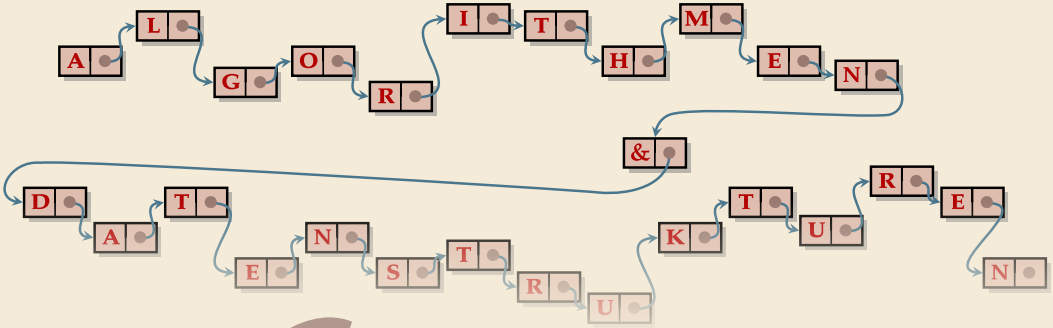


6 Algorithm Science

Algorithmen & Datenstrukturen · Sommersemester 2026

Prof. Dr. Sebastian Wild

CS210 (Sommersemester 2026)
Philipps-Universität Marburg
version `2026-05-18 09:54`

Outline

6 Algorithm Science

6.1 Algorithm Science

Modern take on studying algorithms

algorithm science

The application of the ***scientific method*** *for the design and analysis of algorithms – developing* ***mathematical models*** *for characteristics of algorithms and their inputs, formulating hypotheses relevant to the study of performance on specific classes of computers and systems, running* ***experiments to validate the hypotheses****, and iterating as appropriate with the goal of improving performance.*

https://sedgewick.io/ideas/#algorithm-science

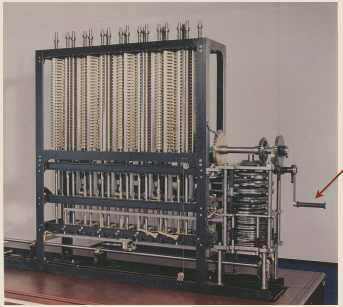
The Scientific Method

Hypothese — Vorhersage — Experimentelle Überprüfung

Feynman on Scientific Method.
https://youtu.be/EYPapE-3FRw

Running time

" As soon as an Analytic Engine exists, it will necessarily guide the future course of the science. Whenever any result is sought by its aid, the question will arise—By what course of calculation can these results be arrived at by the machine in the shortest time? " — *Charles Babbage (1864)*



how many times do you have to turn the crank?

Analytic Engine

Cast of characters

Programmer needs to develop a working solution.

Student might play any or all of these roles someday.

Client wants to solve problem efficiently.

Theoretician wants to understand.

Reasons to analyze algorithms

Predict performance.

Compare algorithms.

Provide guarantees.

Understand theoretical basis.

this course (COS 226)

theory of algorithms (COS 423)

Primary practical reason: avoid performance bugs.

client gets poor performance because programmer did not understand performance characteristics

The challenge

Q. Will my program be able to solve a large practical input?

Insight. [Knuth 1970s] Use scientific method to understand performance.

Scientific method applied to analysis of algorithms

A framework for predicting performance and comparing algorithms.

Scientific method.

- Observe some feature of the natural world.
- Hypothesize a model that is consistent with the observations.
- Predict events using the hypothesis.
- Verify the predictions by making further observations.
- Validate by repeating until the hypothesis and observations agree.

Principles.

- Experiments must be reproducible.
- Hypotheses must be falsifiable.

Feature of the natural world. Computer itself.

