1
14             Swap A[k] and A[ℓ] ;  ℓ := ℓ + 1
15         end if
16     end if
17     k := k + 1
18 end while
19 ℓ := ℓ − 1;    g := g + 1
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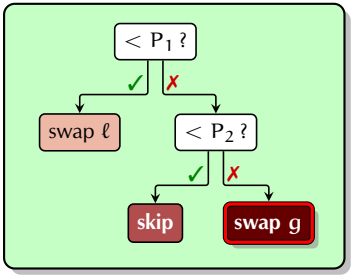
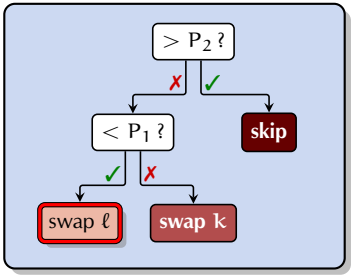
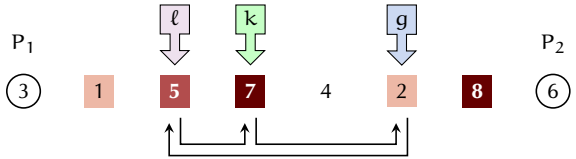
YBB Partitioning Example





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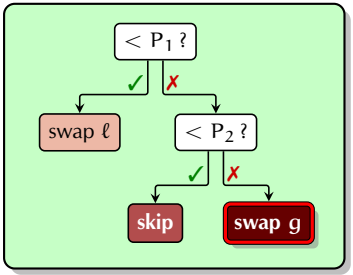
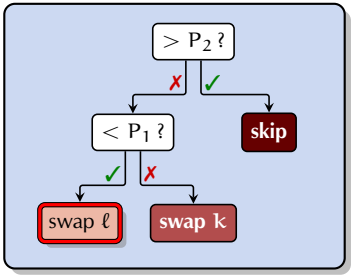
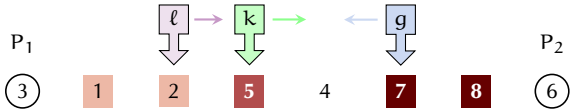
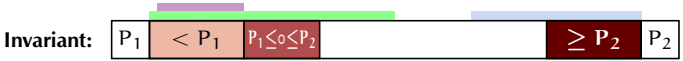
YBB Partitioning Example







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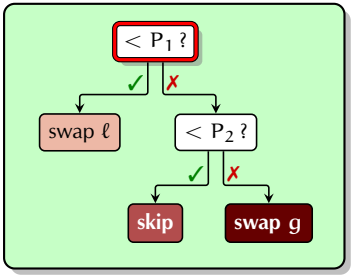
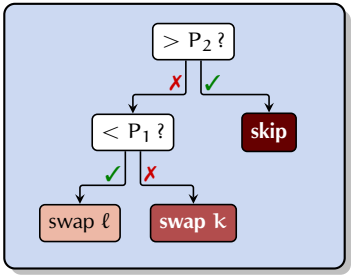
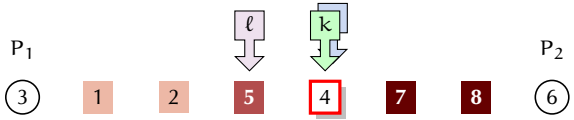
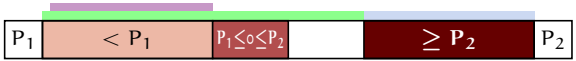
YBB Partitioning Example







