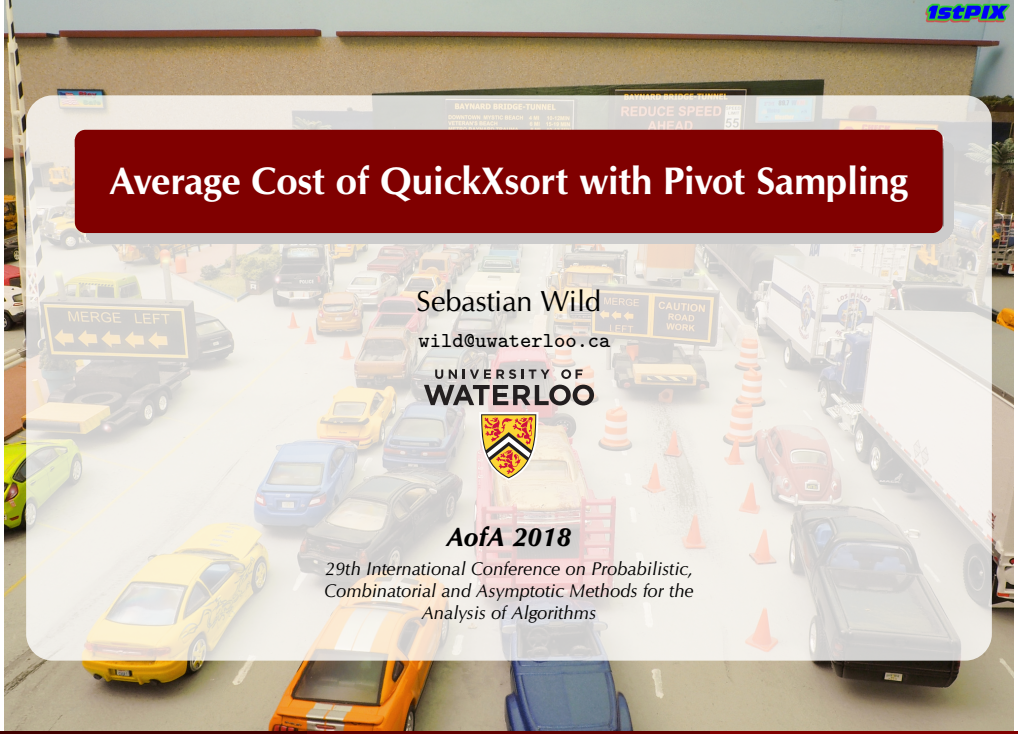

Average Cost of QuickXsort with Pivot Sampling

Sebastian Wild

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UNIVERSITY OF WATERLOO

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29th International Conference on Probabilistic, Combinatorial and Asymptotic Methods for the Analysis of Algorithms

Outline

1 What is QuickXsort?

2 Recurrence

3 Toll function

4 Result

What is QuickXsort?

$$\text{Quick}\mathbf{X}\text{sort} = \text{Quicksort} + \mathbf{X}$$

What is QuickXsort?

QuickXsort = Quicksort + X

QuickXsort(A[l..r])

1. Choose pivot P.
2. i = Partition(A[l..r], P)
   // subproblems: A[l..i − 1] and A[i + 1..r]
3. Sort one subproblem with X.
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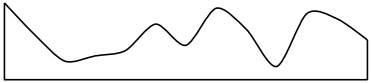

QUICKXSORT($A[l..r]$)

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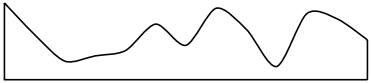

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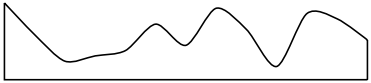

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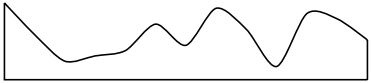

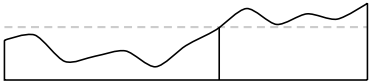

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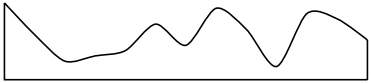

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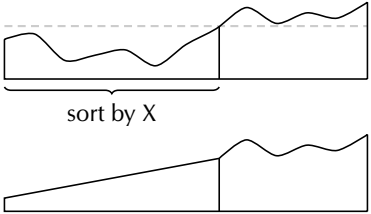

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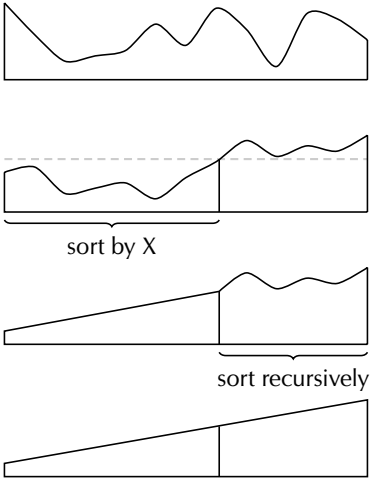

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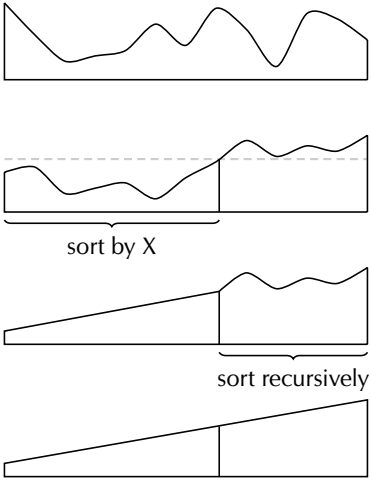

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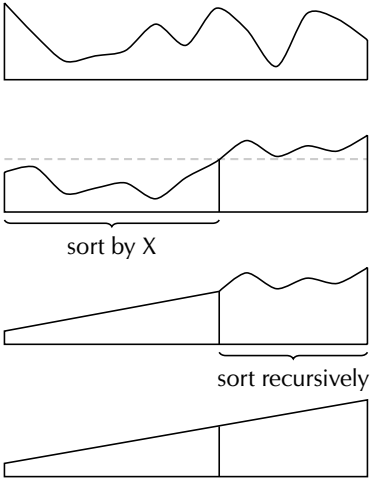

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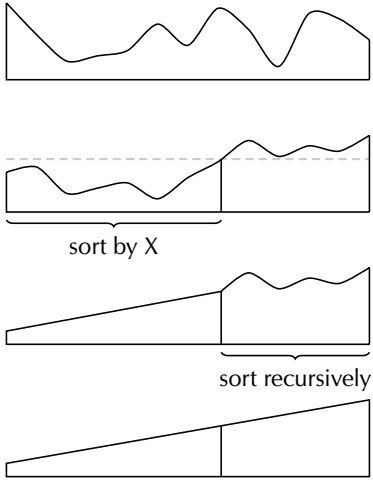

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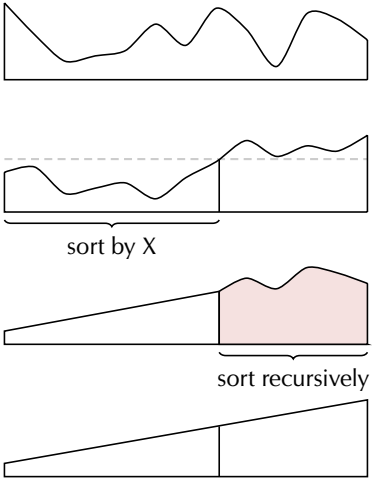

⇝ *QuickXsort can turn X into an in-place sorting method!*

QuickMergesort

Good candidate for X: ***Mergesort***

- #comparisons almost optimal
- but: merging uses extra space as buffer
- merging possible without extra space
  - but not really practical!

