Sesquickselect

One and a Half Pivots for Cache-Efficient Selection

Sebastian Wild

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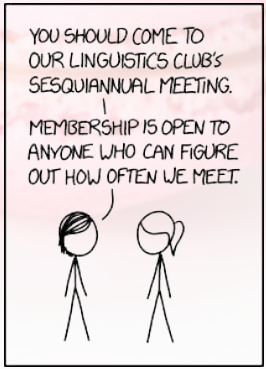
joint work with

Conrado Martínez and Markus Nebel

UNIVERSITY OF WATERLOO



ANALCO 2019

16th Meeting on Analytic Algorithmics and Combinatorics

Outline

1 Introduction

2 Random Ranks

3 Linear Ranks

4 Sesquickselect

5 Conclusion

What happened in quicksort?

- **since Java 7: dual-pivot** quicksort `Arrays.sort(int[])`



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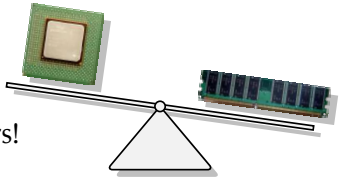

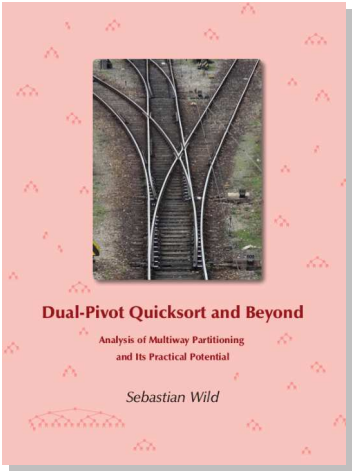

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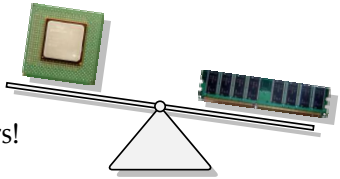

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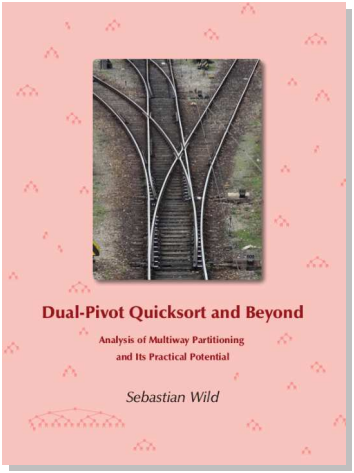

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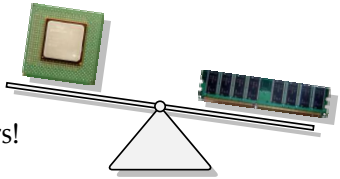

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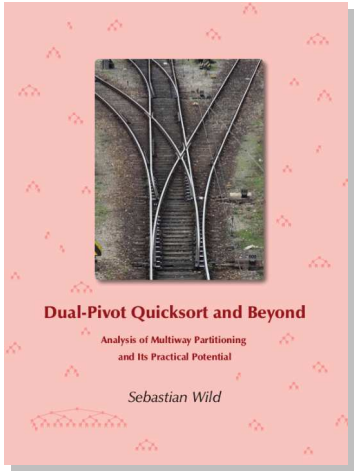



Is multiway partitioning also helpful for quickselect?

$\rightsquigarrow$ consider **memory transfer** volume! ← not comparisons!

Selection

The Selection Problem

Given an **unsorted** array `A[1..n]` of n elements and $m \in [1..n]$, find the **m-th smallest element** in `A`.

`A[m]` after sorting

Algorithms

- (obvious: `sort(A); return A[m];`)
- **Hoare 1961:** quickselect
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- **Floyd & Rivest 1975:** asymptotically optimal #cmps (randomized in expectation)
  - quickselect with pivots from large sample $\Theta(n^{2/3}\log^{1/3}(n))$ elements

quickselect remained method of choice in programming libraries

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YBB-partitioning



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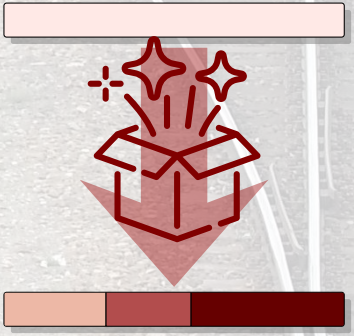

Yaroslavskiy-Bentley-Bloch partitioning

- Split array into 3 regions: small, medium, large

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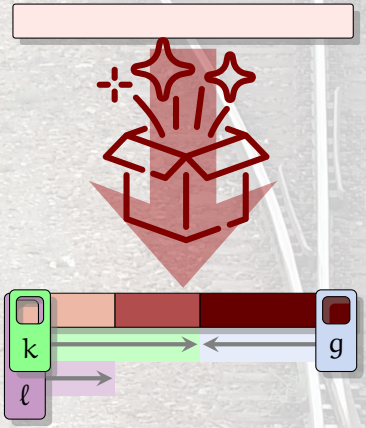

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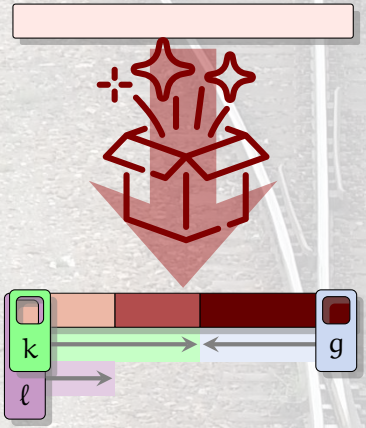

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Costs:

$\propto$ cache misses

- count scanned *elements*

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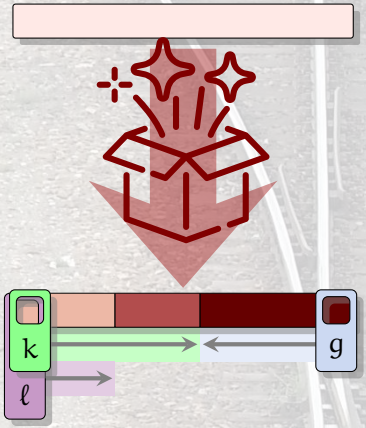

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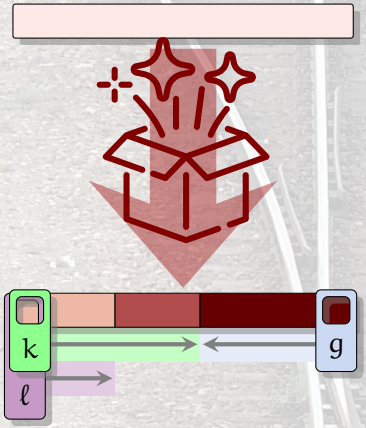

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(comparisons more complicated $\rightsquigarrow$ ignore today)

Recurrence for costs

Dual-pivot quickselect

1. Choose pivots $P_1 < P_2$.
2. Partition around P_1, P_2 $\rightsquigarrow$
3. Recurse in one subproblem
   (or terminate if $m = R_{1,2}$).

Analysis

- $S_{n,m}$: **(random)** number of scanned elements to select mth smallest out of n
- **Recurrence** for the **distribution** of $S_{n,m}$:

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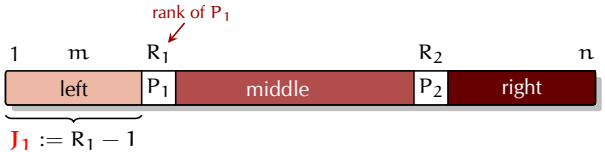

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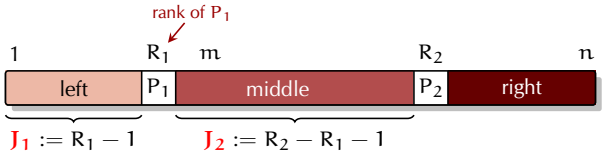

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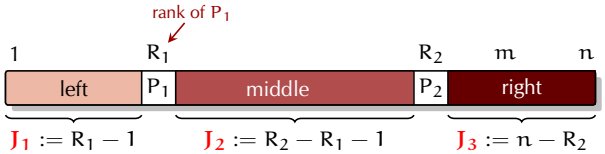

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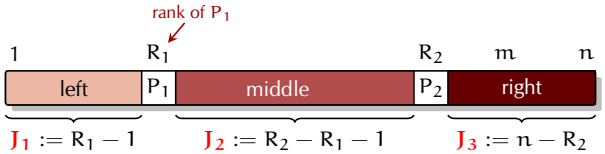

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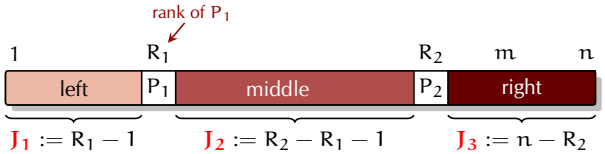