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YBBQUICKSORT(A, left, right)
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2  P1 := A[left];    P2 := A[right]
3  if P1 > P2 then Swap P1 and P2 end if
4  ℓ := left + 1;    g := right − 1;    k := ℓ
5  while k ⩽ g
6      if A[k] < P1
7          Swap A[k] and A[ℓ] ;  ℓ := ℓ + 1
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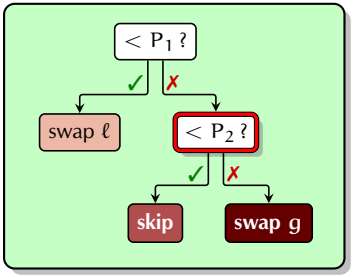
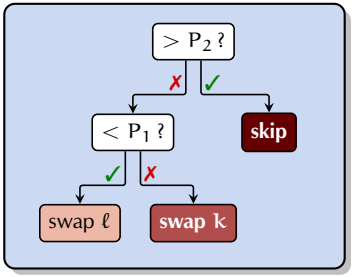
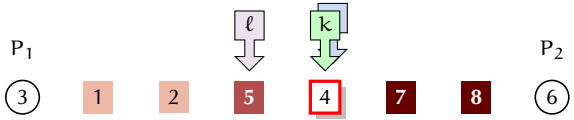
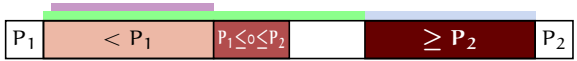
YBB Partitioning Example










```
YBBQUICKSORT(A, left, right)
1  if right < left then return
2  P1 := A[left];    P2 := A[right]
3  if P1 > P2 then Swap P1 and P2 end if
4  ℓ := left + 1;    g := right − 1;    k := ℓ
5  while k ⩽ g
6      if A[k] < P1
7          Swap A[k] and A[ℓ] ;  ℓ := ℓ + 1
8      else if A[k] ⩾ P2
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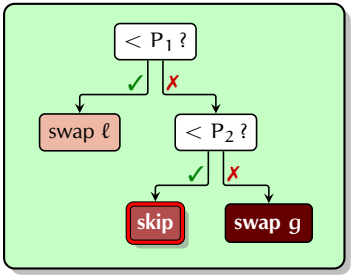
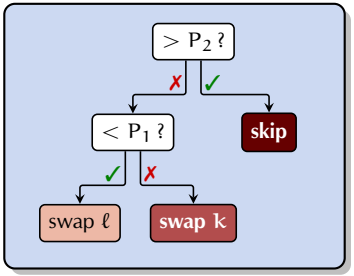
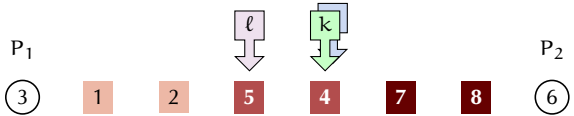
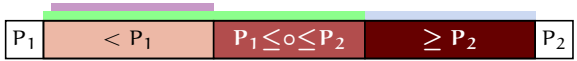
YBB Partitioning Example







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YBBQUICKSORT(A, left, right)
1  if right ⩽ left then return
2  P1 := A[left];    P2 := A[right]
3  if P1 > P2 then Swap P1 and P2 end if
4  ℓ := left + 1;    g := right − 1;    k := ℓ
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7          Swap A[k] and A[ℓ] ;  ℓ := ℓ + 1
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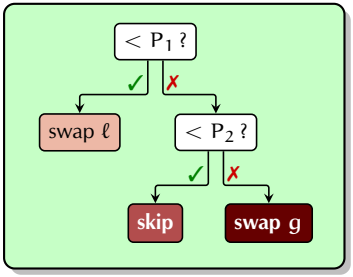

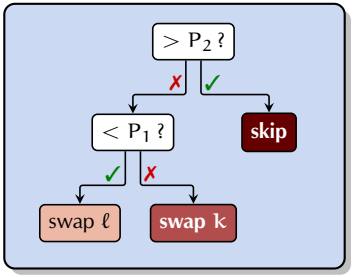



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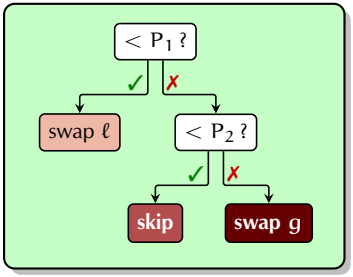

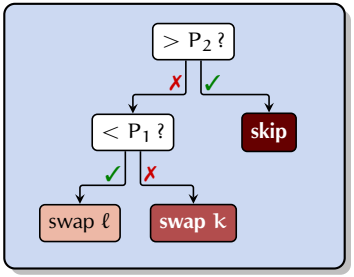



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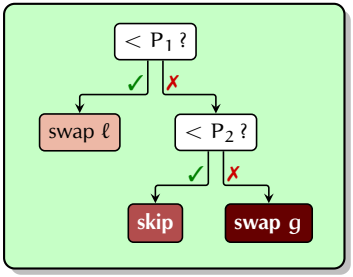

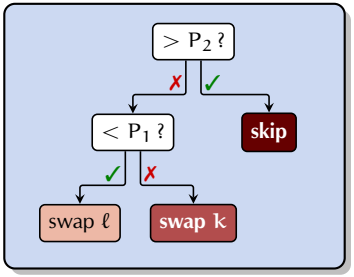



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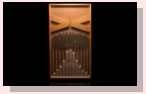
Outline







1 Context

2 Dual-Pivot Quicksort

3 Average-Case Analysis

4 Quicksort + X

5 Conclusion



3 Average-Case Analysis

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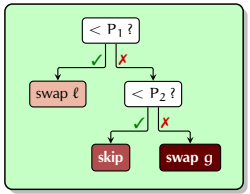

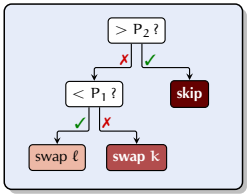

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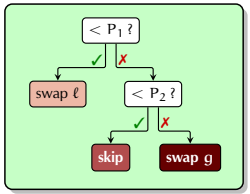

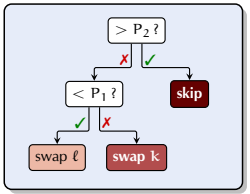

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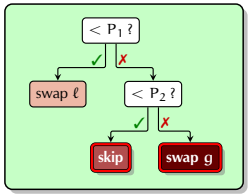

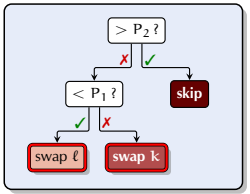

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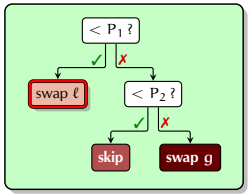

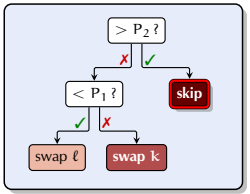

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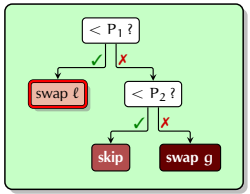

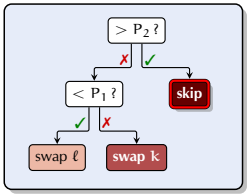

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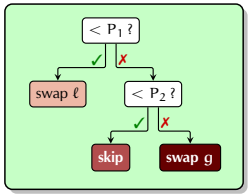

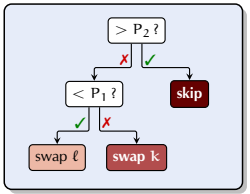

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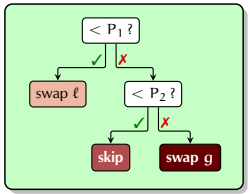

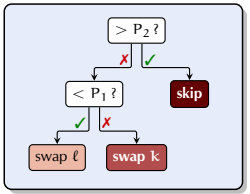

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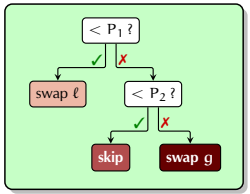

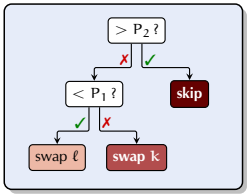

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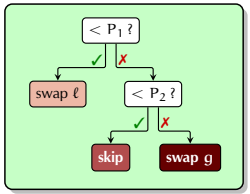

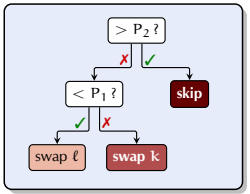

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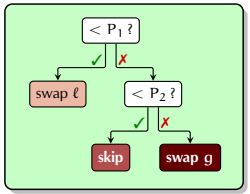

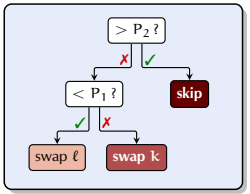

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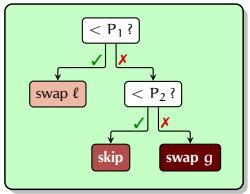

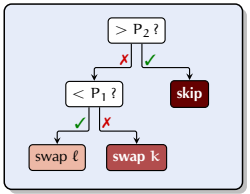

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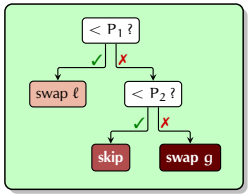

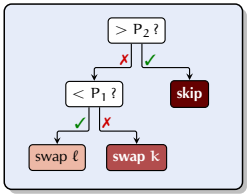

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How many comparisons on average?

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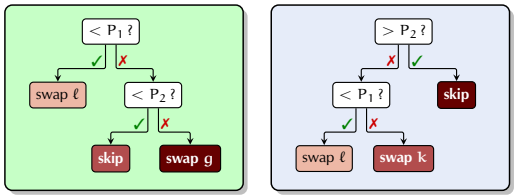

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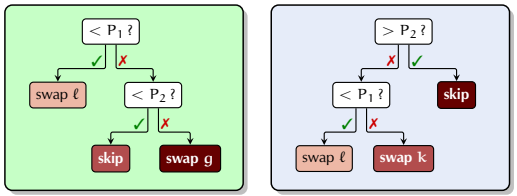

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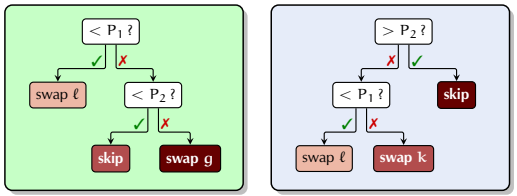

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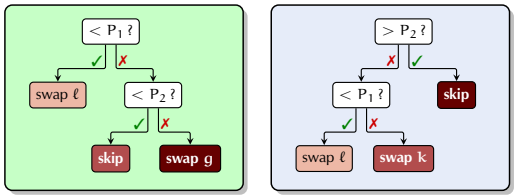

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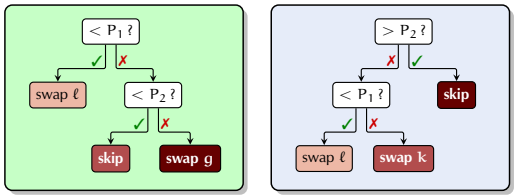

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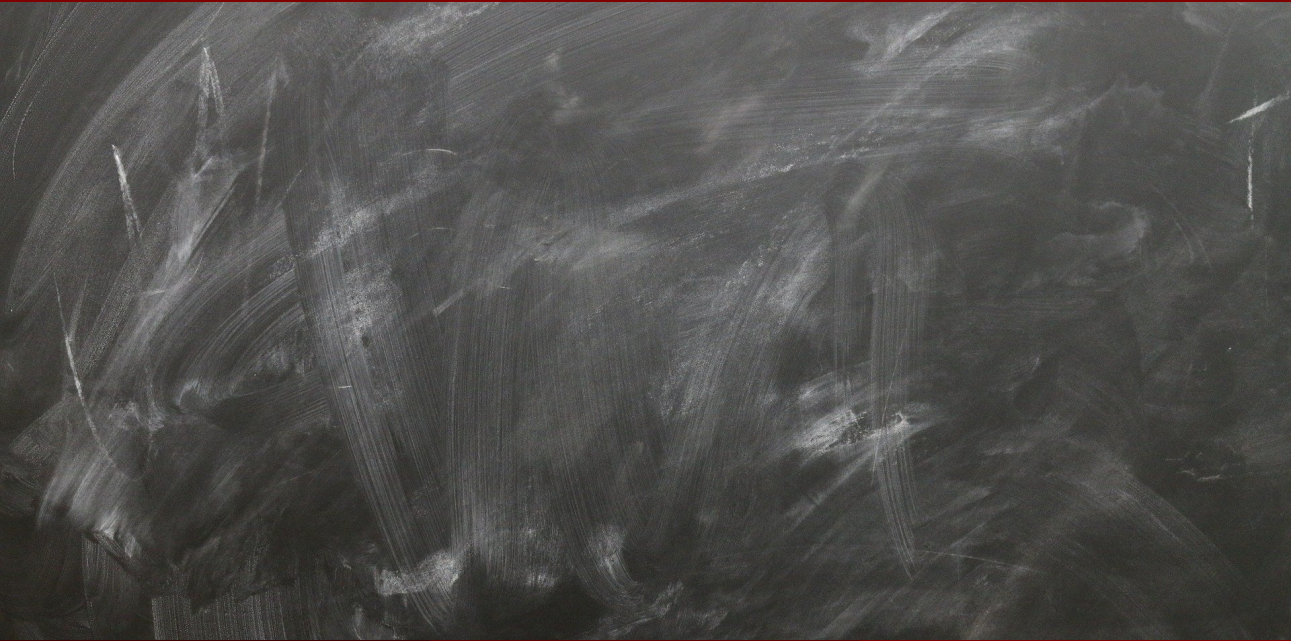
Mathematical Analysis of Algorithms



Mathematical Analysis of Algorithms

Random Model

- n **i.i.d.** elements chosen **uniformly** in $(0,1)$



  - pairwise **distinct** almost surely
  - relative ranking is a **random permutation**
  - $\rightsquigarrow$ equivalent to classic model

- For **dual-pivot** Quicksort with **fixed** pivots $P_1 \leqslant P_2$
  - $\Pr[\ U_i < P_1\ ] = D_1$
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$\vec{D}$ is a uniformly chosen probability distribution!



- Known distribution: $\vec{D} = (D_1, D_2, D_3] \overset{d}{=} \text{Dirichlet}(1,1,1)$.
  $(D_1, D_2]$ uniform in 2D simplex $\{x, y \in (0,1) : x + y < 1\}$



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Comparisons in YBB Partitioning

- $t(n)$: $\mathbb{E}[\#\text{cmps}]$ in **first partitioning step**



- $I_{\geqslant P_2}$: number of small elements
- $I_{\geqslant P_2}$: number of medium elements
- $I_{\geqslant P_2}$: number of large elements

To be precise, we have **conditional** on $\vec{I}$:

$I_{\geqslant S} \triangleq \text{HypG}(n-2, \frac{I_1}{n}, I_3)$

$\rightsquigarrow \mathbb{E}[I_{\geqslant S} \mid \vec{I}] = \frac{I_1}{n-2}$

Draw positions of large elements

$I_{\geqslant S} = 1$

$\rightsquigarrow t(n) = \mathbb{E}[$

$\vec{I} = (I_1, I_2, I_3) \triangleq \text{Mult}(n-2, \vec{D})$

$\rightsquigarrow \vec{I} = \vec{D}n \pm O(\sqrt{n})$ whp