"Worst merge lanes in the D.C. area"

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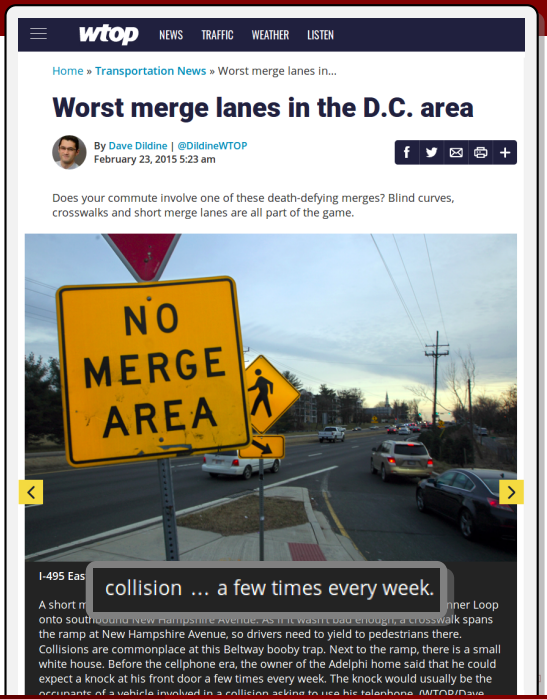

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For this talk: X = Mergesort

Goal: inherit good comparison count, but in-place!

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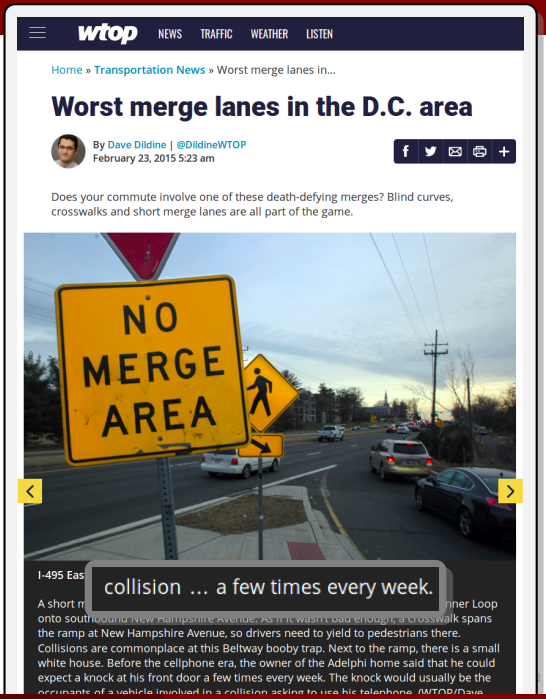

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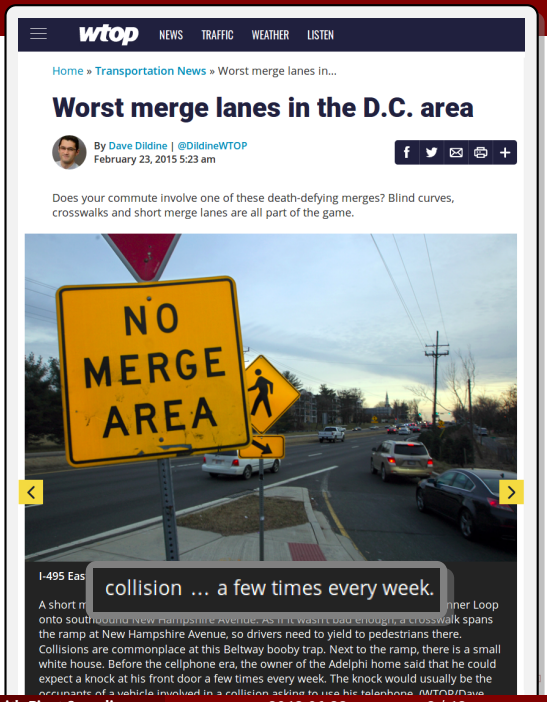

Merging with swaps

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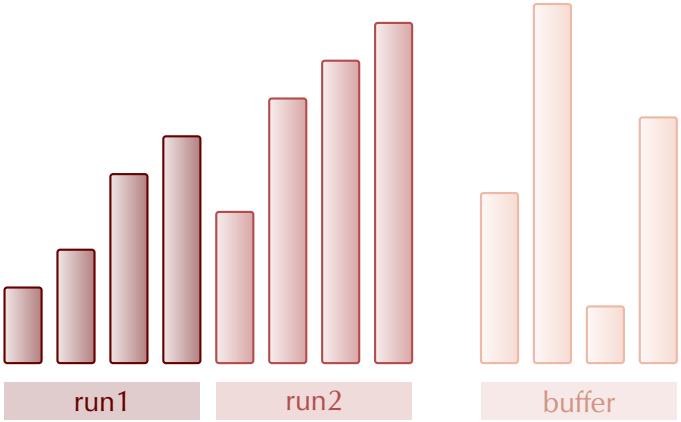

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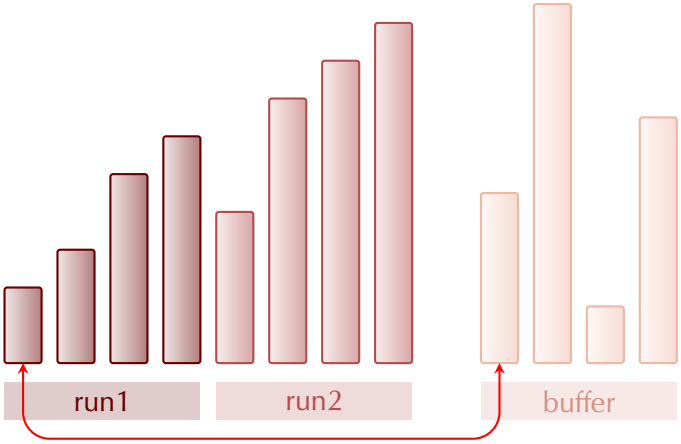