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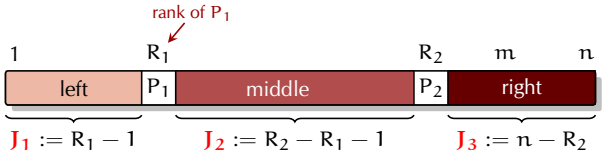

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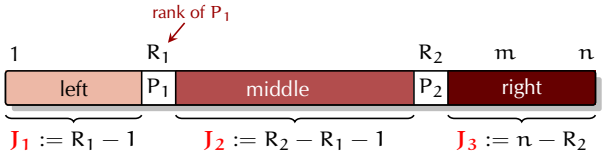

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$$\overset{\text{general case}}{=} T_{n,m} + \sum_{\ell=1}^{s} \mathbb{1}_{\{R_{\ell-1} < m < R_\ell\}} \cdot S_{J_\ell,\, m-R_{\ell-1}} \qquad (R_0 = 0,\ R_s = n+1)$$

(s: s segments)

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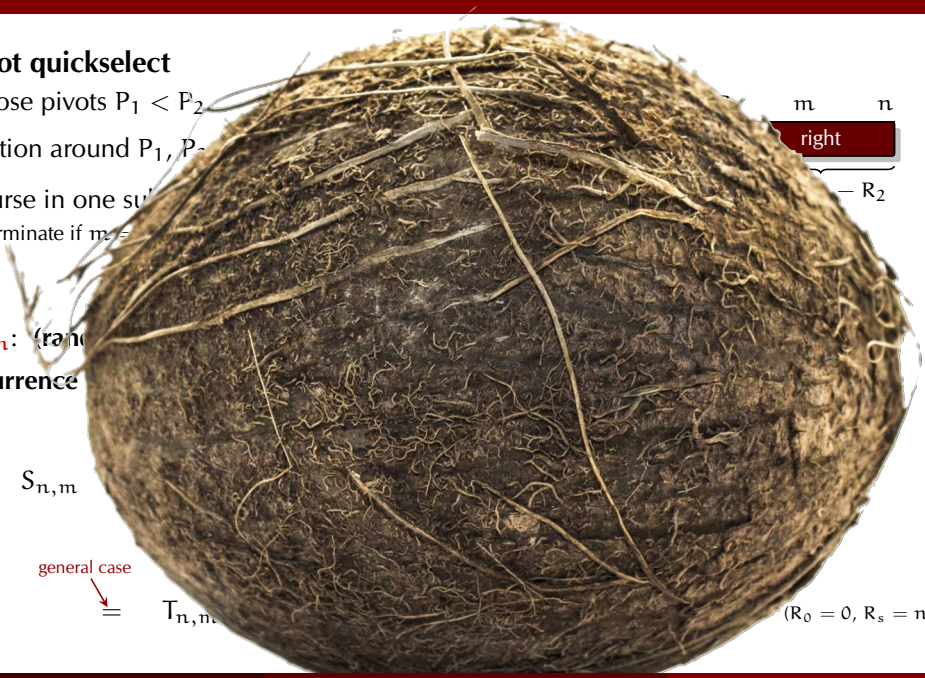

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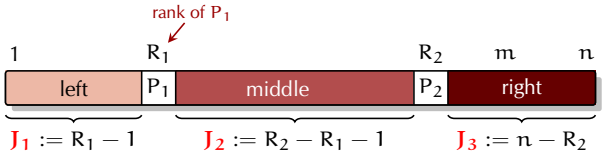

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$$S_{n,m} \overset{\mathcal{D}}{=} \underbrace{(n + J_1)}_{\text{partitioning costs}} + \begin{cases} S_{J_1,m} & \text{if } m \text{ lies in left subarray} \\ S_{J_2,m-R_1} & \text{if } m \text{ lies in middle subarray} \\ S_{J_3,m-R_2} & \text{if } m \text{ lies in right subarray} \end{cases}$$

$$\overset{\text{general case}}{=} T_{n,m} + \sum_{\ell=1}^{s} \mathbb{1}_{\{R_{\ell-1} < m < R_\ell\}} \cdot S_{J_\ell,\, m-R_{\ell-1}} \qquad (R_0 = 0,\ R_s = n+1)$$

(s: s segments)

Outline

1 Introduction

2 Random Ranks

3 Linear Ranks

4 Sesquickselect

5 Conclusion

Random-ranks recurrence



Complication:
2 parameters n, m

$\rightsquigarrow$

Simplification:
Random rank $m = M_n \overset{\mathcal{D}}{=} \text{Uniform}\{1, \dots, n\}$

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conditionally uniform

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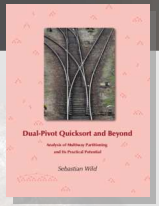
$\rightsquigarrow$ asymptotic for $\mathbb{E}[\bar{S}_n]$ using *distributional master theorem*

Distributional Master Theorem



Distributional Master Theorem

Distributional Master Theorem (DMT) (Theorem 2.76)



Yields asymptotic approximation for $\mathbb{E}[C_n]$ when

Distributional Master Theorem

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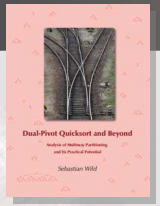

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1. C_n satisfies D & C-recurrence with random subproblem sizes

$$C_n \overset{\mathcal{D}}{=} T_n + \sum_{r=1}^{s} A_r^{(n)} \cdot C_{J_r^{(n)}}^{(r)}$$

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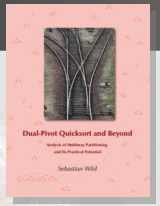

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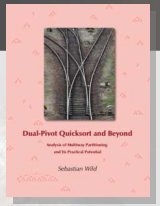

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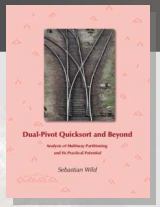

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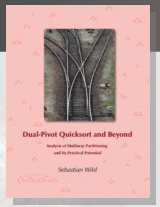

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Hölder continuous

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Results for random ranks

Results

1. YBB quickselect **11% less** scanned elements than standard quickselect (no pivot sampling)
2. more than 2 pivots **not** helpful (Care needed for fair comparison → paper.)

⇝ full results for random ranks available
including pivot sampling, e.g., median of 3

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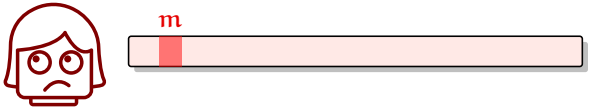

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why shoot for median?



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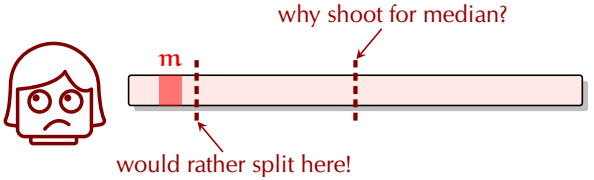

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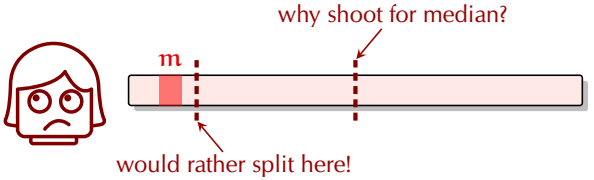

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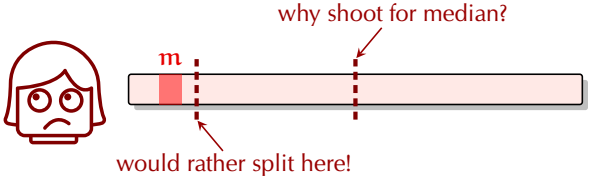

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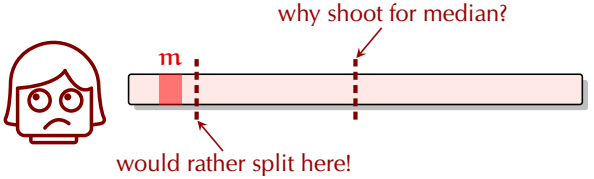

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inevitably learn more about m
than in which subproblem it lies

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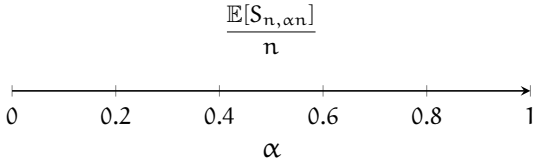
Linear ranks



back to 2 parameters …

consider $\boxed{m = \alpha \cdot n}$ and $n \to \infty$

relative rank $\in (0, 1)$



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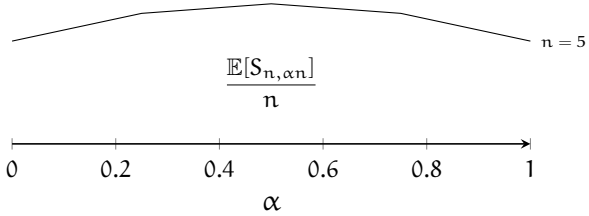