Fraction of elements in $\mathbb{X}$



$\rightsquigarrow \mathbb{E}[I_{\geqslant S}] \sim \mathbb{E}[D_1 \cdot D_3] \cdot n,$ $\mathbb{E}[\#\mathbb{X}] \sim \mathbb{E}[D_1 \cdot (D_1 + D_2)] \cdot n$

Mathematical Analysis of Algorithms

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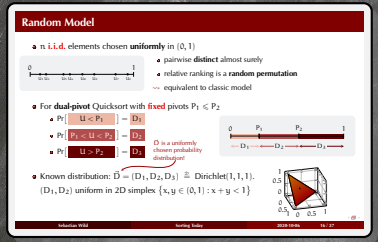

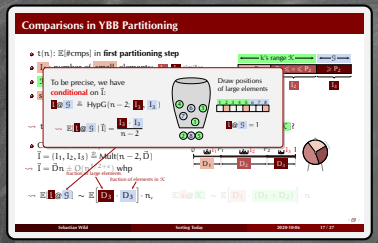

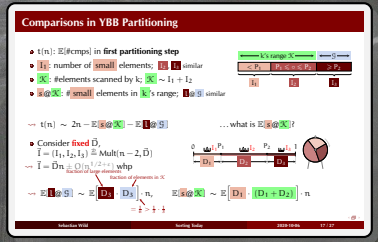

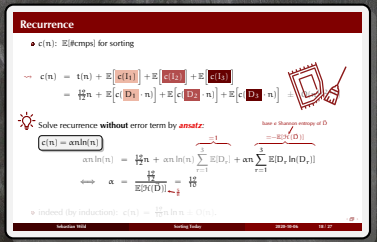

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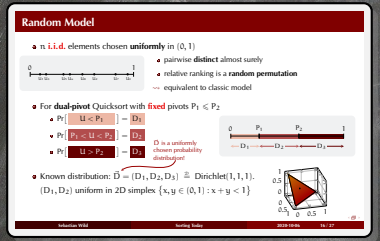

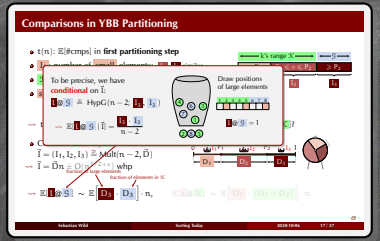

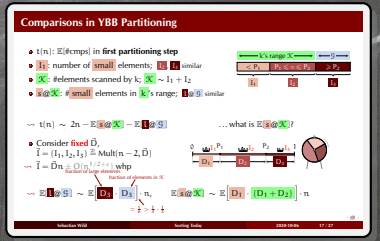

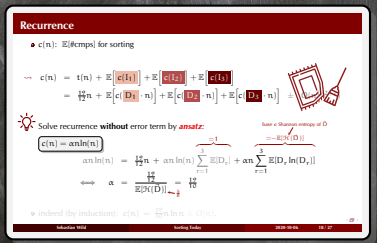

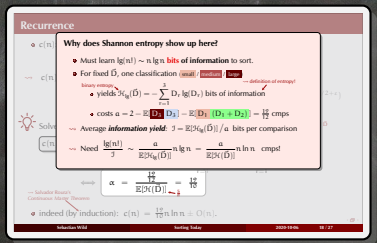

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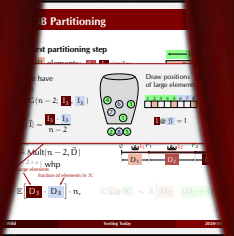
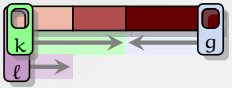


Time for a Competition



Time for a Competition

	Dual-Pivot Quicksort	Standard Quicksort	Relative Difference	Reference
				

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	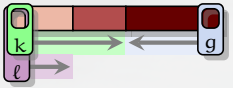	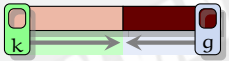		
comparisons	1.9	2	−5%	ESA 2012

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	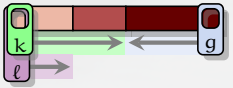			
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	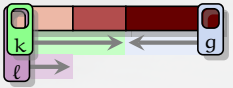	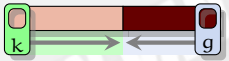		
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	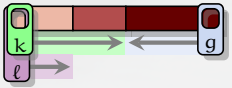	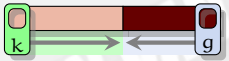		
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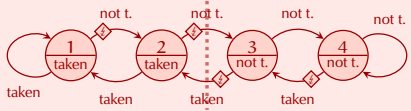

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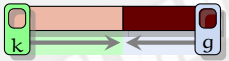		
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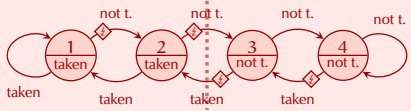

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		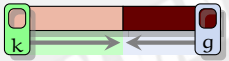		
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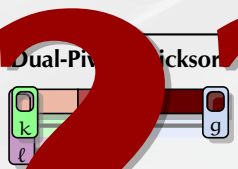

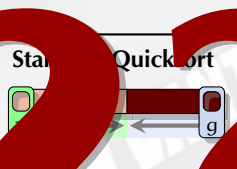

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What happened?

What do comparisons, write accesses, Bytecodes and branch misses have in common?

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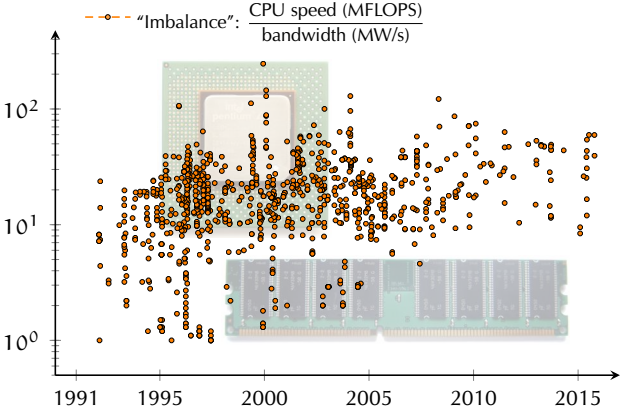

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STREAM Benchmark data with linear regression
www.cs.virginia.edu/stream/by_date/Balance.html

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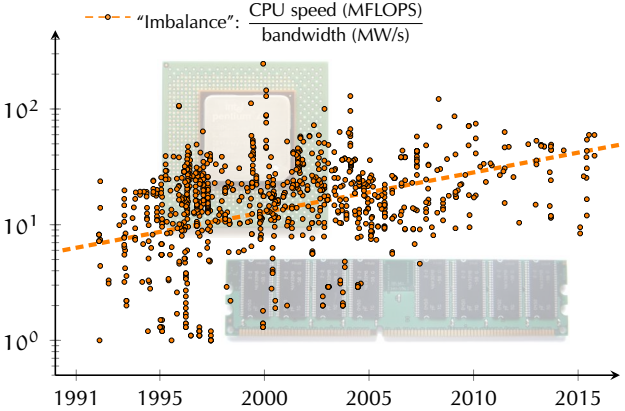

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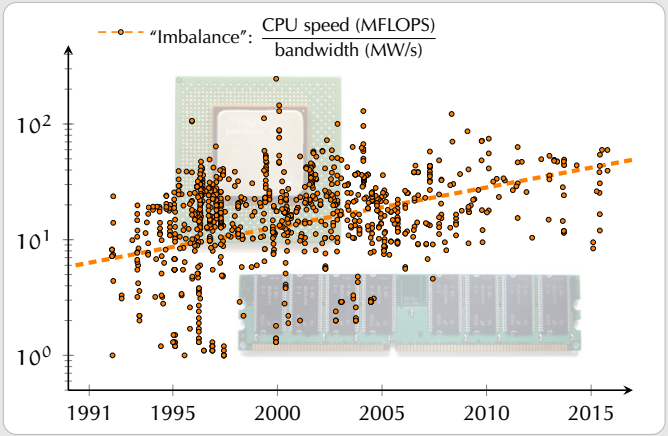
⇝ they happen **inside** the **CPU** …but we need to get the data there, as well!



Average annual growth rates:

CPU speed:	46%
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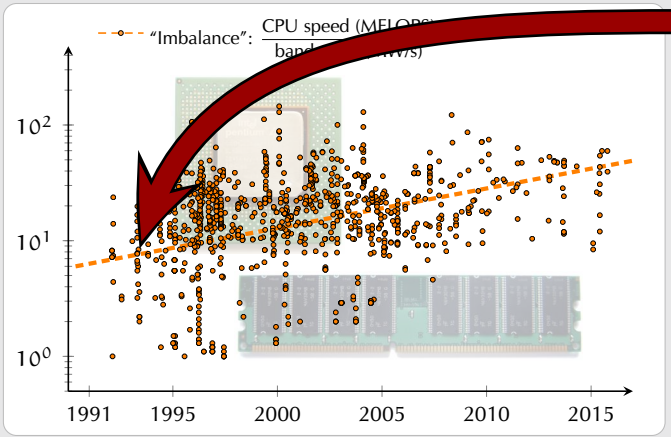

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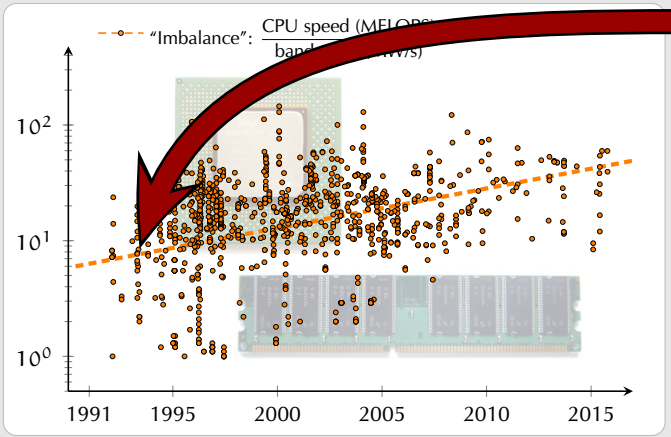

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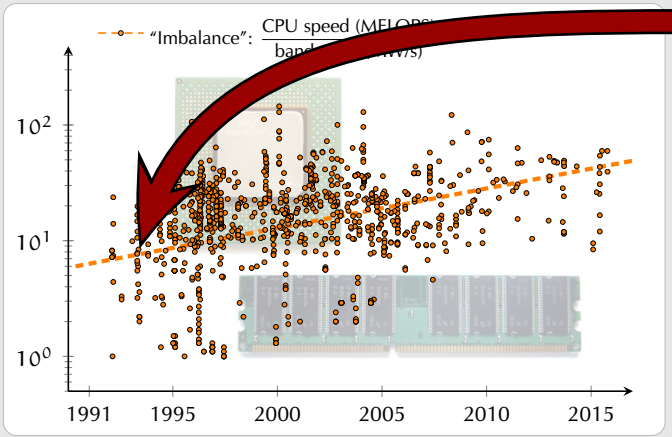

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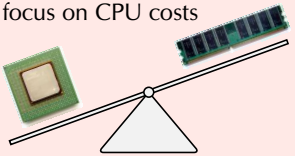

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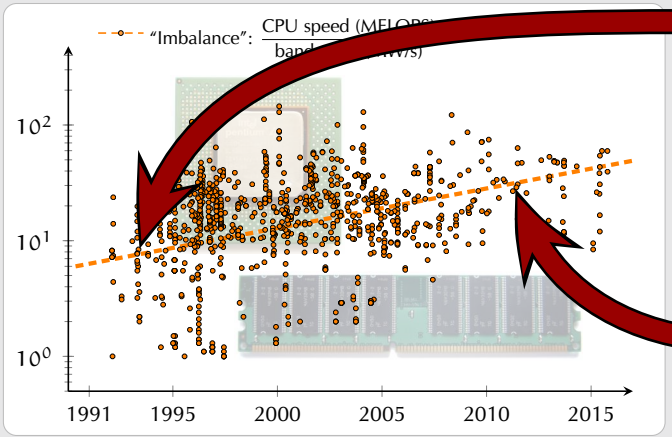

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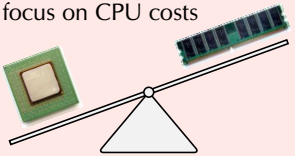

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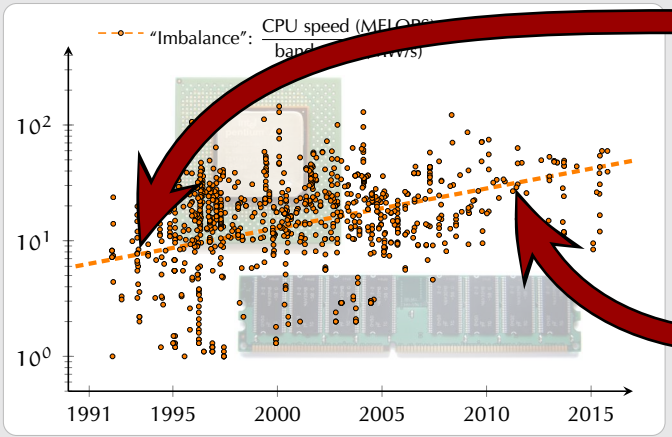

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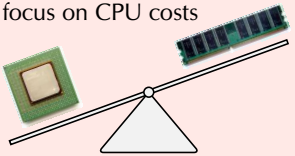

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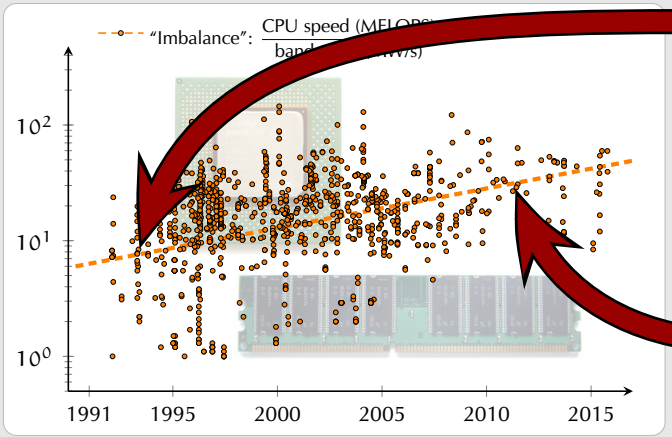

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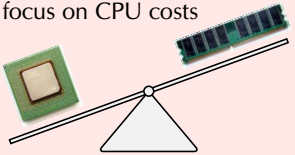

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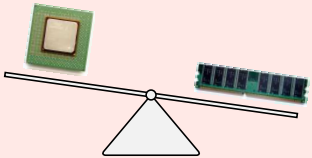
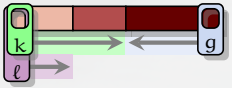
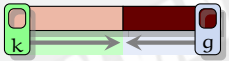


Time for a Competition

	Dual-Pivot Quicksort	Standard Quicksort	Relative Difference	Reference
				
comparisons	1.9	2	−5%	ESA 2012
write accesses	$\frac{11}{10} \approx 1.1$	$\frac{2}{3} \approx 0.\overline{6}$	+65%	ESA 2012
Java Bytecode instr.	23.8	18	+32.2%	TALG 2015
branch mispredictions (2-bit saturating counter)	$\frac{31\pi}{40} - \frac{37}{20} \approx 0.5847$	$\frac{\pi}{2} - 1 \approx 0.5708$	+2.4%	ANALCO 2015

$\cdot n \ln n + O(n)$ *average case results*

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	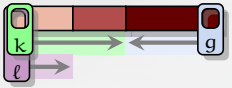	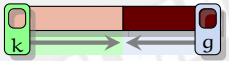		
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	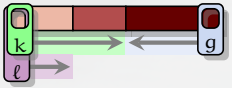	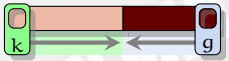		
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Time for a Competition

Summary:

1. Only convincing explanation for running times: **memory transfers**
2. Why was dual-pivot Quicksort not used earlier? *Because it was **not faster** then!*

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Beyond Dual-Pivot: Multiway Partitioning

Can we do even better than YBB Quicksort?

Setup: *Generic s-way one-pass partitioning*

- **s-way partitioning:** split input into $s \geqslant 2$ parts
- **one pass** over input
- m segments grow from left, $s - m$ from right

- classify elements using BSTs: λ_k and λ_g
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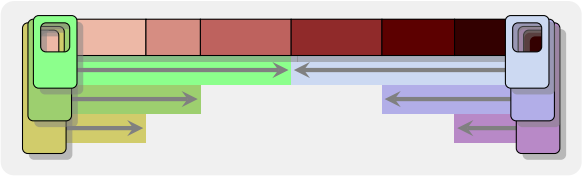

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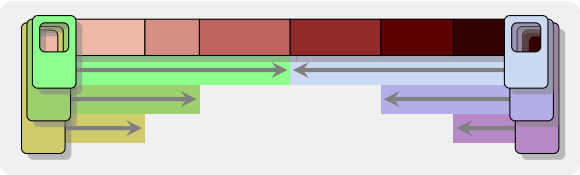

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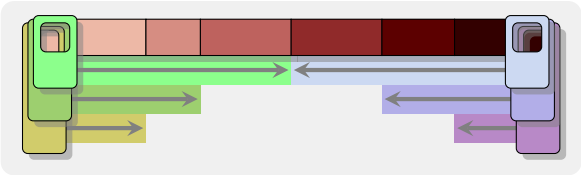

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Known Quicksort variants

Method	s	m	λ_k	λ_g
Classic	2	1		
Lomuto	2	2		
YBB	3	2		
Waterloo	4	2		

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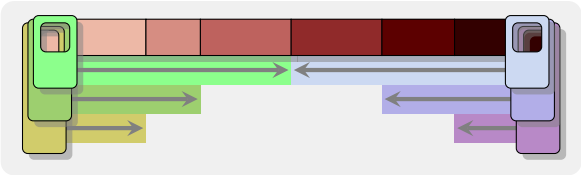