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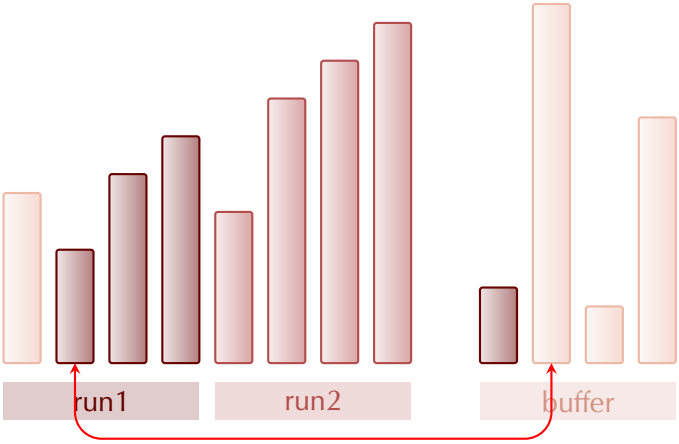

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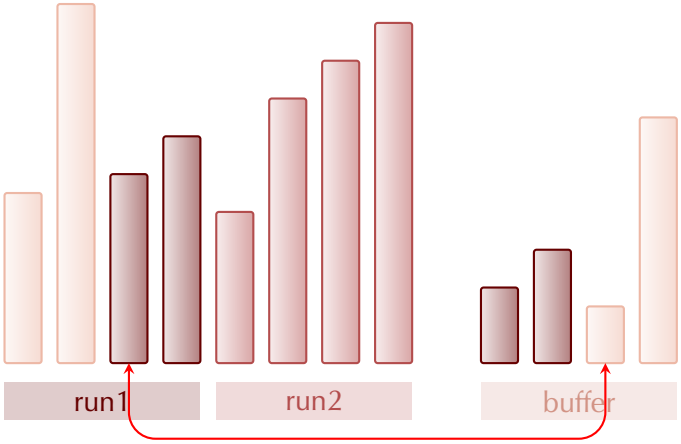

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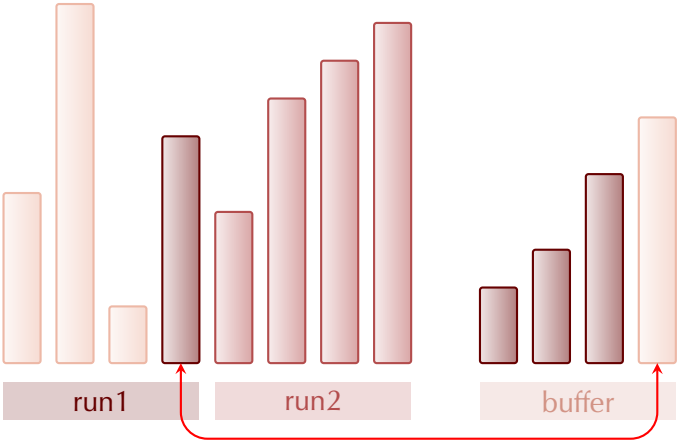

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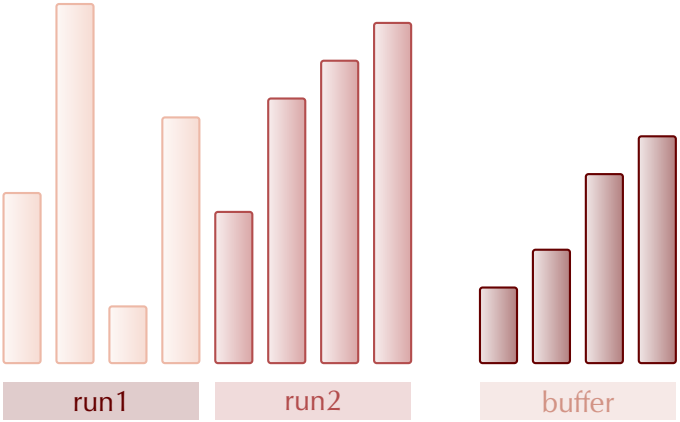

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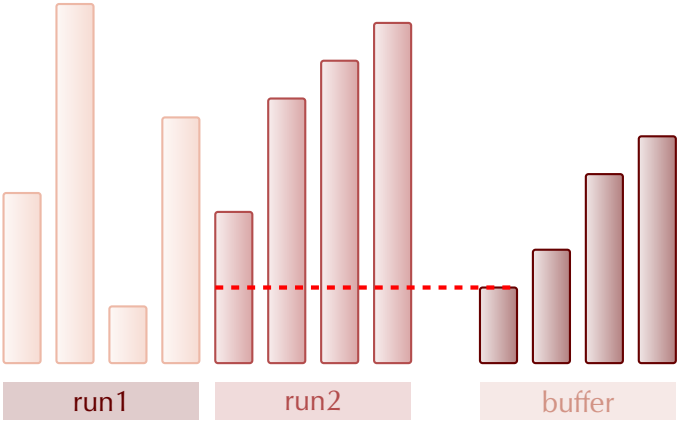

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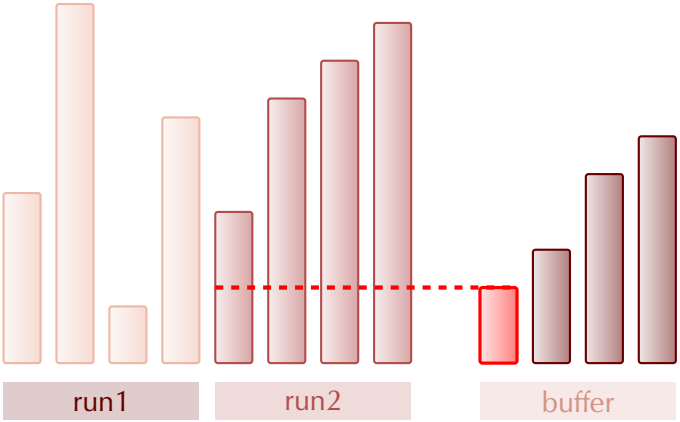

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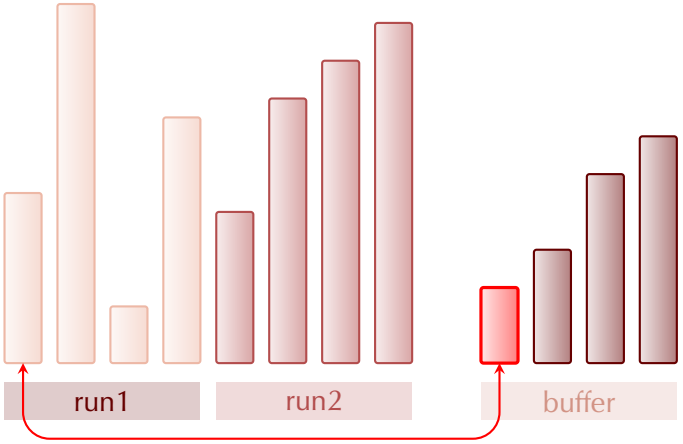