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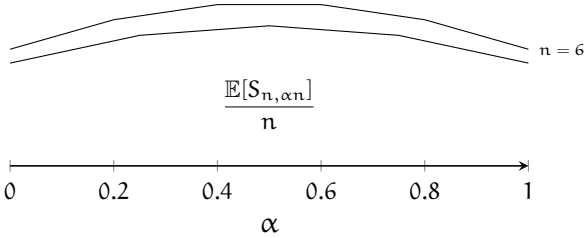

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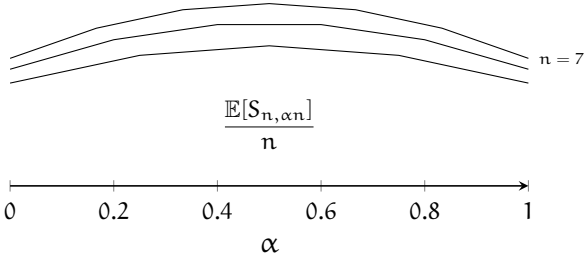

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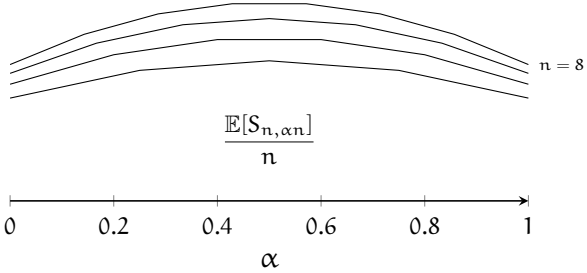

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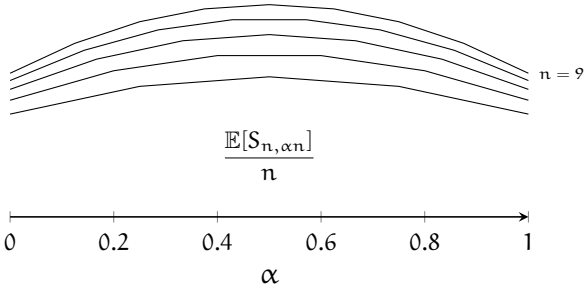

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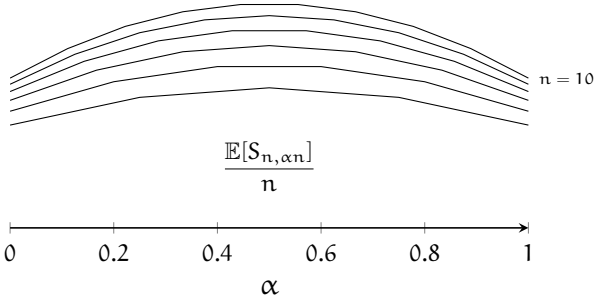

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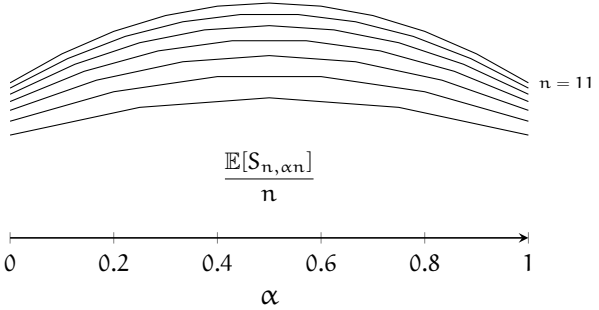

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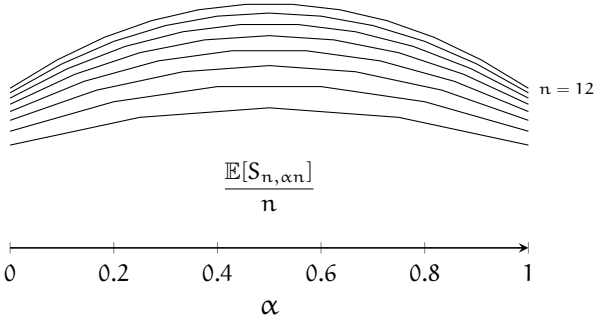

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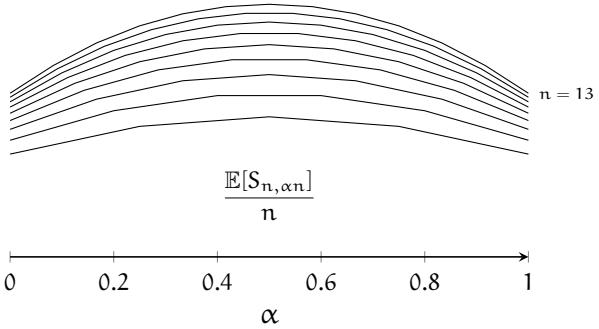

Linear ranks



back to 2 parameters …

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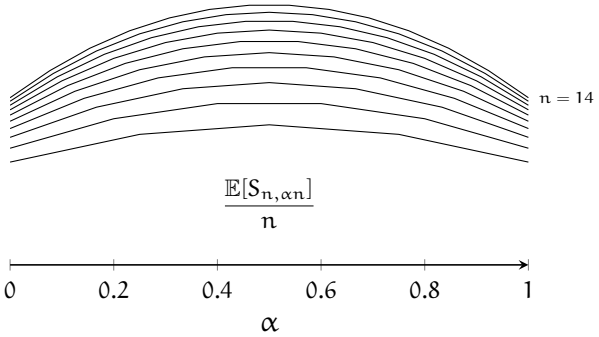

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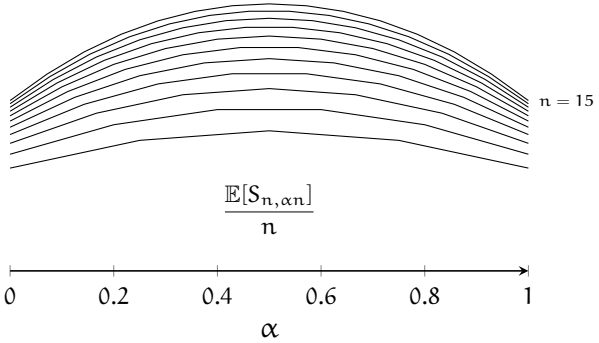

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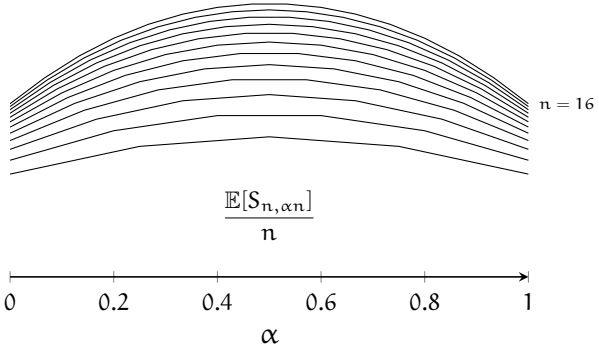

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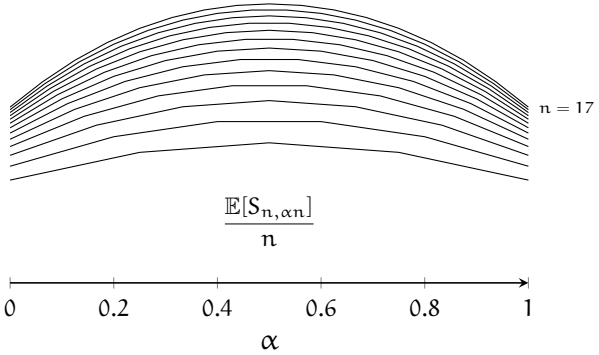

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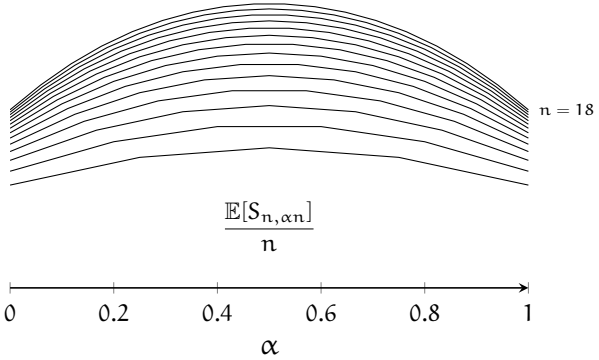