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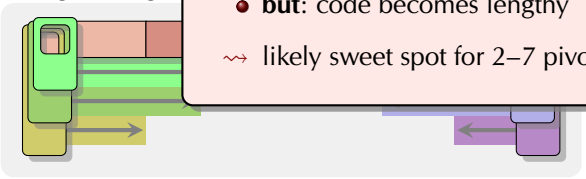

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Findings for dual-pivot Quicksort are typical:

- **comparisons**: negligible improvements
- **scanned elements**: **significant** improvements
- **but**: code becomes lengthy

⇝ likely sweet spot for 2–7 pivots

…n Quicksort variants

	s	m	λ_k	λ_g
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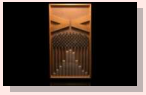
Outline



1 Context



2 Dual-Pivot Quicksort



3 Average-Case Analysis



4 Quicksort + X



5 Conclusion



4 Quicksort + X

What's Wrong with Quicksort?

think: `ints`

- When sorting **simple** objects
  1. **quicksort hard to beat**!
  2. memory transfers a substantial fraction of cost

- What if key **comparisons** are **expensive**?

  ⇝ Quicksort is a factor > 1 off from optimal

  - Heapsort? ↯ cache misses
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Can quicksort help?

What is QuickXsort?

some sorting method

$$\text{Quick}\mathbf{X}\text{sort} = \text{Quicksort} + \mathbf{X}$$

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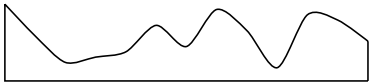
QuickXsort = Quicksort + X

QuickXsort(A[l..r])

1. Choose pivot P.
2. i = Partition(A[l..r], P)
   // subproblems: A[l..i − 1] and A[i + 1..r]
3. Sort one subproblem with X.
4. Sort the other subproblem with QuickXsort.



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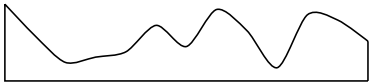

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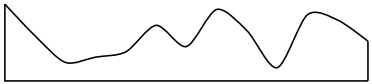

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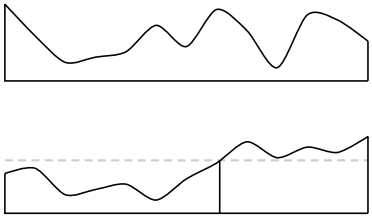

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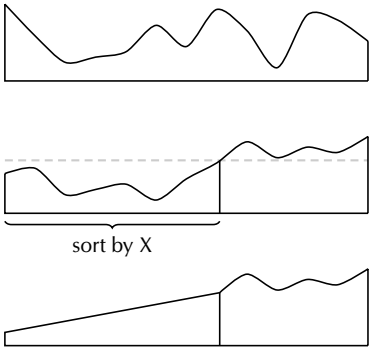

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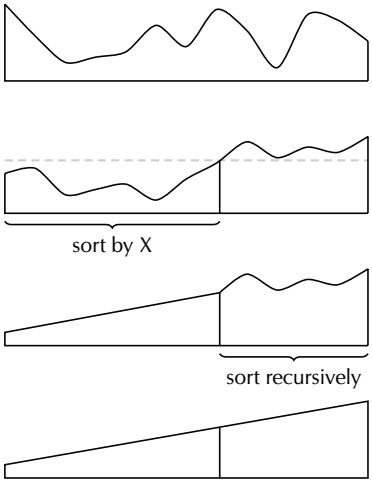

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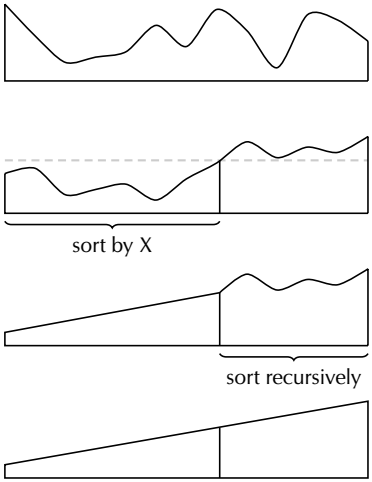

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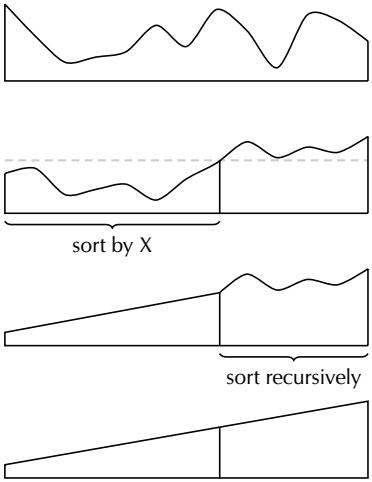

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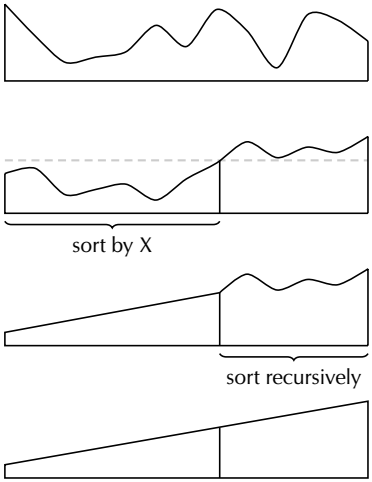

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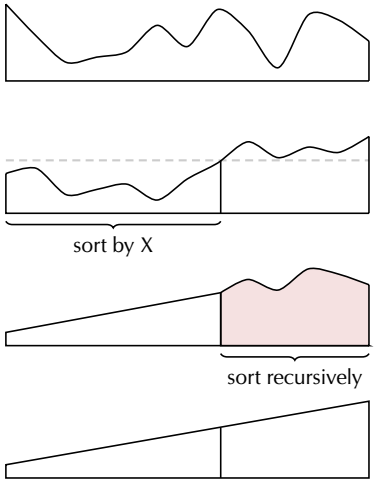

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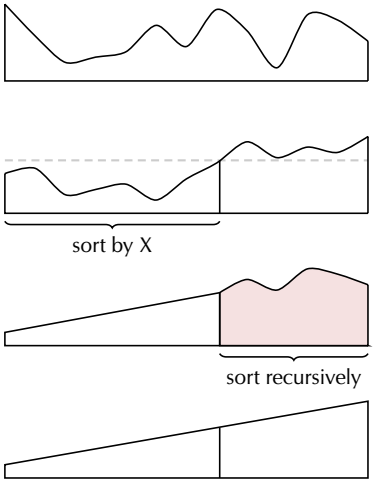

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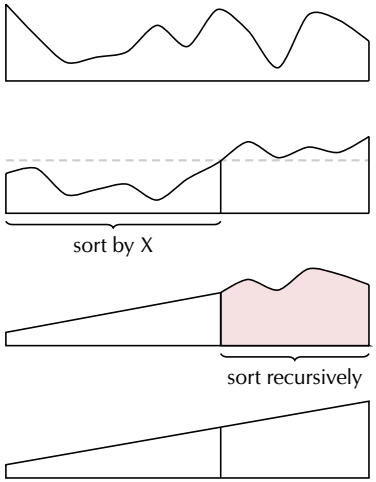

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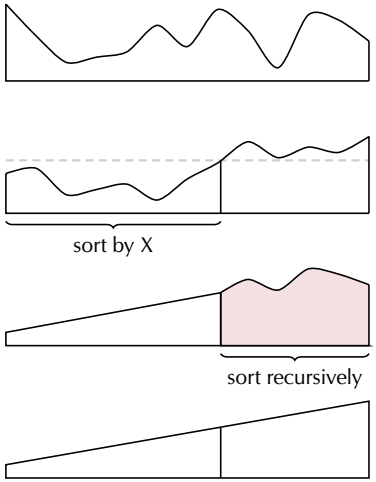

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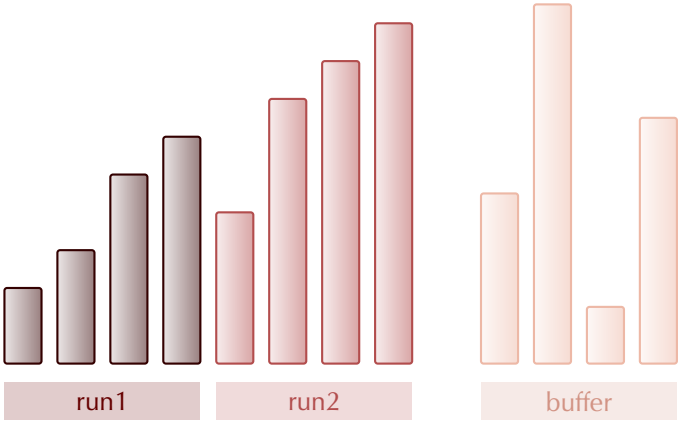

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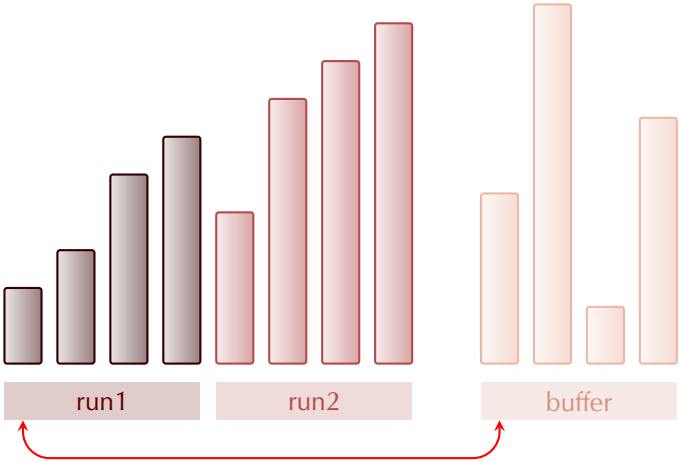

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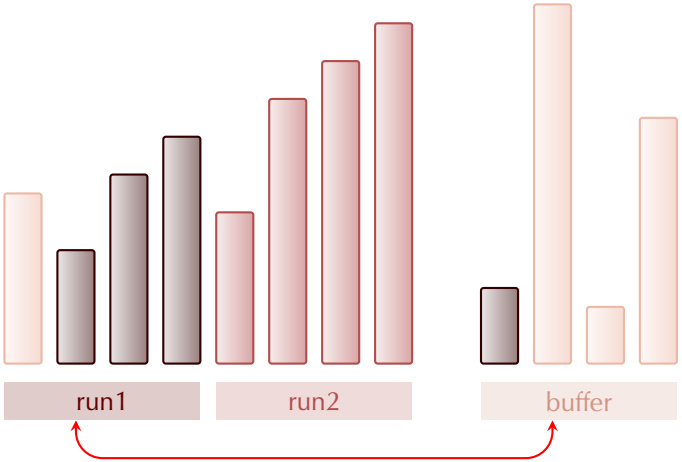

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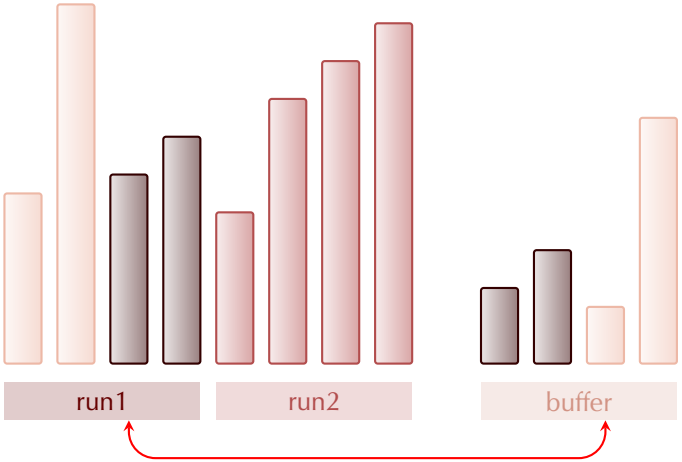

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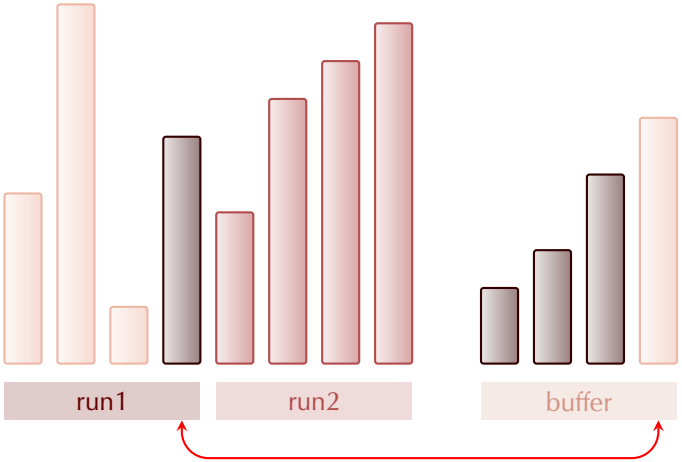

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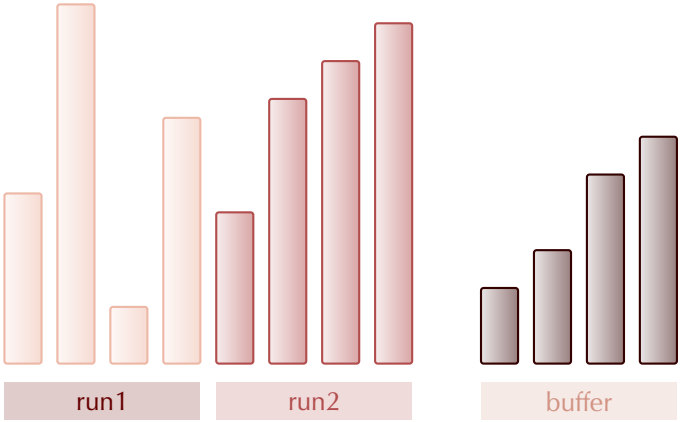

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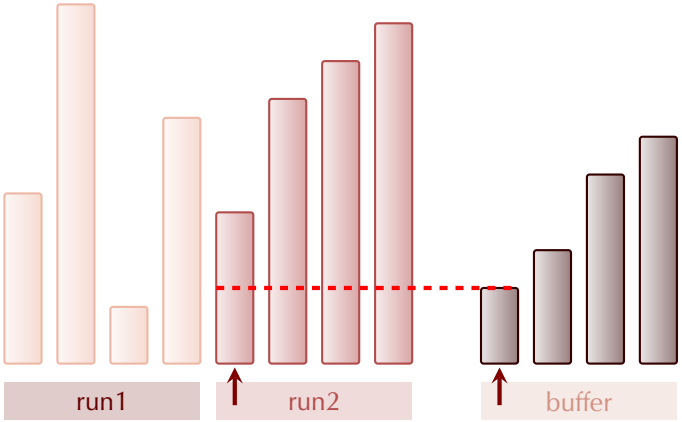

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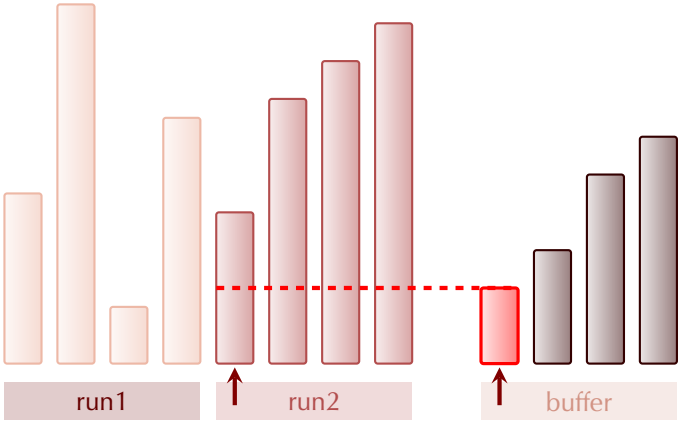

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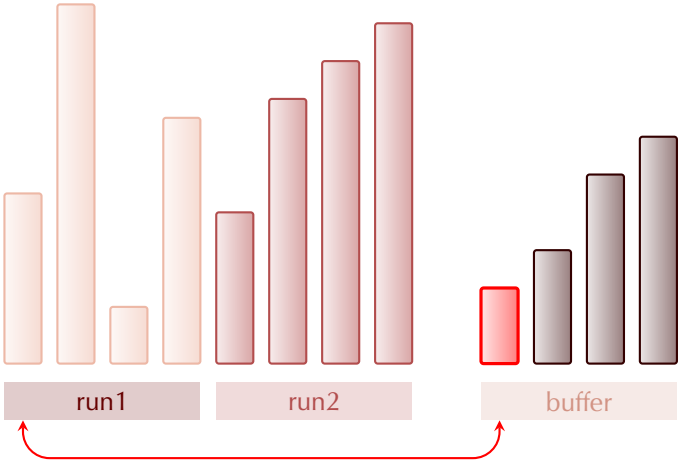

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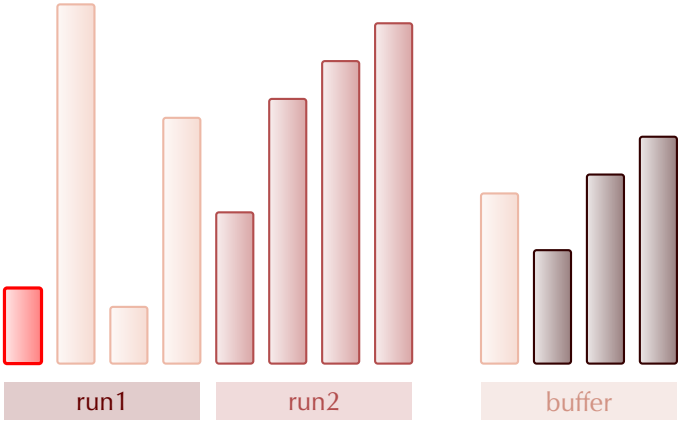

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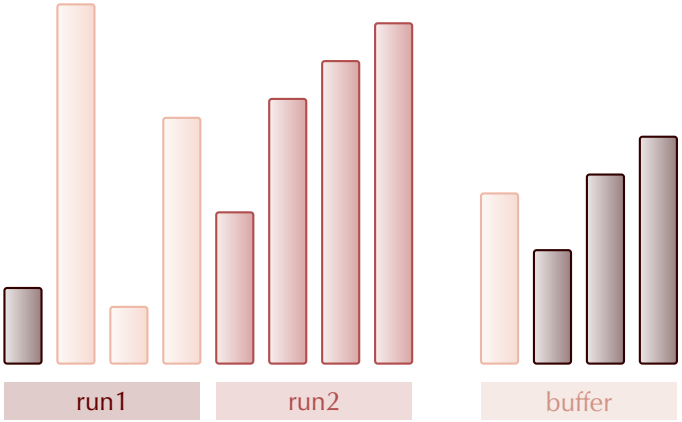

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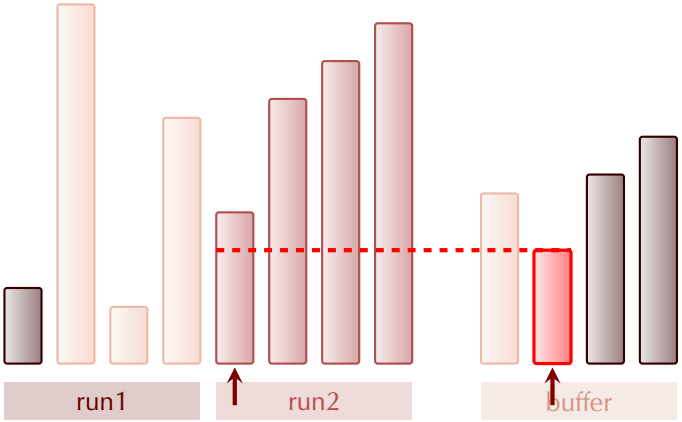

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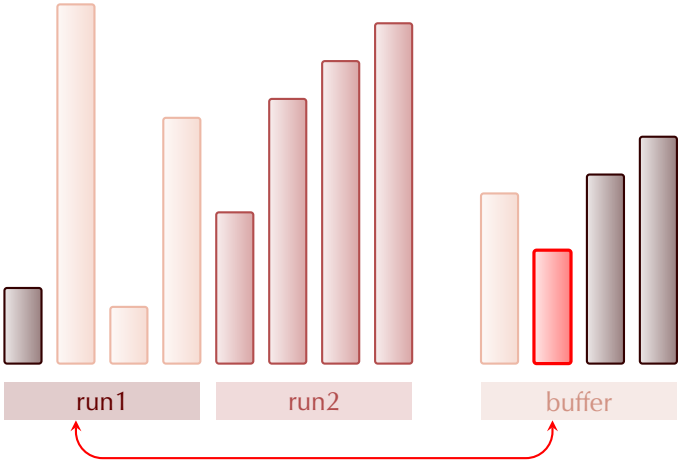

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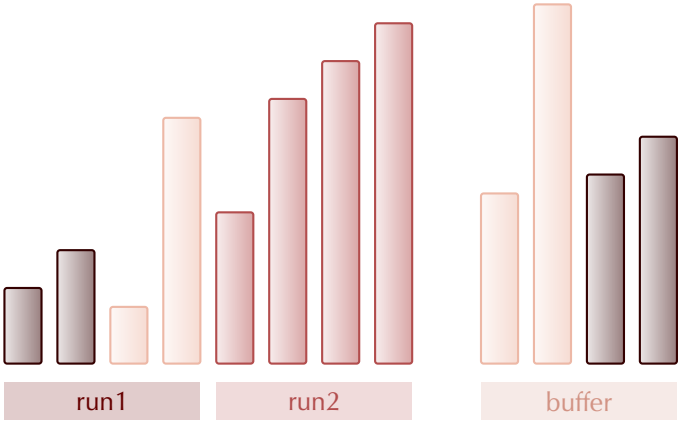

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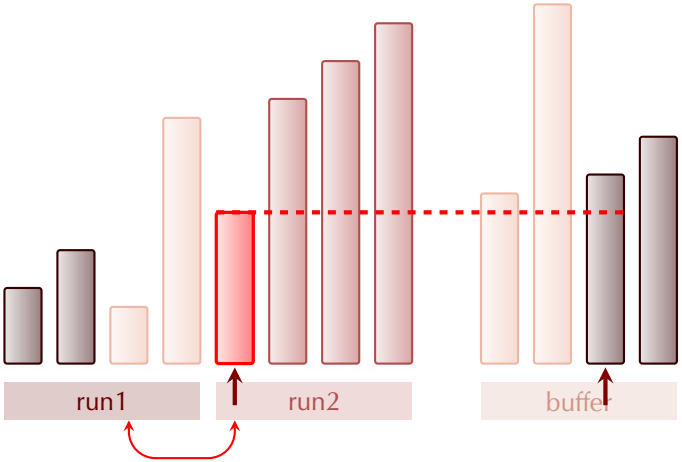

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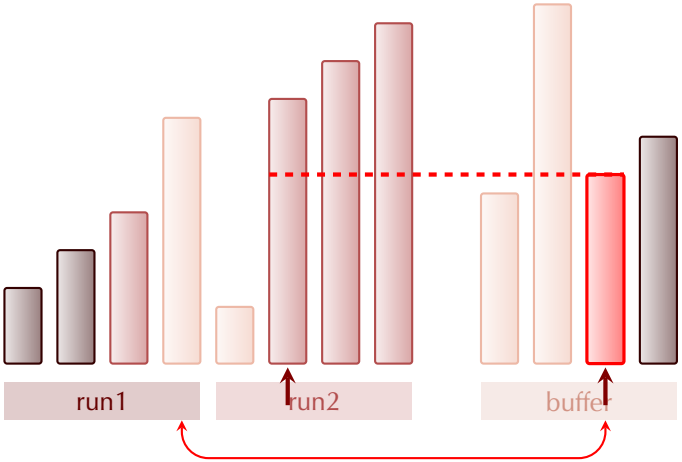

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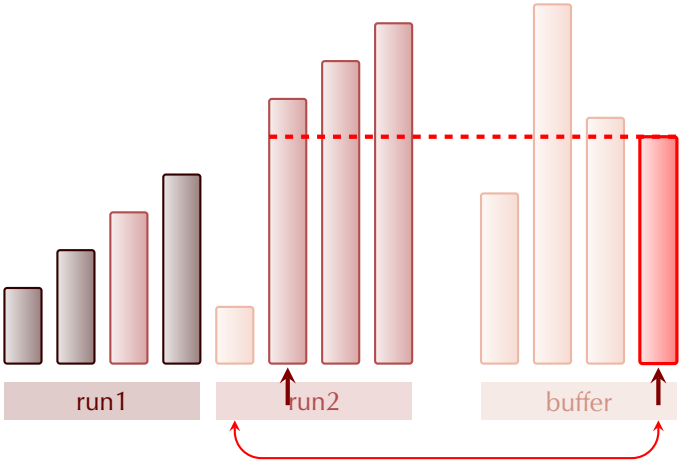

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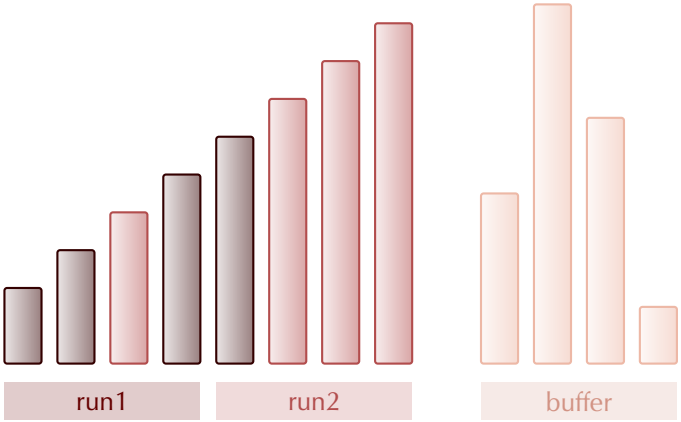

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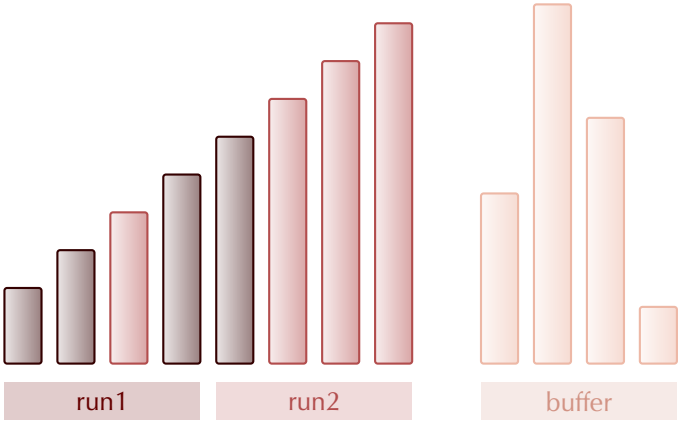

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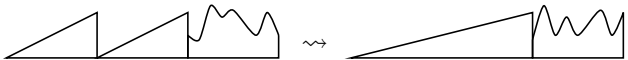

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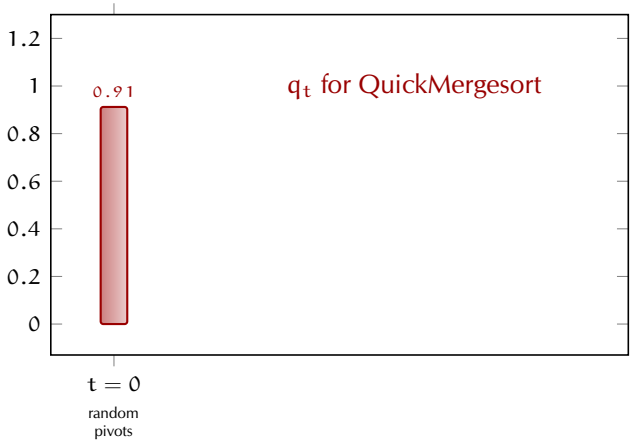

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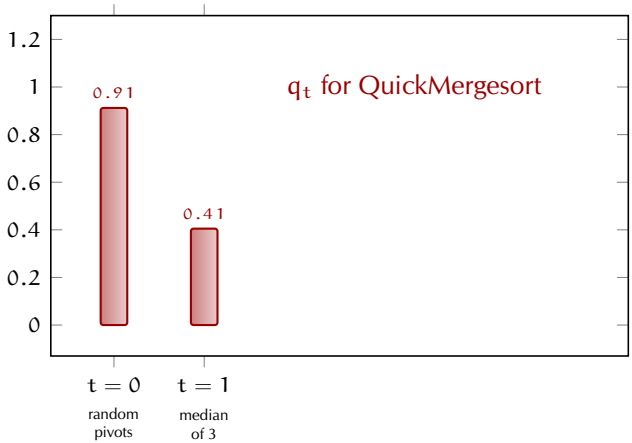

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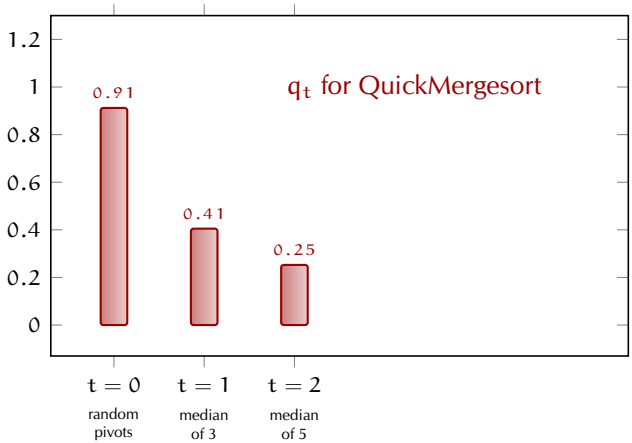

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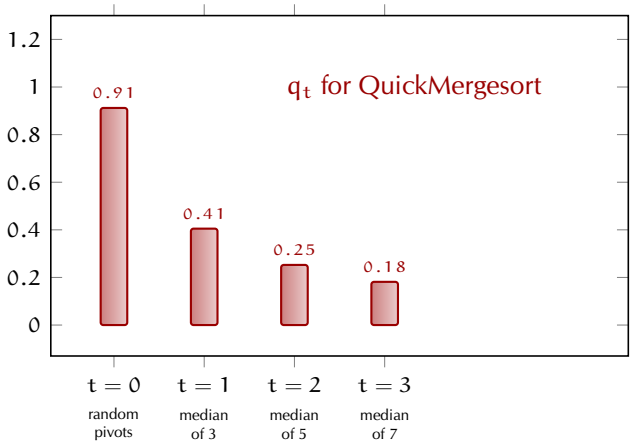

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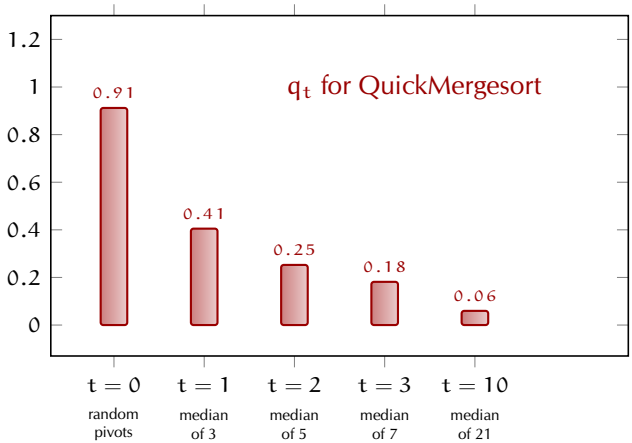

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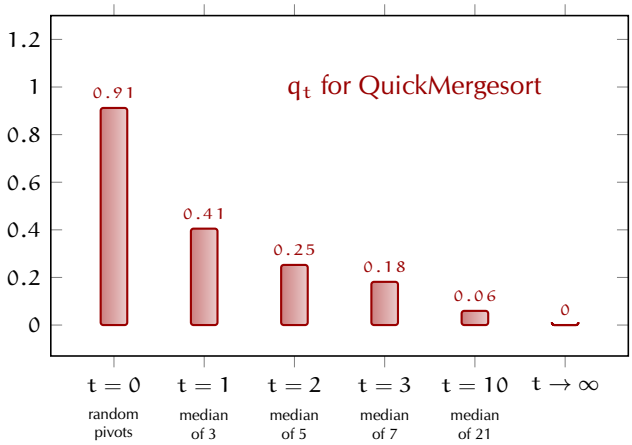

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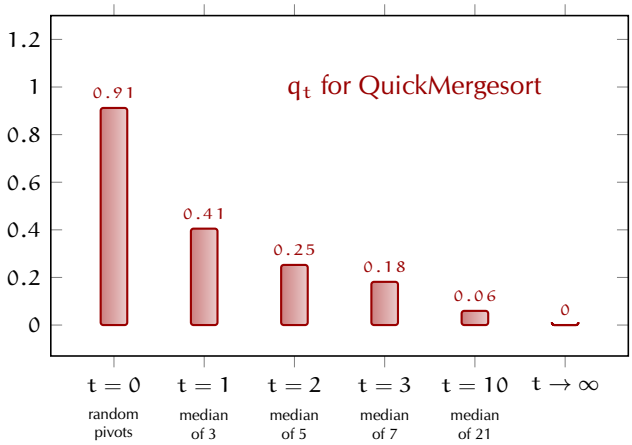

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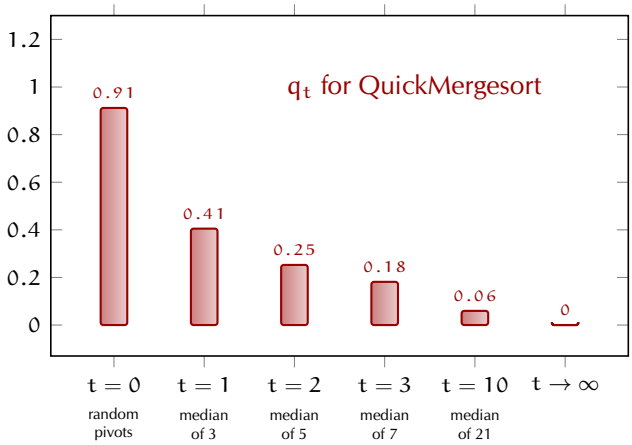

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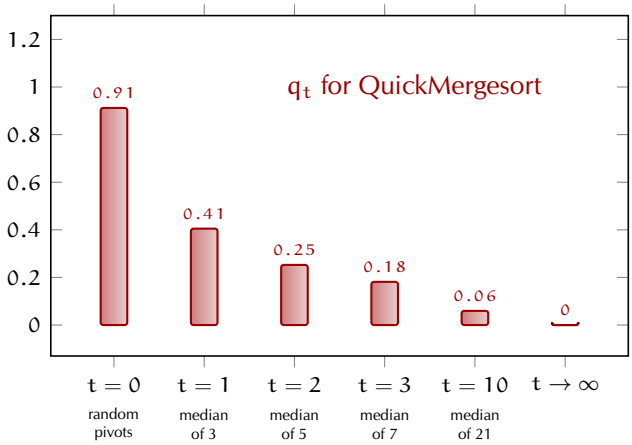

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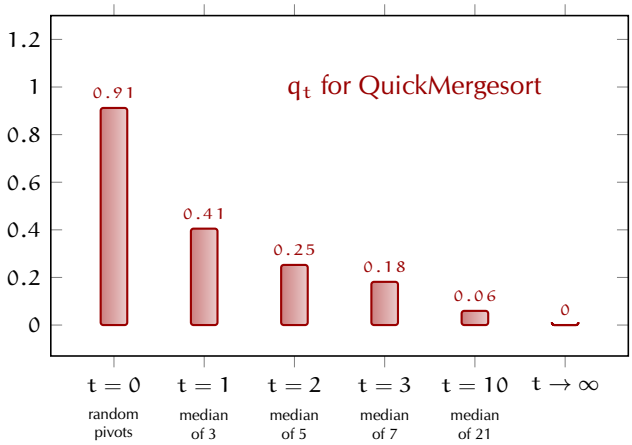

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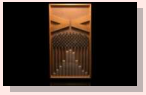
2 Dual-Pivot Quicksort



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4 Quicksort + X



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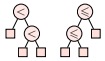

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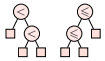