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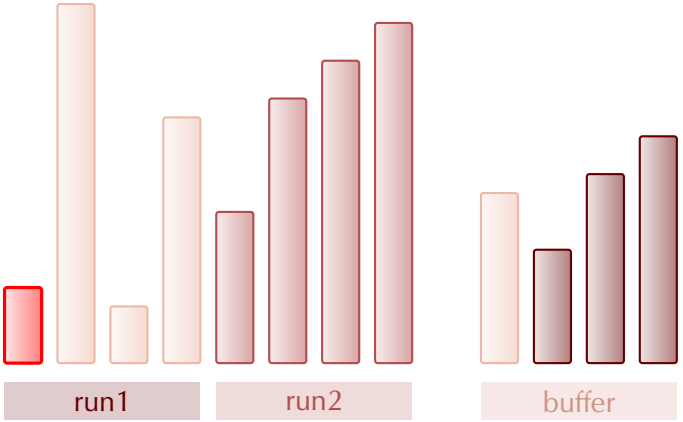

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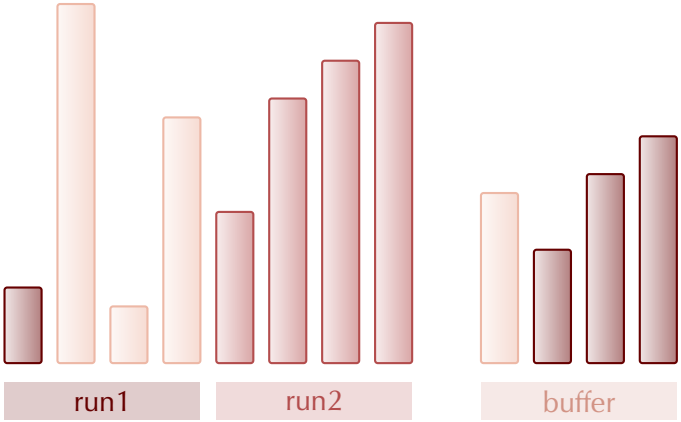

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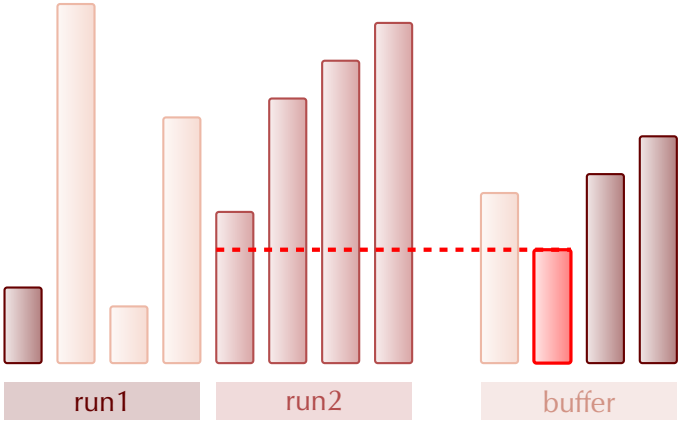

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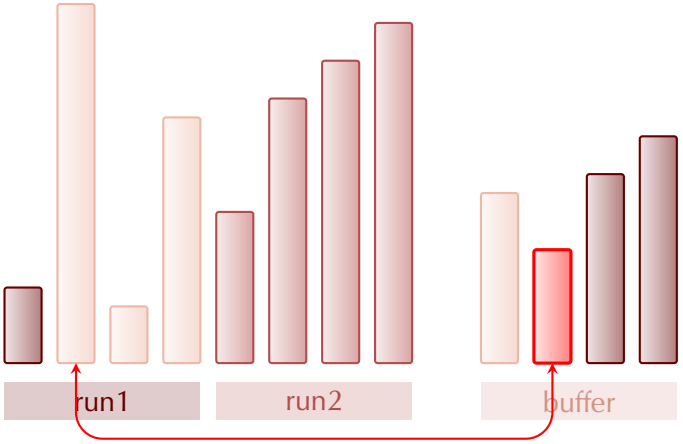

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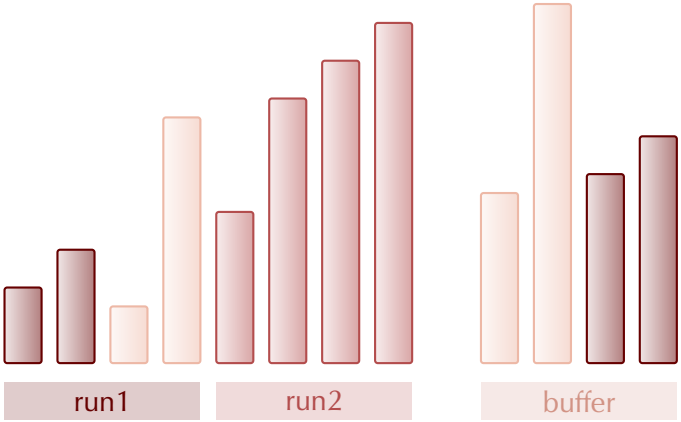

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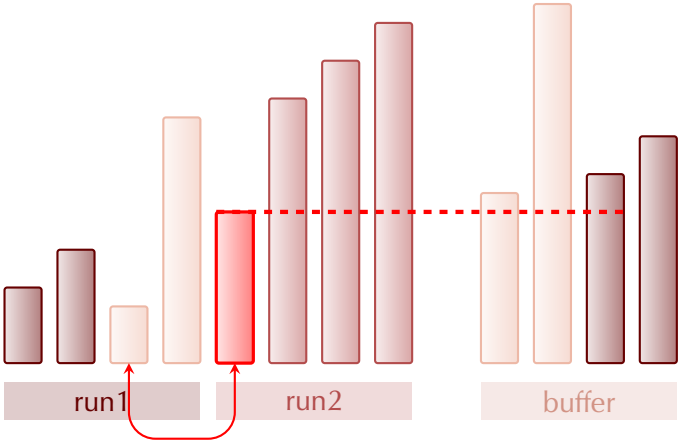

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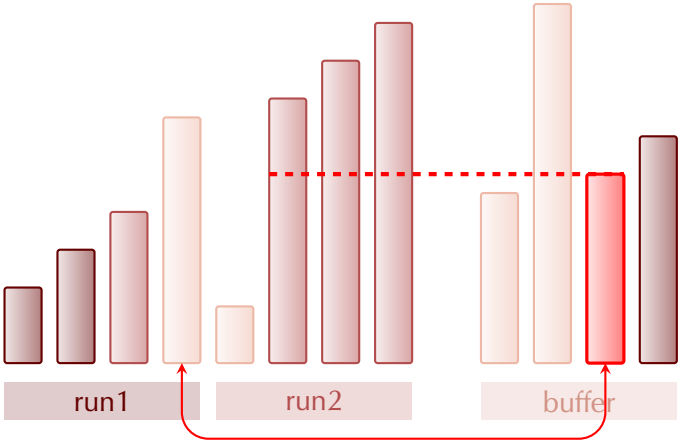