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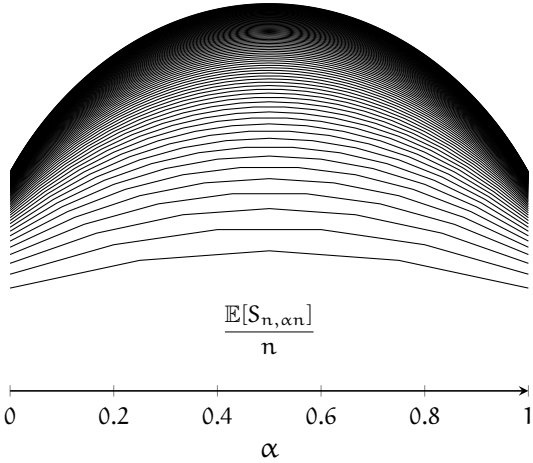

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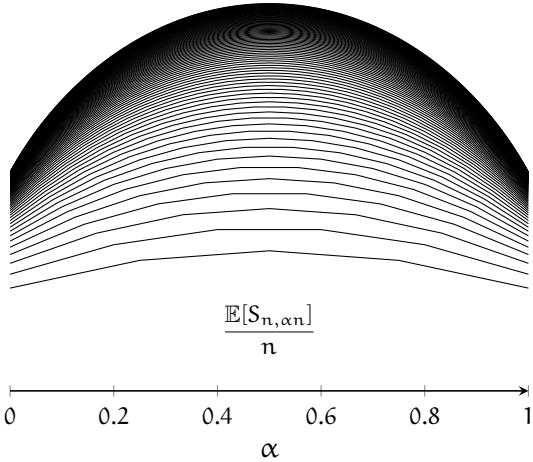

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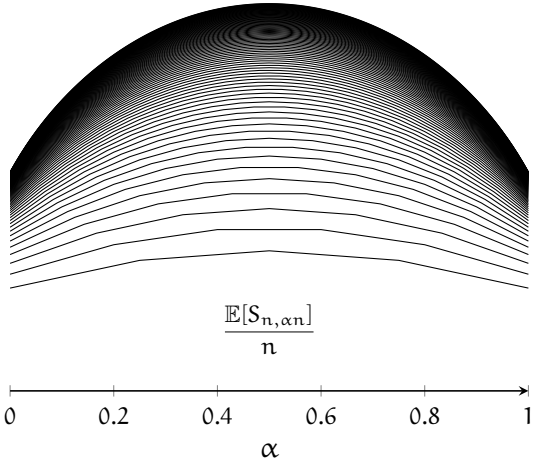

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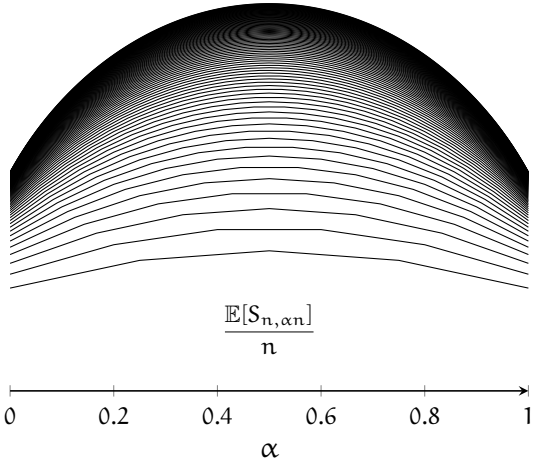

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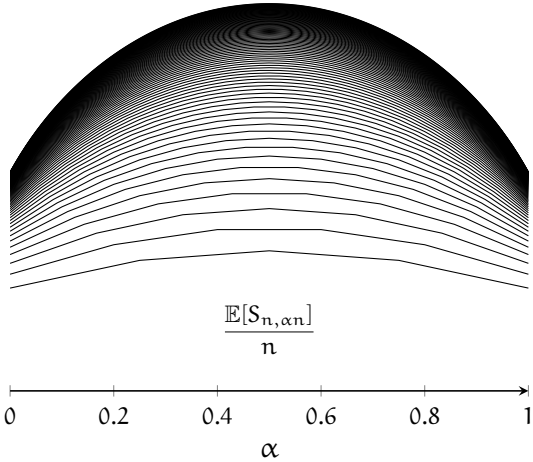

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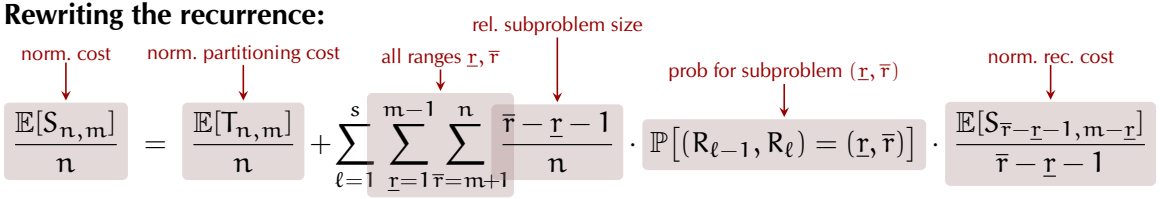

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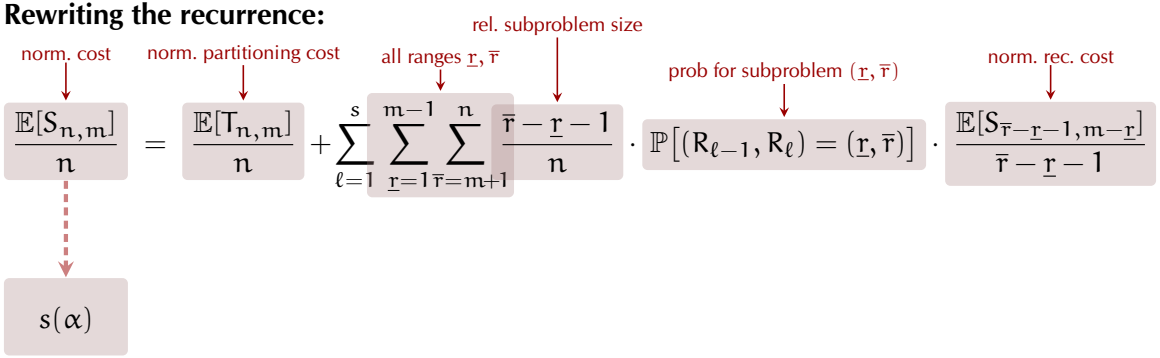

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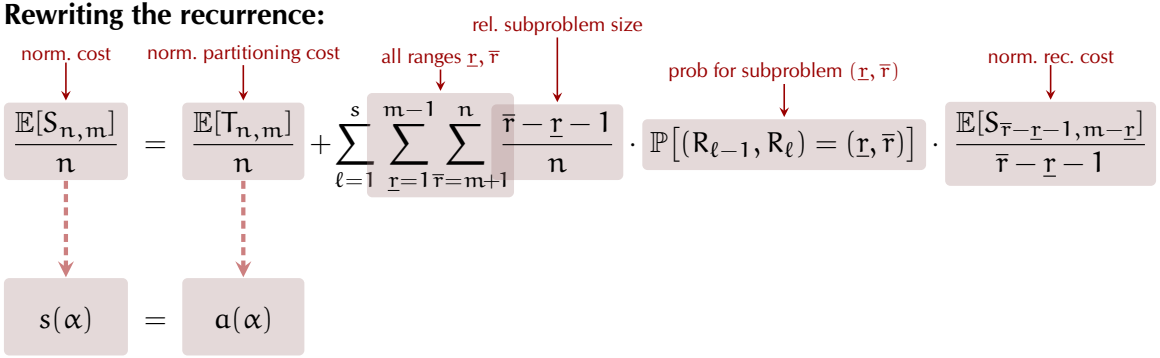

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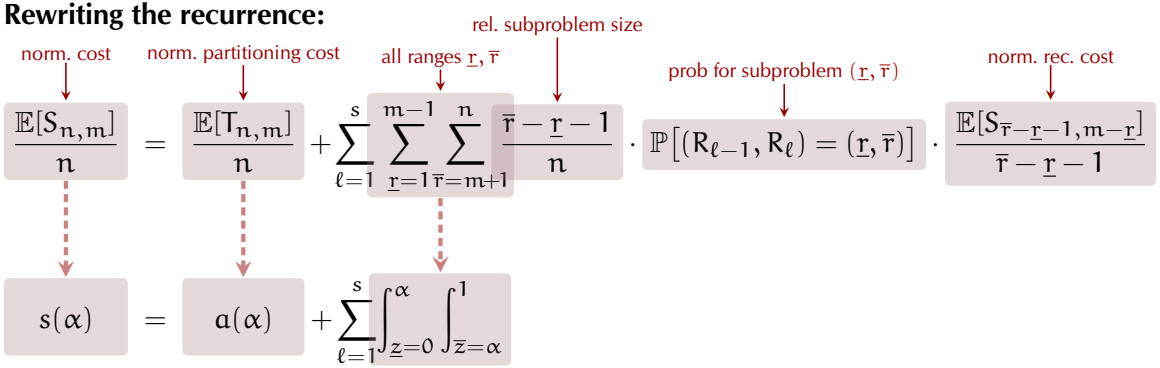