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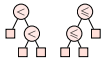

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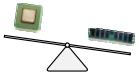

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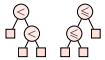

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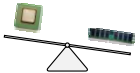

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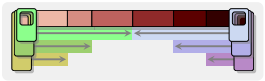

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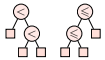

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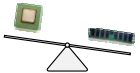

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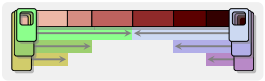

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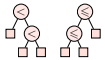

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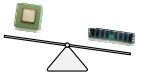

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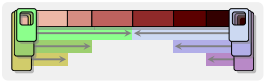

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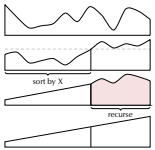

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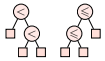

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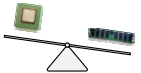

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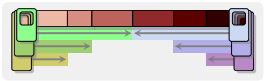

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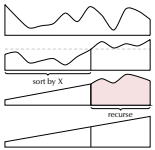

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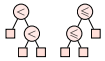

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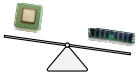

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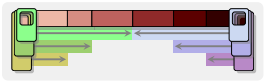

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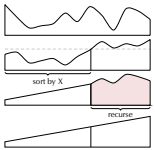

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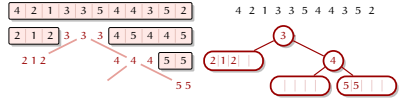

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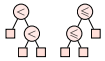

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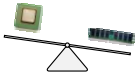

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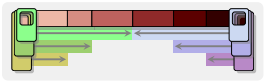

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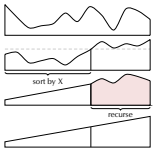

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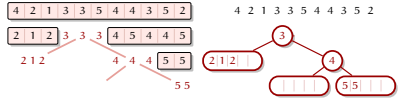

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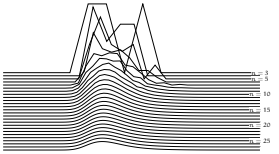

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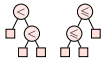

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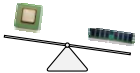

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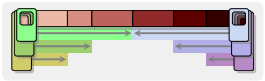

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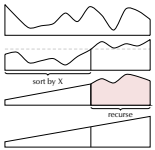

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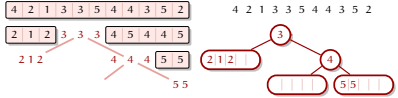

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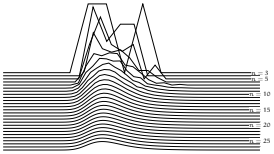

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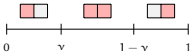

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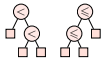

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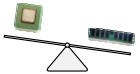

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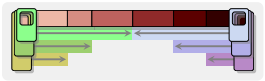

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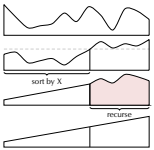
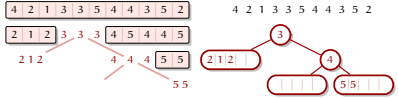
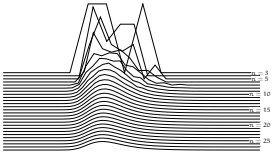


You have *not* seen:

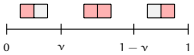
- Treatment of **equal keys**



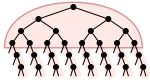
- Limiting **distributions** of costs



- **Sesquickselect** (cache-efficient selection)

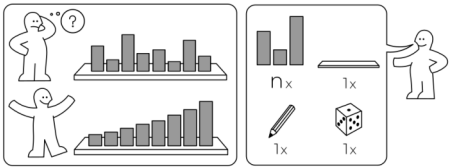


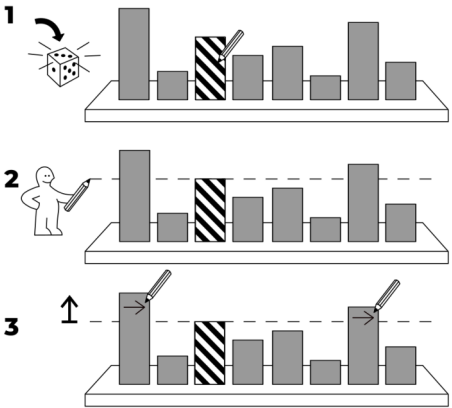
- Intriguing interplay of parameters

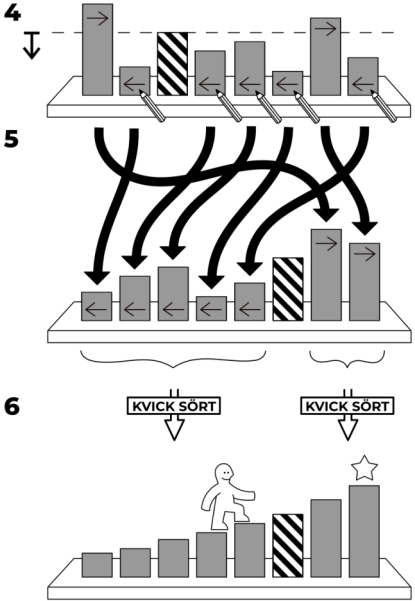


KVICK SÖRT

idea-instructions.com/quick-sort/
v1.2, CC by-nc-sa 4.0 IDEA

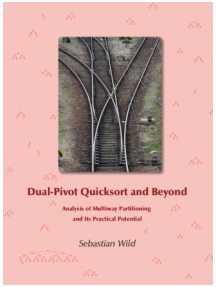








References



Most presented results appear in my thesis:

Dual-Pivot Quicksort And Beyond
Sebastian Wild • TU Kaiserslautern • 2016

Further Articles:

- ***Sesquickselect: One and a half pivots for cache-efficient selection***
  Conrado Martínez, Markus Nebel, Sebastian Wild • ANALCO • 2019
- ***QuickXsort – A Fast Sorting Scheme in Theory and Practice***
  Stefan Edelkamp, Armin Weiß, Sebastian Wild • Algorithmica • 2020
- ***Quicksort Is Optimal for Many Equal Keys***
  Sebastian Wild • ANALCO • 2018
- ***Analysis of Quickselect under Yaroslavskiy's Dual-Pivoting Algorithm***
  Sebastian Wild, Markus Nebel, Hosam Mahmoud • Algorithmica • 2016
- ***Analysis of Pivot Sampling in Dual-Pivot Quicksort***
  Markus Nebel, Sebastian Wild, Conrado Martínez • Algorithmica • 2016
- ***Analysis of Branch Misses in Quicksort***
  Conrado Martínez, Markus Nebel, Sebastian Wild • ANALCO • 2015
- ***Average Case and Distributional Analysis of Dual-Pivot Quicksort***
  Sebastian Wild, Markus Nebel, Ralph Neininger • ACM Trans. Alg. • 2015
- ***Average-Case Analysis of Java 7's Dual-Pivot Quicksort***
  Sebastian Wild, Markus Nebel • ESA • 2012

Basic Algorithms

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- **Classic Quicksort**
  - *crossing-pointer* scheme
  - symmetric treatment

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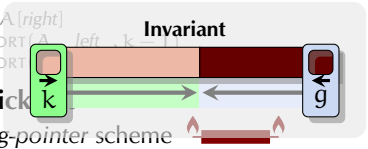

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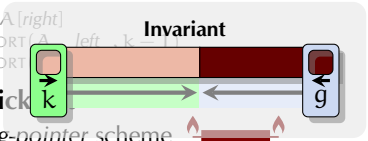

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- **YBB Quicksort**
  - . . . looks messy

YbbQuicksort(A, *left*, *right*)