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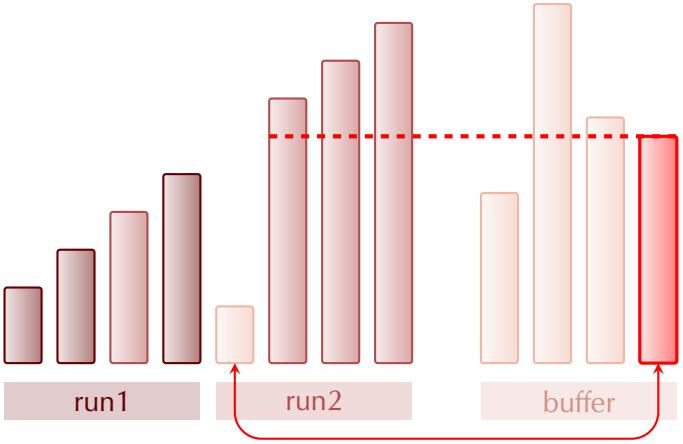

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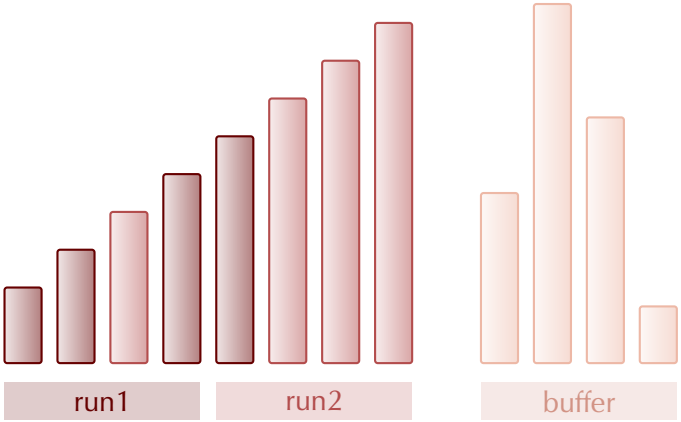

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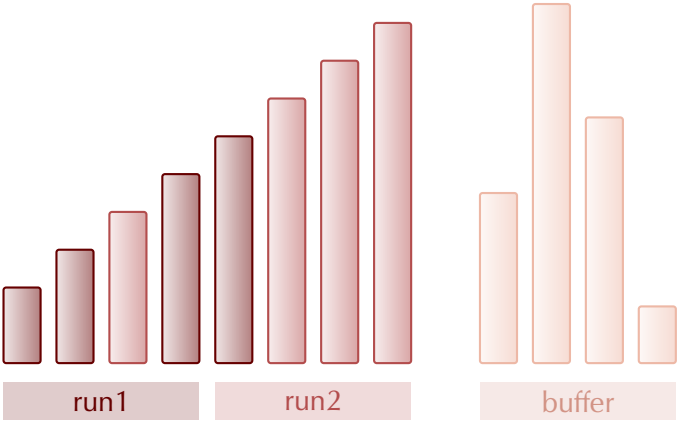

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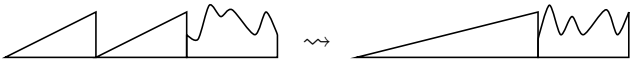
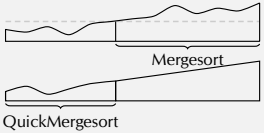


Which side with Mergesort?

QuickMergesort($A[l..r]$)

```
1  Choose pivot P.
2  i = Partition(A[l..r], P)
5  if ...
6      Mergesort(A[i+1..r]) with buffer A[l..i-1]
7      QuickMergesort(A[l..i-1]).
8  else
9      Mergesort(A[l..i-1]) with buffer A[i+1..r]
10     QuickMergesort(A[i..r]).
11 end if
```



Which side with Mergesort?

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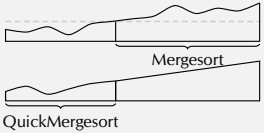

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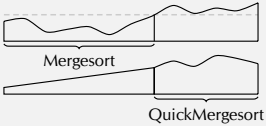

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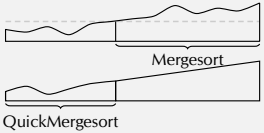

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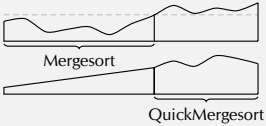

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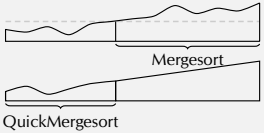

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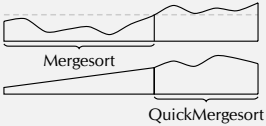

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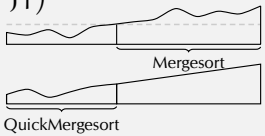

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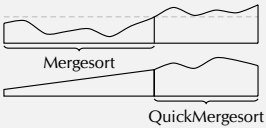

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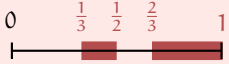

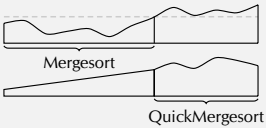

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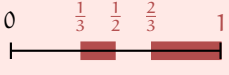
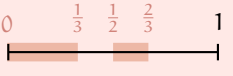
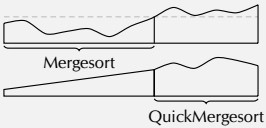


Similarly: **mergesort** iff





Outline

1 What is QuickXsort?

2 Recurrence

3 Toll function

4 Result

Costs of QuickXsort

Assumptions: #comparisons

- average costs for X: $x(n) = an \lg n + bn \pm O(n^{1-\varepsilon})$ — αn buffer space
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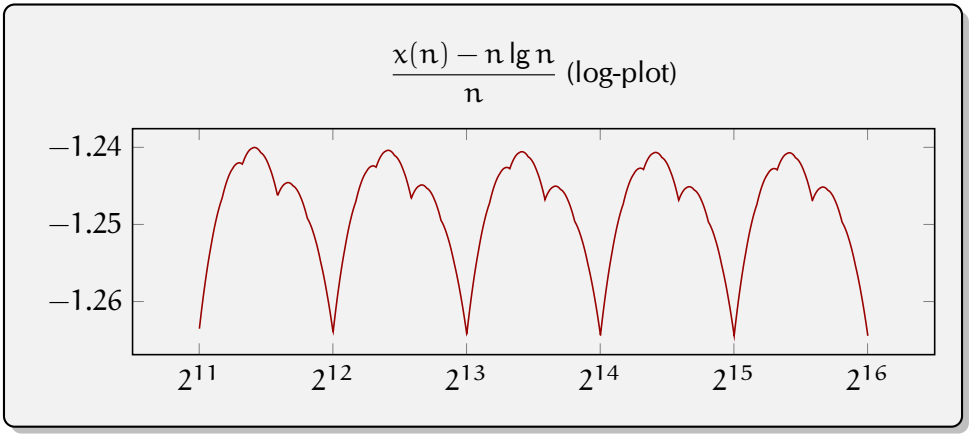