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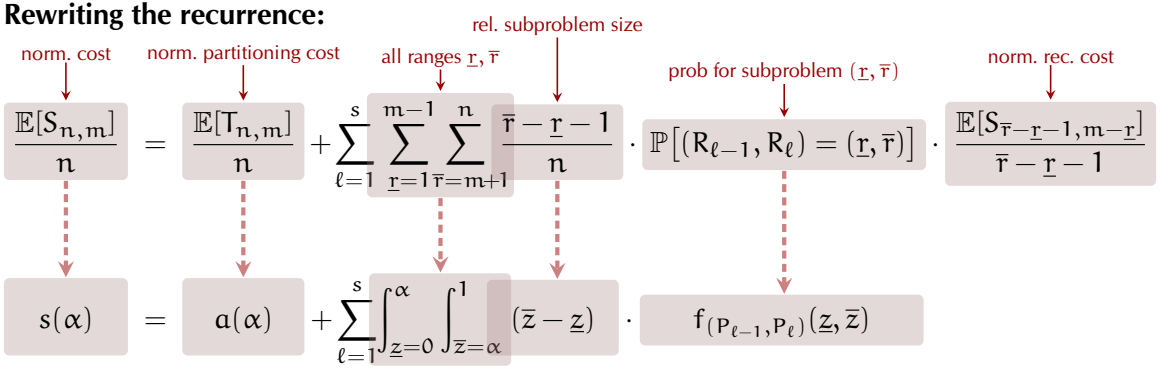

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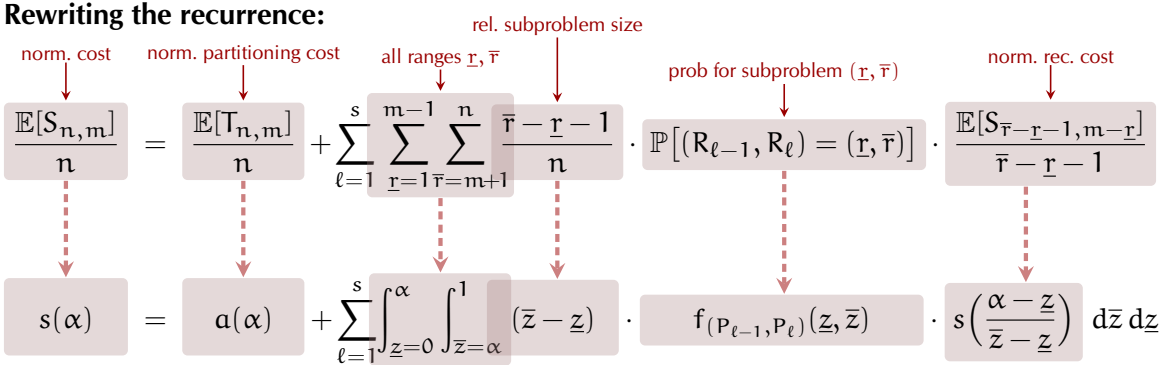

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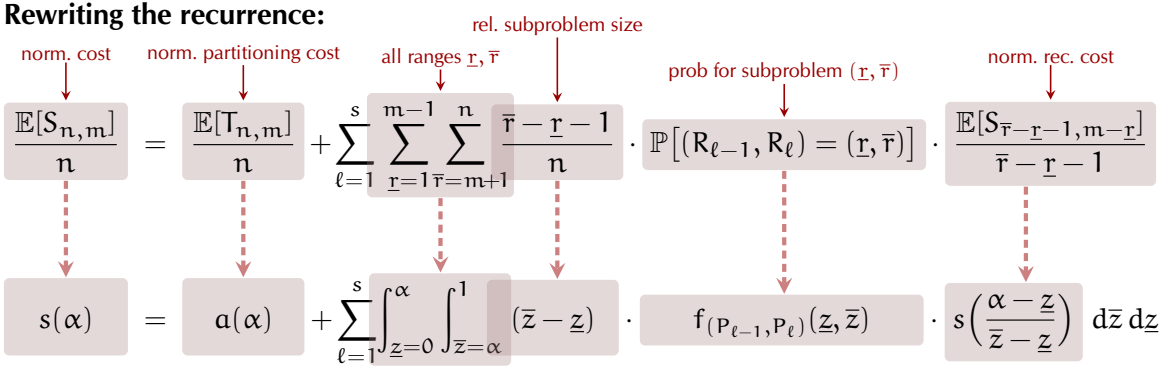

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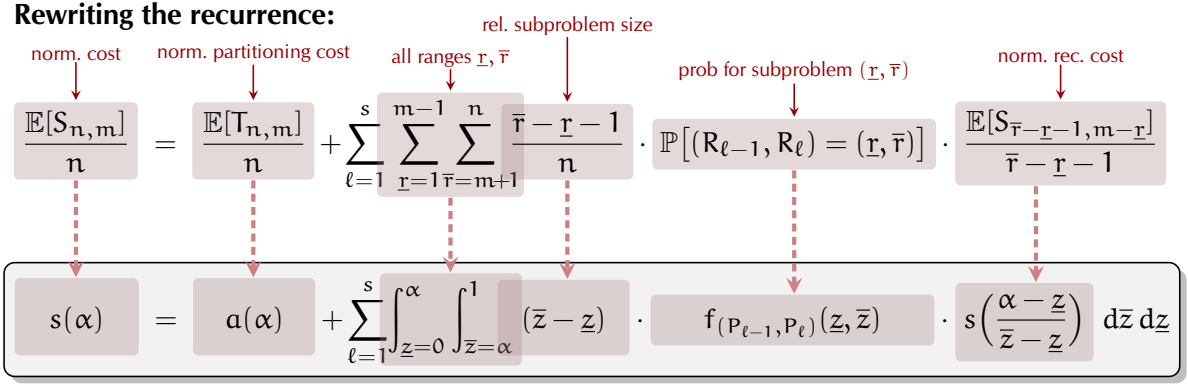

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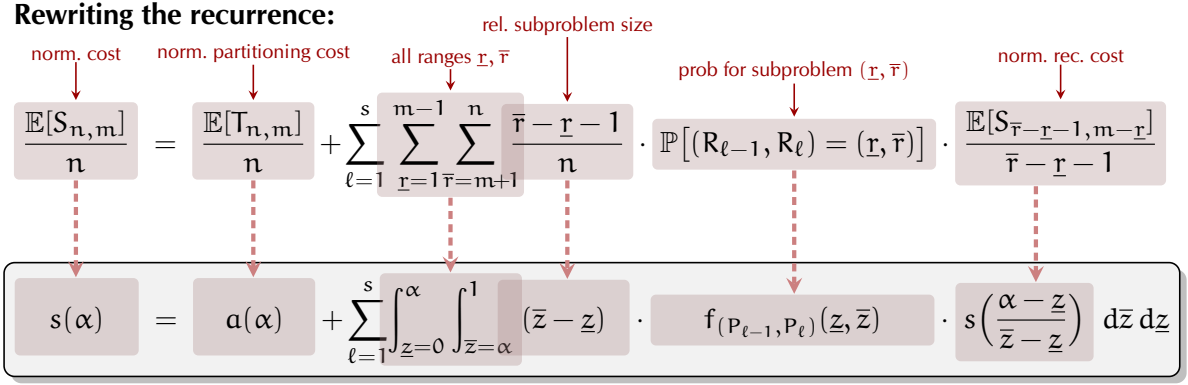

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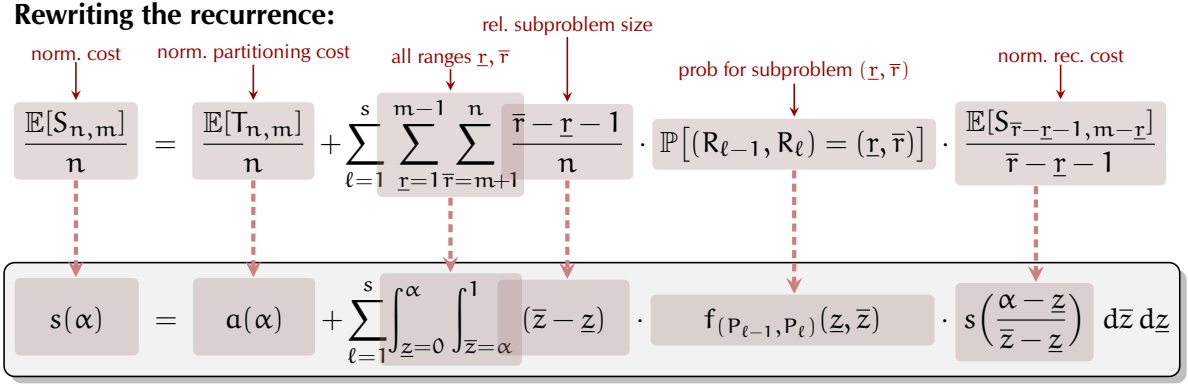