```
1  if right ⩽ left then return
2  P1 := A[left];   P2 := A[right]
3  if P1 > P2 then Swap P1 and P2 end if
4  ℓ := left + 1;   g := right − 1;   k := ℓ
5  while k ⩽ g
6      if A[k] < P1
7          Swap A[k] and A[ℓ] ;  ℓ := ℓ + 1
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9          while A[g] > P2 and k < g
10             g := g − 1
11         end while
12         Swap A[k] and A[g] ;  g := g − 1
13         if A[k] < P1
14             Swap A[k] and A[ℓ] ;  ℓ := ℓ + 1
15         end if
16     end if
17     k := k + 1
18 end while
19 ℓ := ℓ − 1;   g := g + 1
20 Swap A[left] and A[ℓ] ; Swap A[right] and A[g]
21 YbbQuicksort(A,  left  , ℓ − 1 )
22 YbbQuicksort(A, ℓ + 1, g − 1)
23 YbbQuicksort(A, g + 1,  right  )
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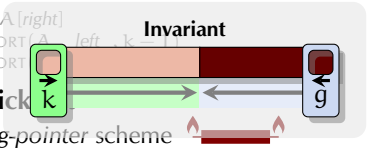

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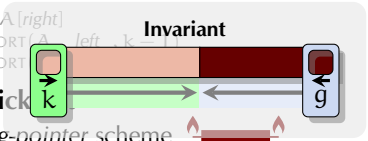