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(partition)

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(Mergesort right; recurse left)

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Mergesort left — recurse right

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+ base cases

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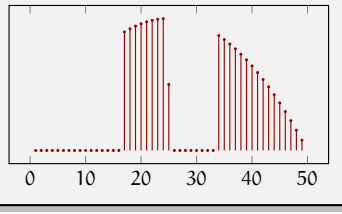
$w_{51,j}$ for median-of-3 QuickMergesort



Shape Function

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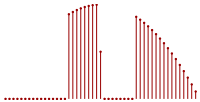

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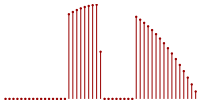

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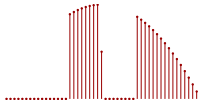

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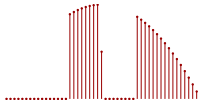

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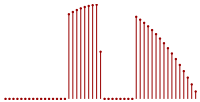

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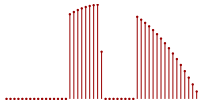

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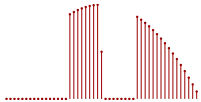

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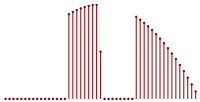

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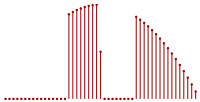

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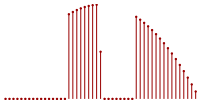

- precise criterion:

$w(z) \approx n w_{n,zn}$ … with guaranteed convergence rate

the "shape function"

$$\exists w : [0,1] \to \mathbb{R}_{\geqslant 0} \;:\; \sum_{j=0}^{n-1} \left| w_{n,j} - \int_{j/n}^{(j+1)/n} w(z)\,dz \right| \;=\; O(n^{-1}) \qquad (n \to \infty).$$

- subproblem size distribution: $J_1 \overset{\mathcal{D}}{=} J_2 \overset{\mathcal{D}}{=} \text{Binomial}(n, \text{Beta}(t+1, t+1)) \pm O(1)$

beta-binomial distribution

- binomial distribution highly concentrated $\rightsquigarrow$ $\frac{J_1}{n} \overset{\mathcal{D}}{\approx} P \overset{\mathcal{D}}{=} \text{Beta}(t+1, t+1)$
  formally: $n\mathbb{P}\big[J_1 = \lfloor zn \rfloor\big] = f_{\text{Beta}}(z) \pm O(n^{-1})$

Chernoff etc.

median of $2t+1$ i.i.d. $\text{Uniform}(0,1)$ r.v.

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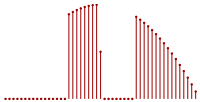

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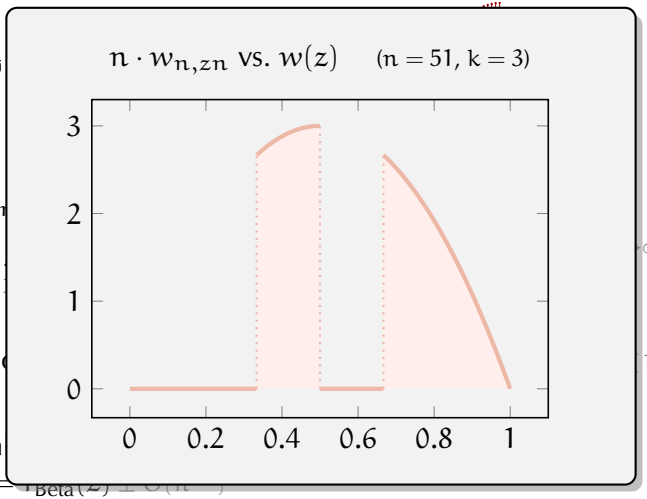

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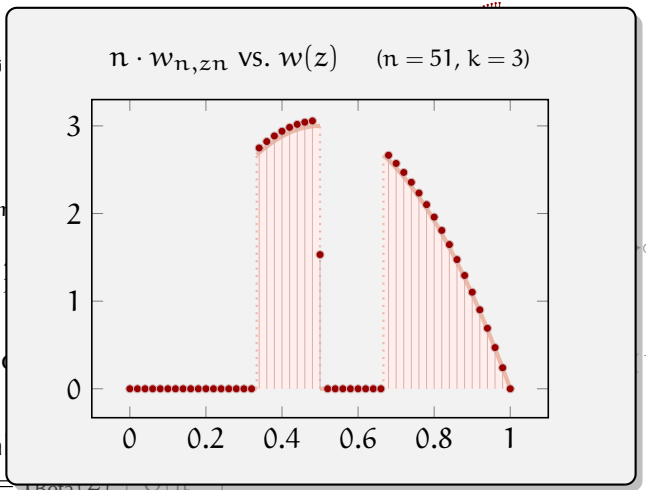

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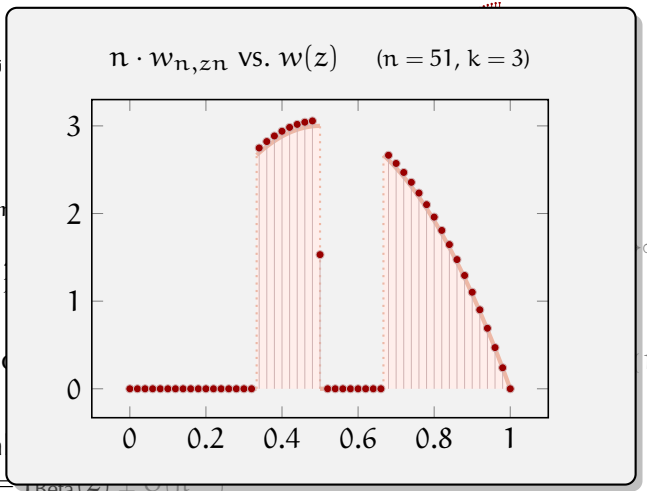

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Intuition behind shape function? $w(z) = 2\big[z \in$  $\big] \cdot f_{\mathrm{Beta}(t+1,t+1)}(z)$

- $w(z)$ is **limiting density** of **relative subproblem size** of recursive call $\left(\int_0^1 w(z)\,dz = 1\right)$
- expected fraction of elements in recursive call:

$$r := \int_0^1 z \cdot w(z)\,dz = 2\int_{z\in}$$ $$z \cdot \frac{z^t(1-z)^t}{B(t+1,t+1)}\,dz = 2\mathbb{E}\Big[\big[P \in$$ $$\big] \cdot P\Big]$$

incomplete beta integral

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median of 7	**46.6%** mergesort / **53.4% recursive**		

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median of 41	**55.1%** mergesort / **44.9% recursive**		

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median of 7	**46.6%** mergesort / **53.4% recursive**		
median of 41	**55.1%** mergesort / **44.9% recursive**		
exact median	**50%** mergesort / **50% recursive**		