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Goal: Bound $r_{n,m} := \left|\mathbb{E}[S_{n,m}] - s(\frac{m}{n})n\right|$ (we show: $O(n^{1-\varepsilon})$)

- find recurrence for $r_{n,m}$
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The gory details



1. bounding difference of sum and integral
   - split range of integration
     - where transform varies to quickly
     - where we hit discontinuities
   - use local limit law for Dirichlet-multinomial distribution
2. bound $r_{n,m}$
   - bound expected subproblem size (geometrically decreasing)

   $\rightsquigarrow$ dominated by toll function

Solving the integral equation

2 Compute limiting function

$$s(\alpha) \;=\; a(\alpha) \;+\; \sum_{\ell=1}^{s} \int_{\underline{z}=0}^{\alpha} \int_{\overline{z}=\alpha}^{1} (\overline{z}-\underline{z}) \cdot f_{(P_{\ell-1},P_\ell)}(\underline{z},\overline{z}) \cdot s\Big(\frac{\alpha-\underline{z}}{\overline{z}-\underline{z}}\Big)\, d\overline{z}\, d\underline{z}$$

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general case quite intractable, but

- analytic solutions for simplest cases
- good numeric approximation

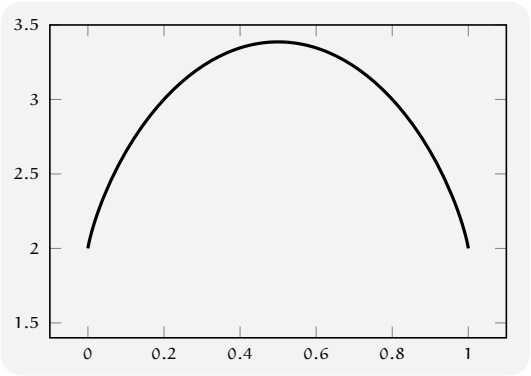
Linear-ranks Results



Analytic solutions for simplest cases

- **standard quickselect** $s(\alpha) = 2\,h(\alpha) + 2$

$$h(\alpha) = -\alpha \ln(\alpha) - (1-\alpha)\ln(1-\alpha)$$



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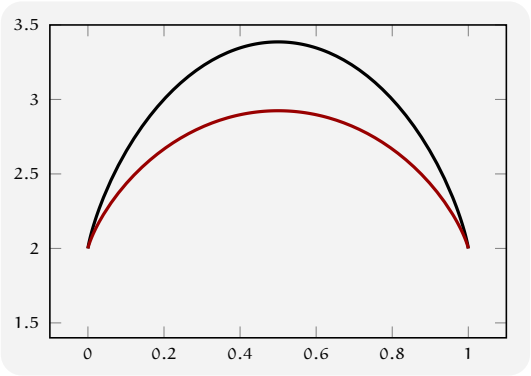

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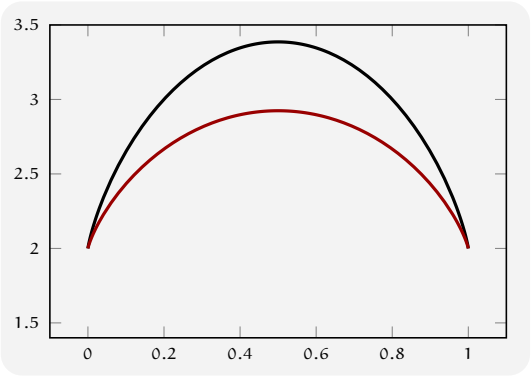

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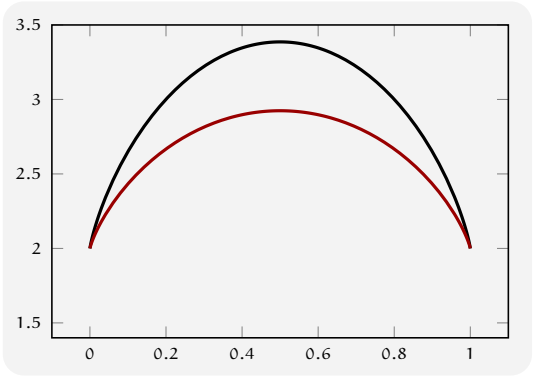
Linear-ranks Results



Analytic solutions for simplest cases



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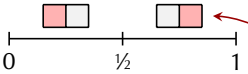
$$h(\alpha) = -\alpha \ln(\alpha) - (1-\alpha)\ln(1-\alpha)$$



- **YBB-quickselect** $s(\alpha) = 1.\overline{3}\,h(\alpha) + 2$

- **proportion from 2**

$$s(\alpha) = \begin{cases} -4.29394\alpha\ln(1-\alpha) + 4.29394\ln(1-\alpha) - 1.04899\alpha^3 - 3.14697\alpha^2 + 6.65176\alpha - 2\alpha\ln(\alpha) + 1.5 & \alpha \leqslant 0.5 \\ 2\alpha\ln(1-\alpha) - 2\ln(1-\alpha) + 1.04899\alpha^3 - 6.29394\alpha^2 + 2.78914\alpha + 4.29394\alpha\ln(\alpha) + 3.95581 & \alpha > 0.5 \end{cases}$$



sample $k = 2$ elements;
use smaller if $\alpha \leqslant 0.5$,
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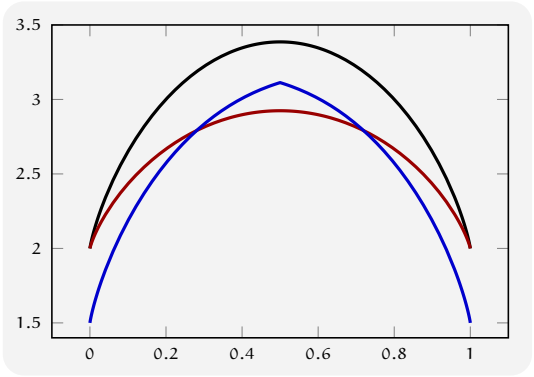

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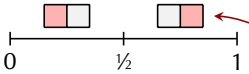

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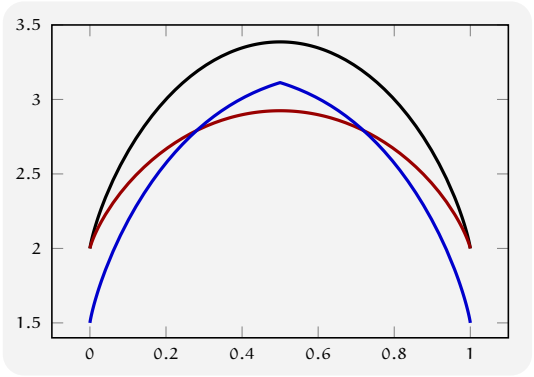

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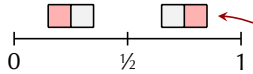

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Can we get the best of both worlds?