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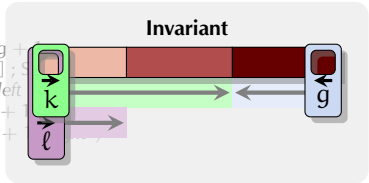

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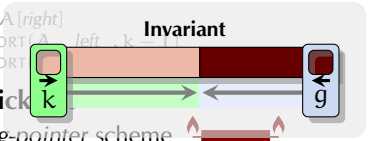

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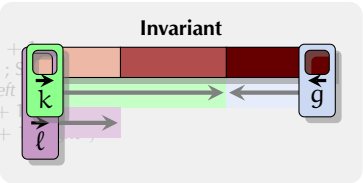

QuickMergesort

QUICKMERGESORT($A[l..r]$)

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1  Choose pivot P.
2  i = PARTITION(A[l..r], P)

5  if ...
6      MERGESORT(A[i+1..r]) using buffer A[l..i-1]
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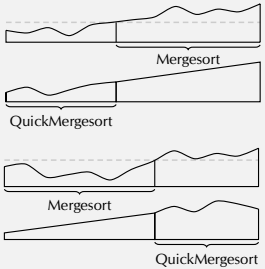

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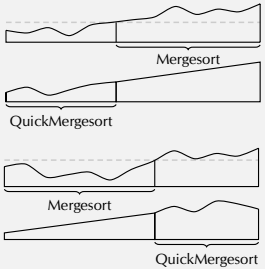

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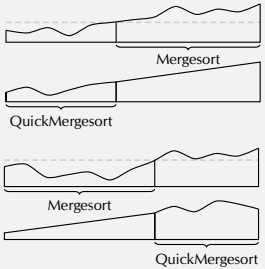

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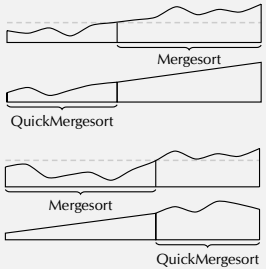

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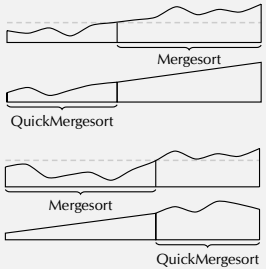

What happened? – Details

What do comparisons, write accesses, Bytecodes and branch misses have in common?

$\rightsquigarrow$ they happen **inside** the **CPU**

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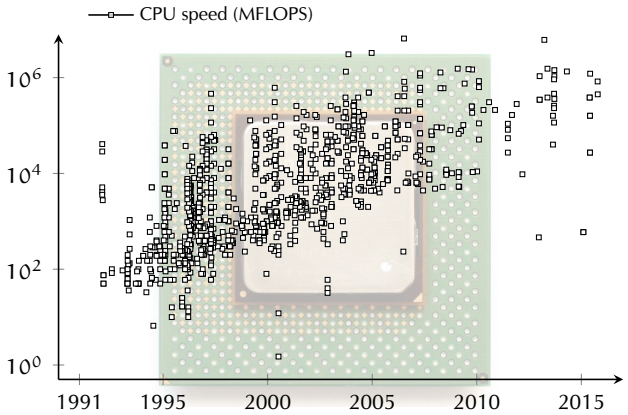

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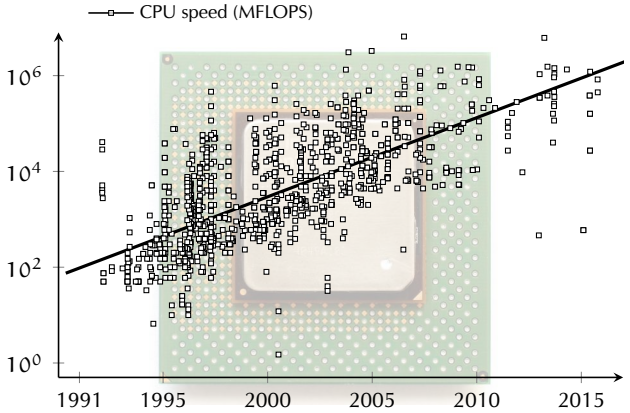

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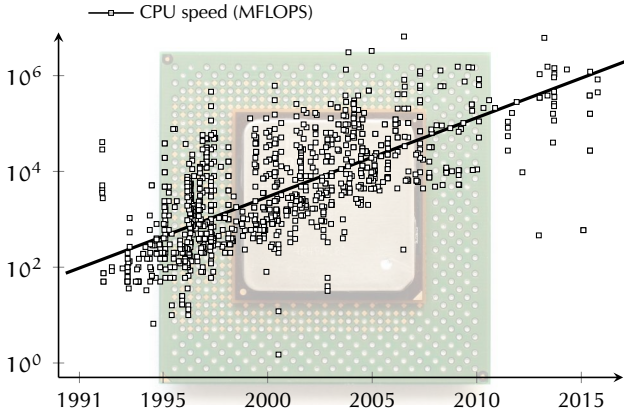

Average annual growth rate:
CPU speed: 46%

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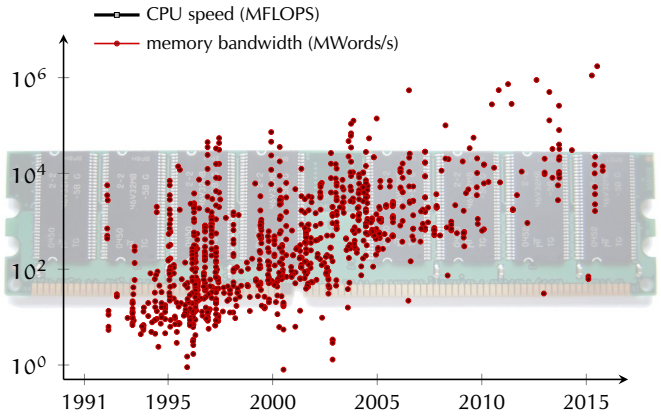