Outline

1 What is QuickXsort?

2 Recurrence

3 Toll function

4 Result

The leading term

$$\boxed{t(n) \sim K n^{\alpha} \log^{\beta} n}$$

$$t(n) = \mathbb{E}\left[\left[\frac{J_1}{n-1} \in \text{[redacted]}\right] x(J_1)\right] + \mathbb{E}\left[\left[\frac{J_2}{n-1} \in \text{[redacted]}\right] x(J_2)\right] + (n-k) + b(k)$$

Recall: $x(n) = n \lg n - 1.24n \pm O(1)$

$J_1 \overset{\mathcal{D}}{\approx} Pn$ where $P \overset{\mathcal{D}}{=} \text{Beta}(t+1, t+1)$

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$\sim \mathbb{E}\left[\left[\ \ \in \text{[redacted]}\right] Pn \lg(Pn)\right]$

$\sim \mathbb{E}\left[\left[\ \ \in \text{[redacted]}\right] Pn \lg(\ n\ )\right] = \mathbb{E}\left[\left[P \in \text{[redacted]}\right] \cdot P\right] \cdot n \lg n$

$= \left(\mathbb{E}\left[\left[P \in \text{[redacted]}\right] \cdot (1-P)\right]\right) \cdot n \lg n$

Recall: $\gamma = 2\mathbb{E}\left[\left[P \in \text{[redacted]}\right] \cdot P\right]$

$= \frac{1}{2}(1-\gamma) \cdot n \lg n$

$\rightsquigarrow$ $c(n) \sim$ solution of $c(n) = t(n) + \int w(z) c(zn) dz$

$\gamma \cdot n \lg n = (1-\gamma) \cdot n \lg n + \gamma\gamma \cdot n \lg n$

Ansatz: $c(n) \sim \gamma \cdot n \lg n$

The leading term

$t(n) \sim Kn^{\alpha} \log^{\beta} n$

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Ansatz: $c(n) \sim \gamma \cdot n \lg n$

The leading term

$t(n) \sim Kn^{\alpha}\log^{\beta} n$

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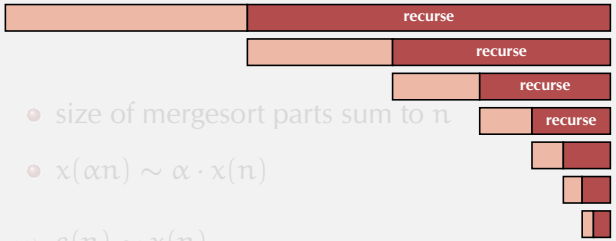

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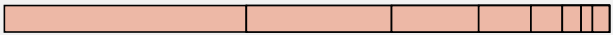

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The linear term

Linear term depends on

- pivot sampling
- partitioning method
- linear term of mergesort
- but **not** on how **small subproblems** solved!

unlike for Quicksort!

Consider $c'(n) = c(n) - n \lg n$
Can show:

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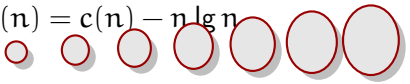

$$
\begin{aligned}
& c'(n) \\
&= c(n) - an\lg n \\
&= \sum_{r=1}^{2} \mathbb{E}\big[A_r(J_r)c(J_r)\big] - an\lg n + t(n) \\
&= \sum_{r=1}^{2} \Big(\mathbb{E}\big[A_r(J_r)(c(J_r) - aJ_r\lg J_r)\big] - a\,\mathbb{E}\big[A_r(J_r)J_r\lg J_r\big]\Big) - an\lg n \\
&\quad + (n-1) + \mathbb{E}\big[A_2(J_2)\cdot x(J_1)\big] + \mathbb{E}\big[A_1(J_1)\cdot x(J_2)\big] \\
&= \sum_{r=1}^{2} \mathbb{E}\Big[A_r(J_r)c'(J_r)\Big] + (n-1) - an\lg n \\
&\quad + a\,\mathbb{E}\Big[\big(A_1(J_1)+A_2(J_2)\big)J_1\lg J_1\Big] + b\,\mathbb{E}[A_2(J_2)J_1] \\
&\quad + a\,\mathbb{E}\Big[\big(A_2(J_2)+A_1(J_1)\big)J_2\lg J_2\Big] + b\,\mathbb{E}[A_1(J_1)J_2] \;\pm\; O(n^{1-\varepsilon})
\end{aligned}
$$

Since $J_1 \triangleq J_2$ we can simplify

$$
\begin{aligned}
& \mathbb{E}\Big[\big(A_1(J_1)+A_2(J_2)\big)J_1\lg J_1\Big] + \mathbb{E}\Big[\big(A_2(J_2)+A_1(J_1)\big)J_2\lg J_2\Big] \\
&= \mathbb{E}\Big[\big(A_1(J_1)+A_2(J_2)\big)J_1\lg J_1\Big] + \mathbb{E}\Big[\big(A_2(J_1)+A_1(J_2)\big)J_1\lg J_1\Big] \\
&= \mathbb{E}\Big[J_1\lg J_1\cdot\big(\big(A_1(J_1)+A_1(J_2)\big)+\big(A_2(J_1)+A_2(J_2)\big)\big)\Big] \\
&= 2\mathbb{E}[J\lg J] \\
&= 2\,\mathbb{E}[\tfrac{J}{n}]\cdot n\lg n + 2\cdot\tfrac{1}{\ln 2}\mathbb{E}[\tfrac{J}{n}\ln\tfrac{J}{n}]\cdot n \\
&= n\lg n - \tfrac{1}{\ln 2}\big(H_{k+1}-H_{t+1}\big)n \;\pm\; O(n^{1-\varepsilon}).
\end{aligned}
$$

Plugging this back into our equation for $c'(n)$, we find

$$
\begin{aligned}
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&\quad + b\Big(I_{0,\frac{1}{3}}(t+2,t+1) + I_{\frac{1}{3},\frac{2}{3}}(t+2,t+1)\Big)\cdot n \;\pm\; O(n^{1-\varepsilon}) \\
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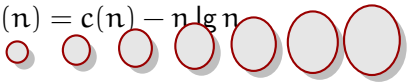

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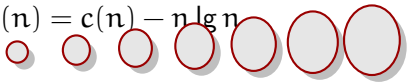

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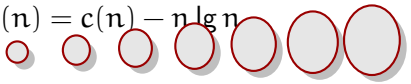

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1 What is QuickXsort?