Outline

1 Introduction

2 Random Ranks

3 Linear Ranks

4 Sesquickselect

5 Conclusion

Sesqui-pivoting

⇝ *Use min, max **or** both as pivots*

"1.5 pivots"

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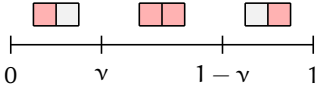

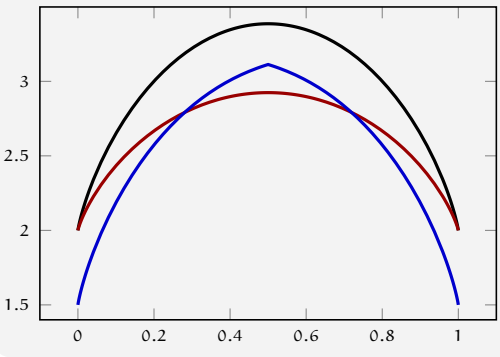

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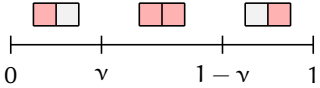
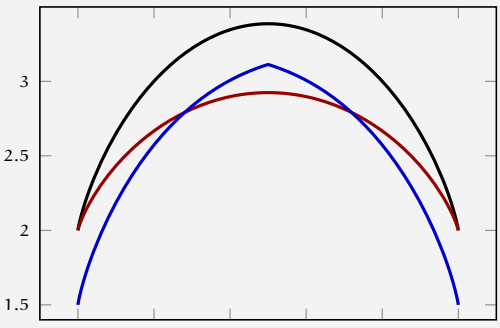


- when to switch? $\nu \in (0, \frac{1}{2})$



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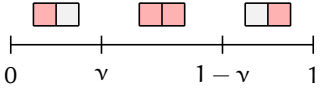

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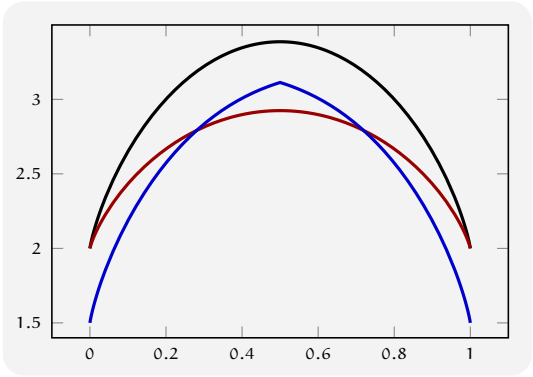
solve $\int$-eq symbolically in ν



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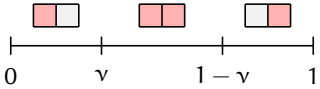

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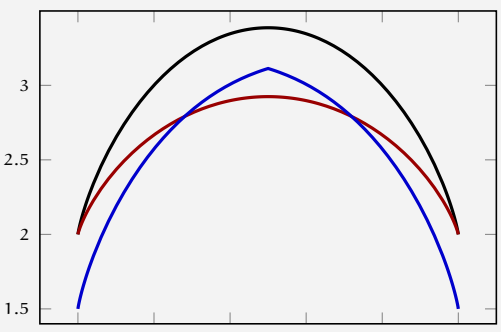

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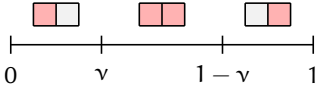

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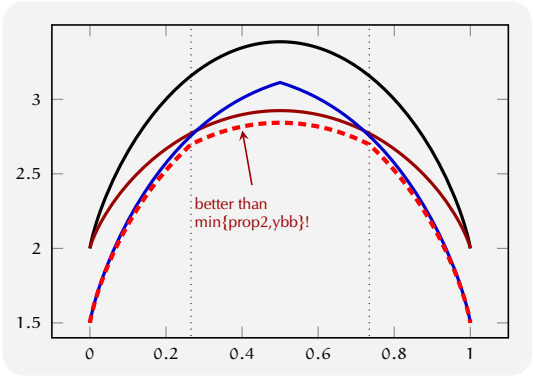

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Larger samples sizes

Sample size $k = 2$ *very small* ...

SQSk: Allow to choose 1 or 2 pivots out of k (depending on α)

code on github

- Solving differential equations seems intractable $\rightsquigarrow$ numeric approximations

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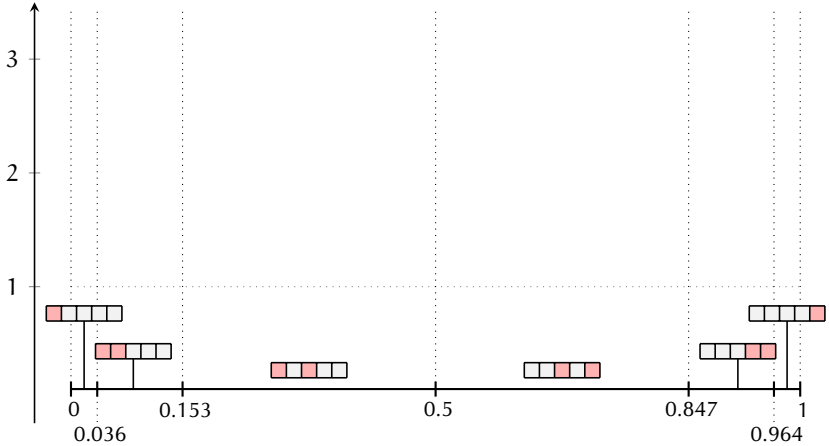
Conjectured (approx) optimal for $k = 5$:



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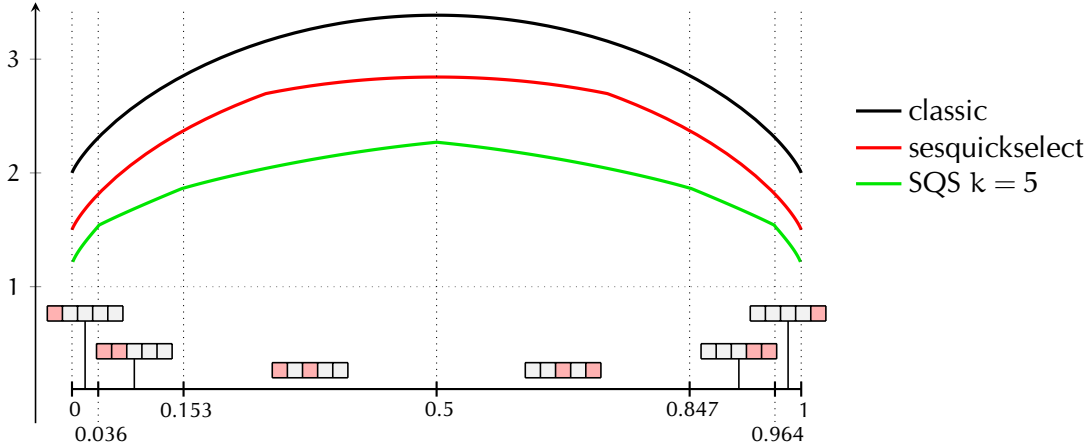

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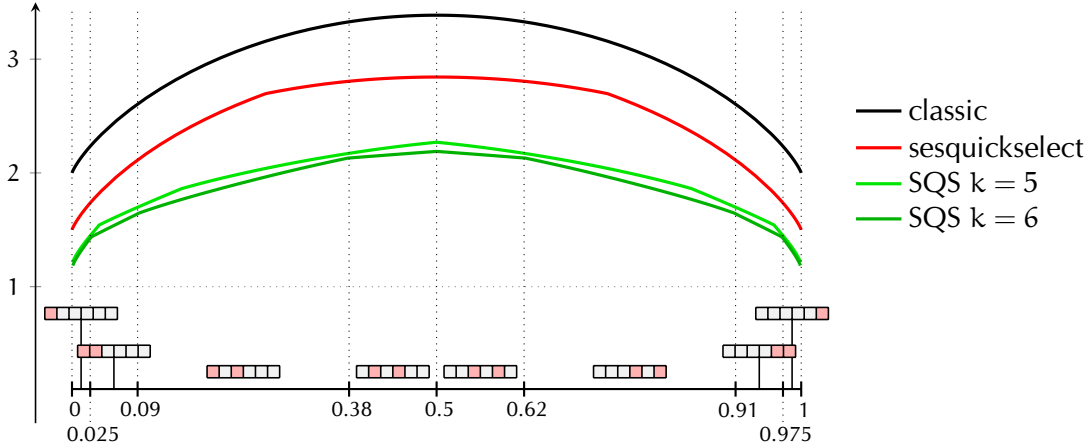

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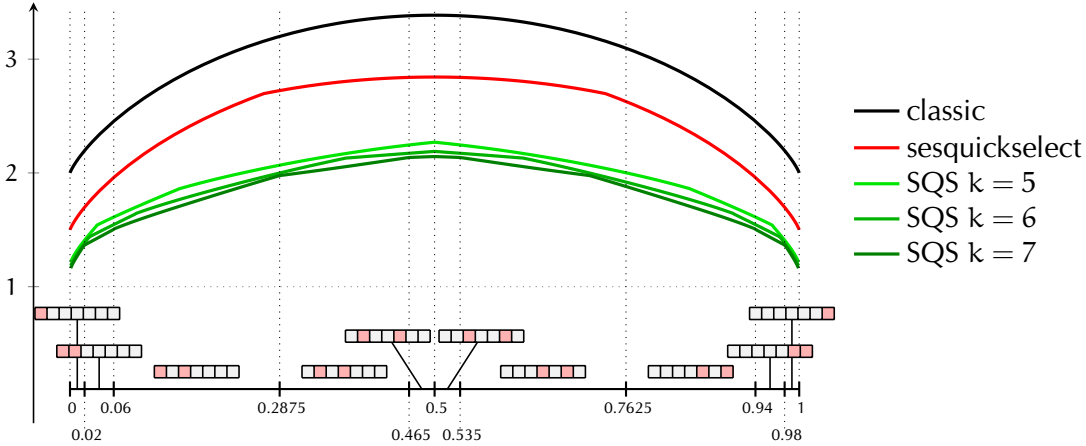