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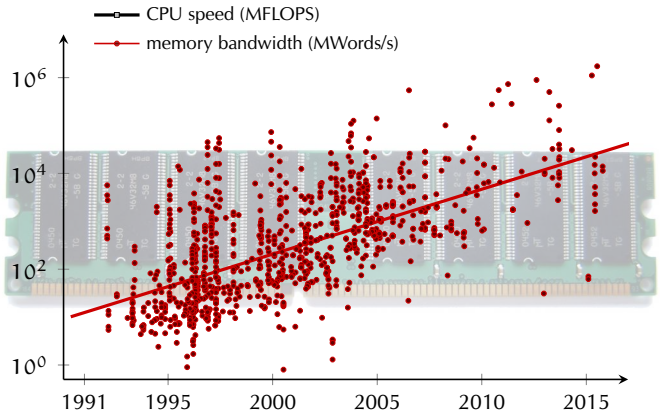

Average annual growth rate:

CPU speed: 46%
Memory Bandwidth: 37%

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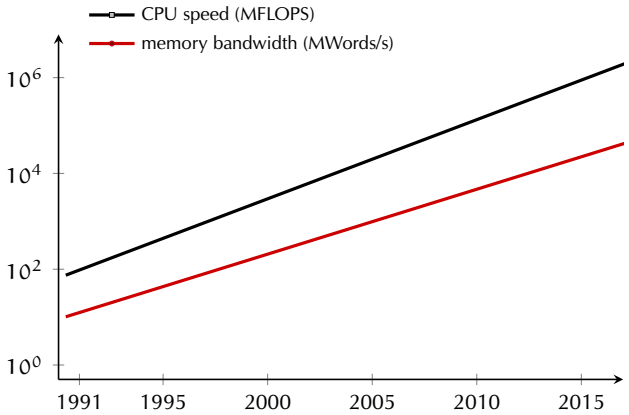

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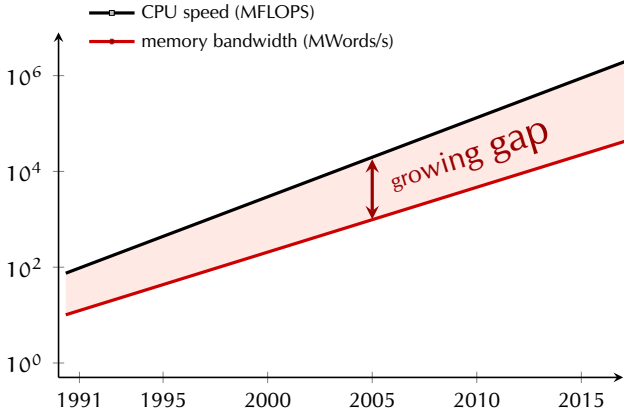

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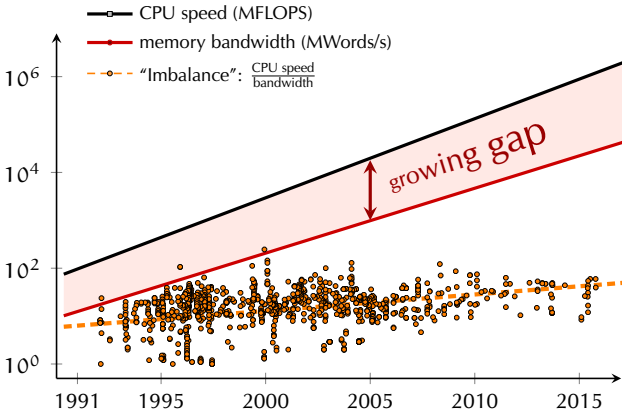

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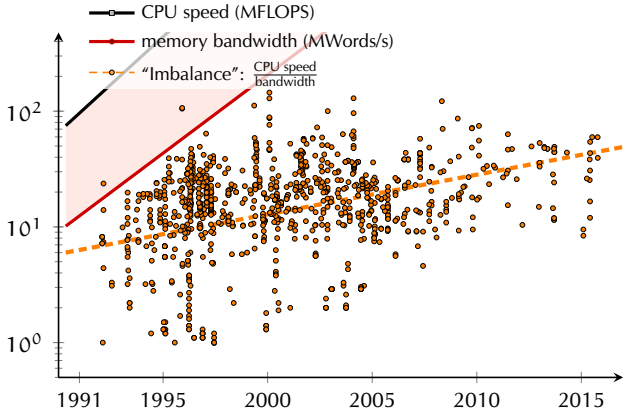

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Average annual growth rate:

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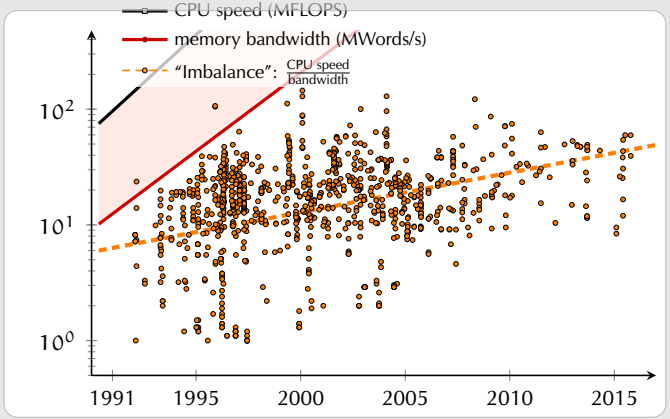

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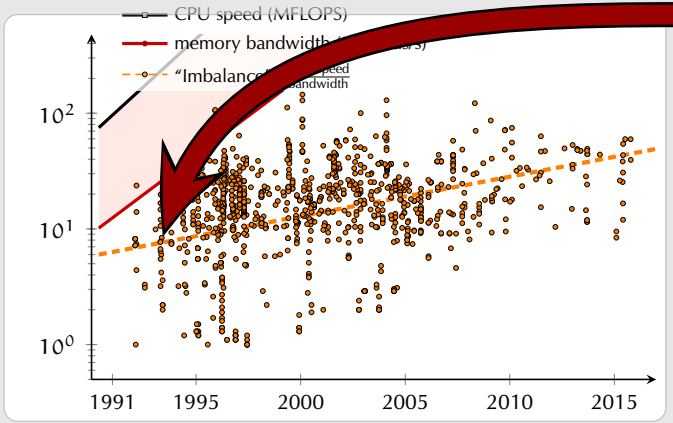

Classic Quicksort (Bentley & McIlroy) **1993**

- 1 mem transfer ≈ 7 instructions

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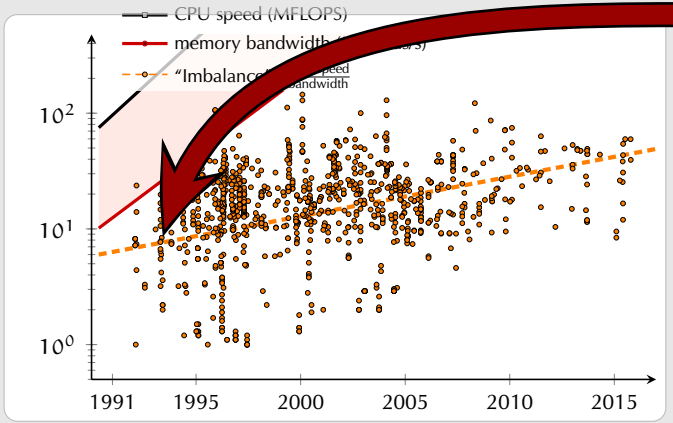

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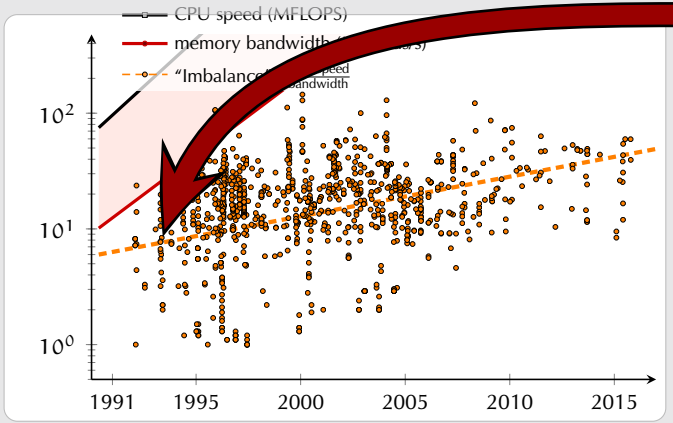

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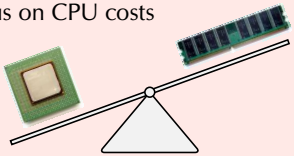

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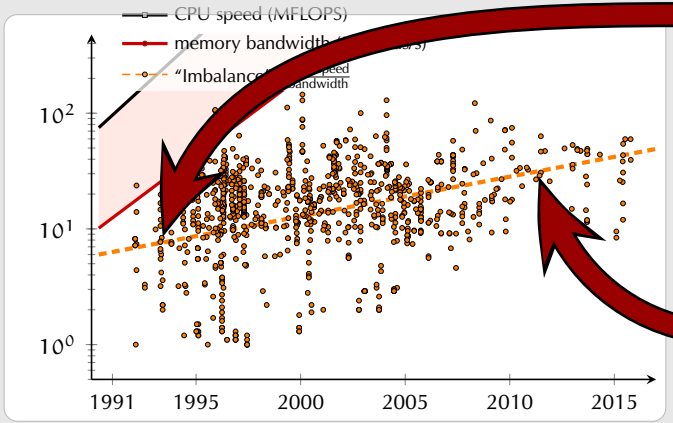

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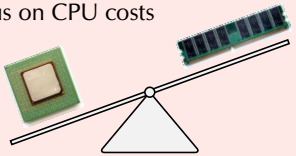

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YBB Quicksort, **2009 – today**

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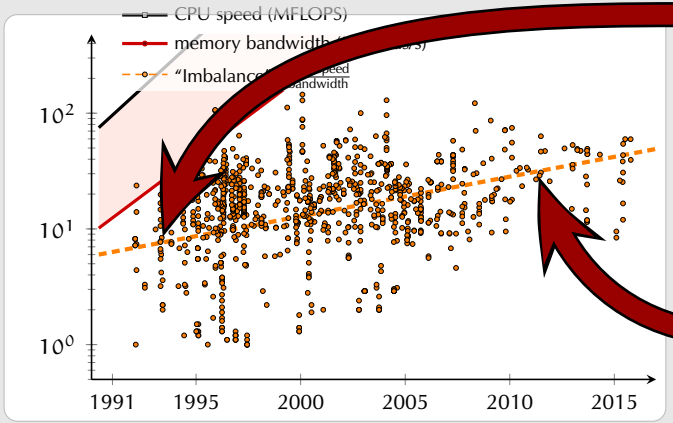

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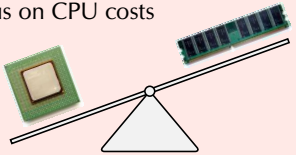

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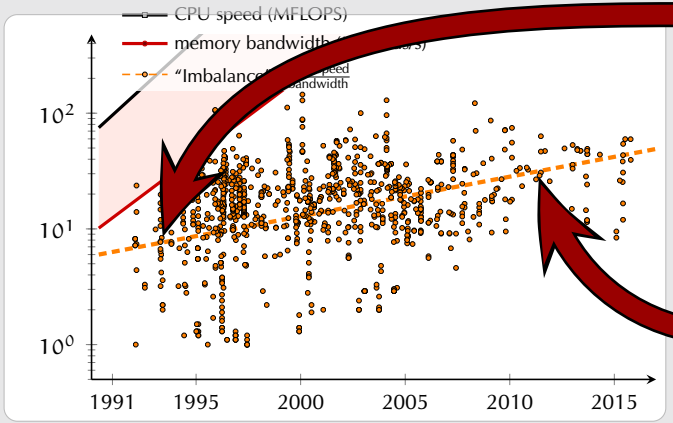

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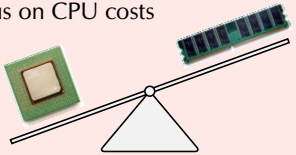

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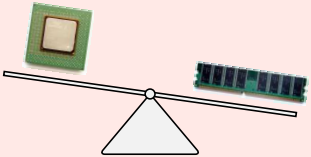

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