2 Recurrence

3 Toll function

4 Result

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Comparisons in QuickMergesort:

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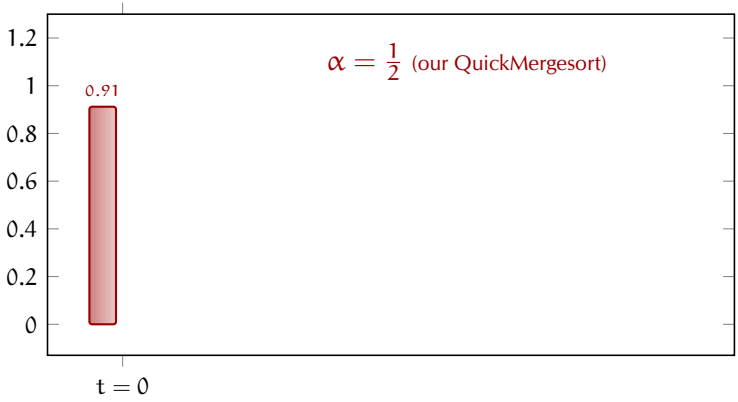

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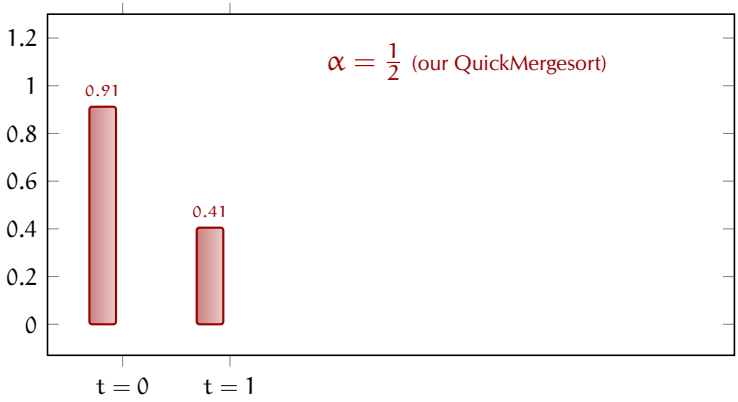

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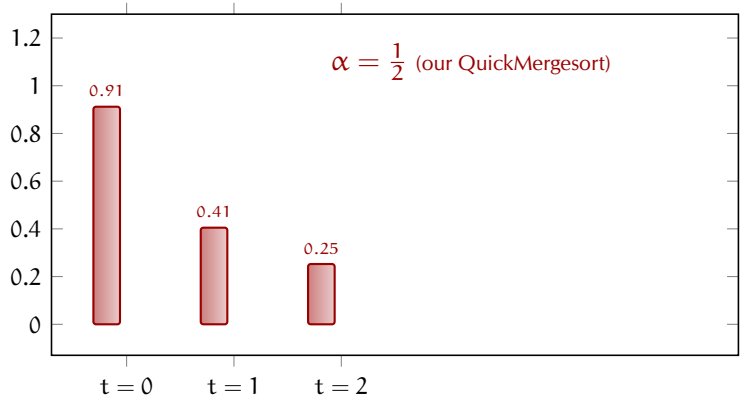

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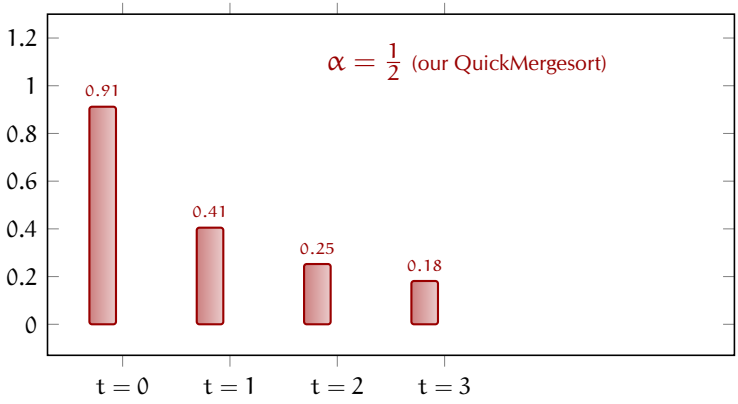

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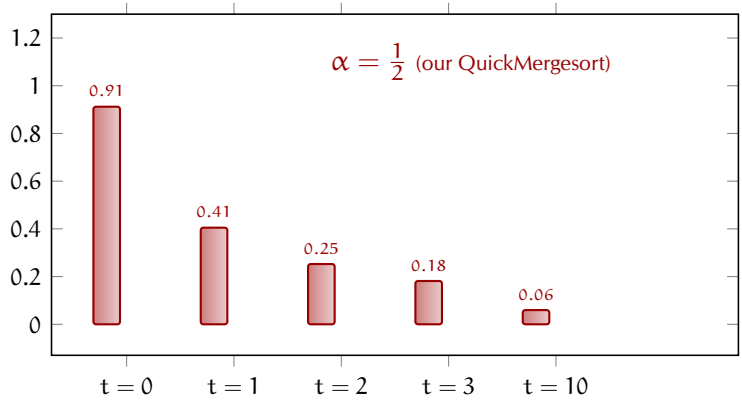

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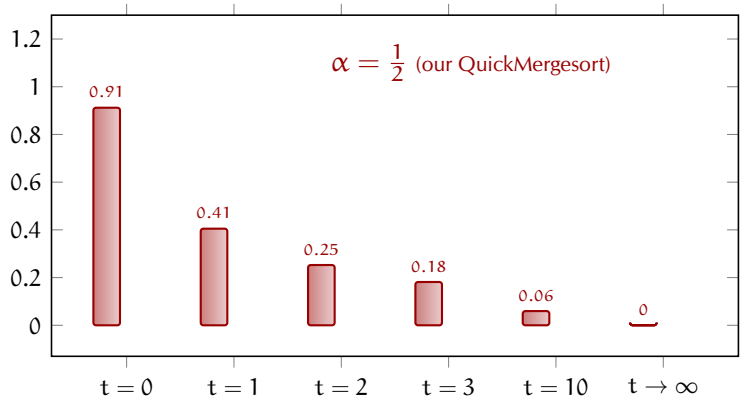

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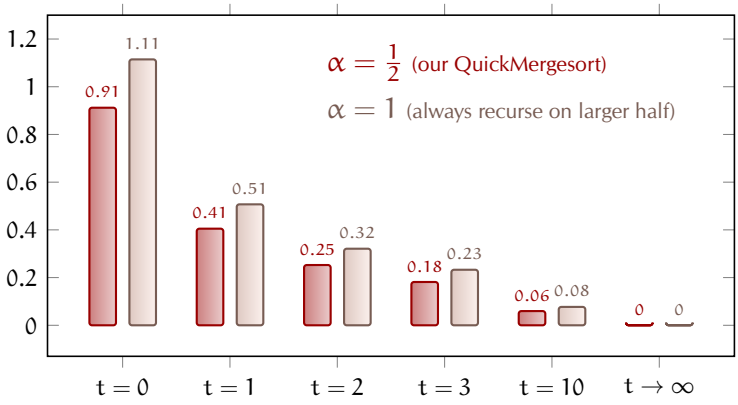

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- QuickMergesort sorts **in-place** with close-to-**optimal #comparisons** (but not a stable sort)
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- variance
- influence of fluctuations in Mergesort?
- limit distributions?

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NO
MERGE
AREA

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