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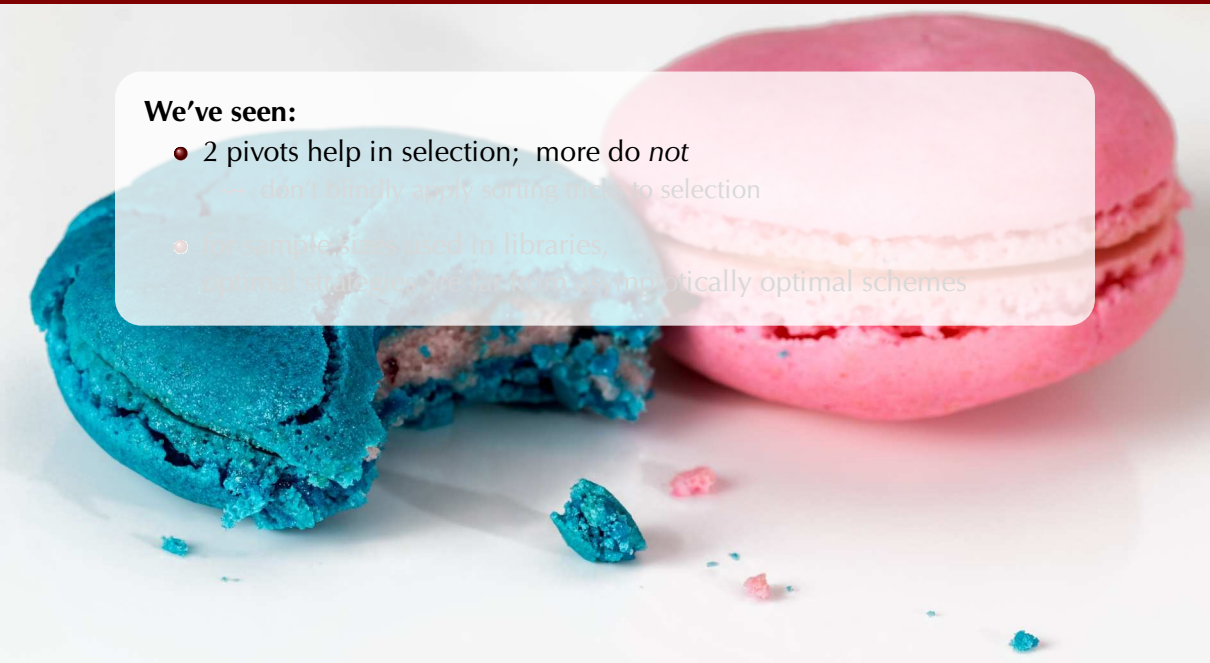

We've seen:

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  ↝ don't blindly apply sorting tricks to selection
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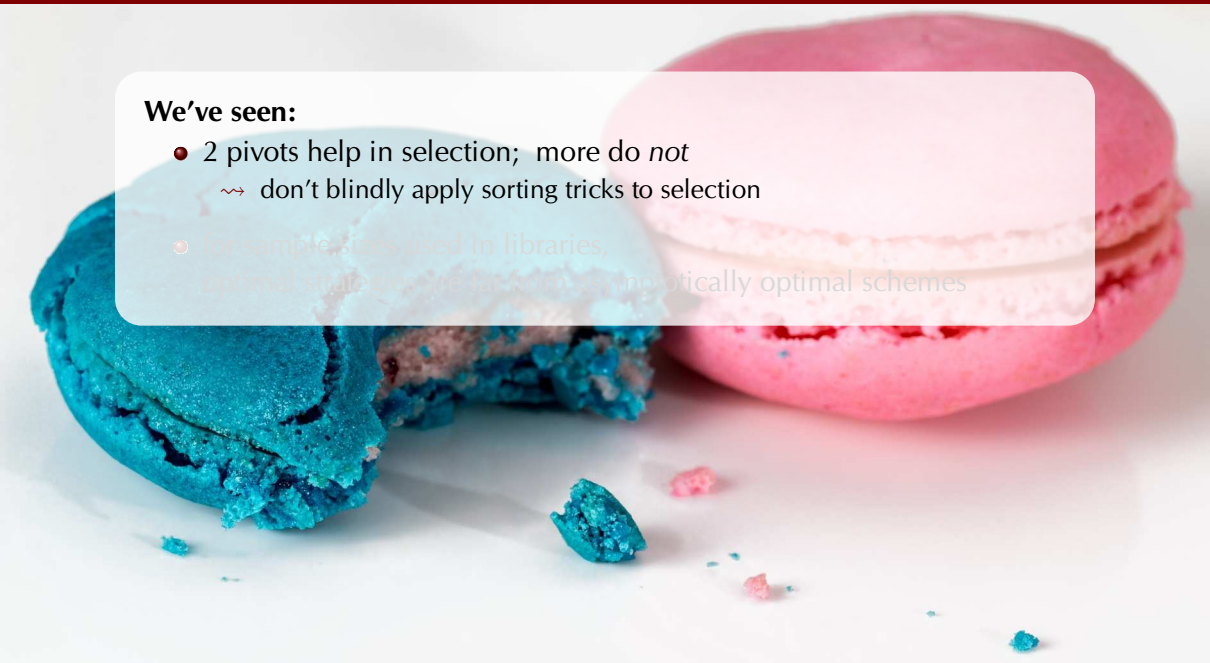

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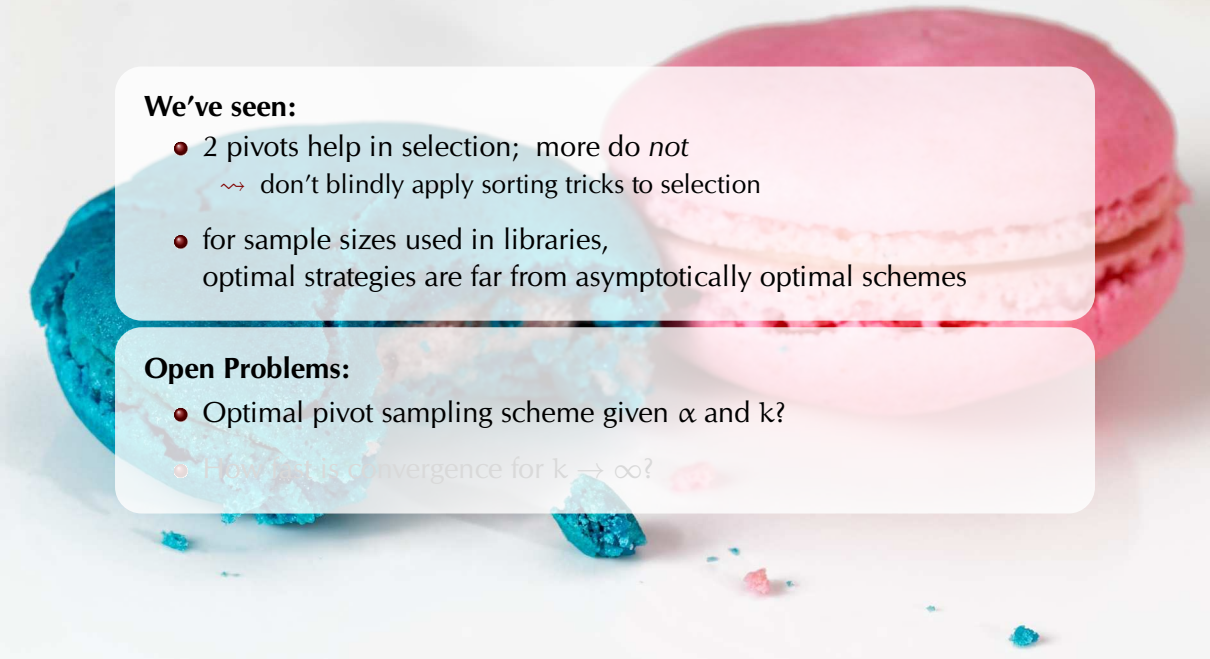

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- Optimal pivot sampling scheme given α and k?
- How fast is convergence for $k \to \infty$?

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KEEP
OFF
MEDIAN

Icons made by *Freepik*, *Gregor Cresnar*, *Those Icons*, *Smashicons*, *Good Ware*, *Pause08*, and *Madebyoliver* from `www.flaticon.com`.

3-way railway switch by *Asiano*
processor by *Konstantin Lanzet*
RAM module by *Utente:Sassospicco*
keep off median sign by *Michael Rivera*
Comic by *Randall Munroe*: xkcd 1602
Other photos from `www.pixabay.com`.