6.2 Laufzeitexperimente

Ein (erstes) Wiedersehen: 3-SUM

3-SUM-Problem

- **Gegeben:** $A[0..n)$ with $A[i] \in \mathbb{Z}$ for $0 \leq i < n$.
- **Ziel:** Anzahl c von Tripeln aus A mit Summe 0

Parameter:

- Eingabegröße: # Zahlen n

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Mit der einfachsten Lösung hatten wir auch schon vorgearbeitet:

1. **Algorithmische Idee**
   Alle Tripel durchprobieren
2. **Pseudocode**
   (3 geschachtelte for-Schleifen)
   siehe nächste Folie
3. **Korrektheitsbeweis**
   direkt nach Definition von c
4. **Analyse**
   # Ausführungen von Zeilen 5–6

$$\frac{(n-2)(n-1)n}{6} \sim \frac{1}{6}n^3 = \Theta(n^3)$$

Brute-Force 3-SUM Code

10^{-9} s $\frac{1}{6} \cdot (1000)^3 = \frac{1}{6} \cdot 10^9$

Pseudocode

```
1  c = 0
2  for i = 0, ..., n − 3                    (i, j, k)
3      for j = i + 1, ..., n − 2
4          for k = j + 1, ..., n − 1
5              t = A[i] + A[j] + A[k]
6              if t == 0 then c := c + 1
7          end for
8      end for
9  end for
10 return c
```

Java-Implementierung

```
1  import edu.princeton.cs.algs4.*;
2
3  public class ThreeSum {
4      public static int count(int[] a) {
5          int n = a.length;
6          int count = 0;
7          for (int i = 0; i < n-2; i++)
8              for (int j = i+1; j < n-1; j++)
9                  for (int k = j+1; k < n; k++)
10                     if (a[i] + a[j] + a[k] == 0)
11                         count++;
12         return count;
13     }
14
15     public static void main(String[] args) {
16         In in = new In(args[0]);
17         int[] a = in.readAllInts();
18         StdOut.println(count(a));
19     }
20 }
```

Clicker Question

Wie lange dauert es, mit `ThreeSum` für $n = 1\,000$ Zahlen die Tripel zu zählen, die sich zu 0 summieren?

A	1.3 ns	**E**	130 ms	**I**	130 min
B	1.3 µs	**F**	1.3 s	**J**	13 h
C	13 µs	**G**	1.3 min	**K**	1.3 Tage
D	13 ms	**H**	13 min	**L**	1.3 Jahre

→ *sli.do/cs210*

Clicker Question

Wie lange dauert es, mit `ThreeSum` für $n = 1\,000$ Zahlen die Tripel zu zählen, die sich zu 0 summieren?

A	~~1.3 ns~~	E	130 ms ✓	I	~~130 min~~
B	~~1.3 µs~~	F	~~1.3 s~~	J	~~13 h~~
C	~~13 µs~~	G	~~1.3 min~~	K	~~1.3 Tage~~
D	~~13 ms~~	H	~~13 min~~	L	~~1.3 Jahre~~

→ *sli.do/cs210*

Measuring the running time

Q. How to time a program?

A. Manual.

```
% java ThreeSum 1Kints.txt
```

tick tick tick

```
70
% java ThreeSum 2Kints.txt
```

tick tick tick tick tick tick tick tick
tick tick tick tick tick tick tick tick
tick tick tick tick tick tick tick tick

```
528
% java ThreeSum 4Kints.txt
```

tick tick tick tick tick tick tick tick
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```
4039
```

Measuring the running time

Q. How to time a program?

A. Automatic.

public class Stopwatch (part of stdlib.jar)	
Stopwatch()	*create a new stopwatch*
double elapsedTime()	*time since creation (in seconds)*

```
public static void main(String[] args)
{
   In in = new In(args[0]);
   int[] a = in.readAllInts();
   Stopwatch stopwatch = new Stopwatch();
   StdOut.println(ThreeSum.count(a));
   double time = stopwatch.elapsedTime();
   StdOut.println("elapsed time " + time);
}
```

Empirical analysis

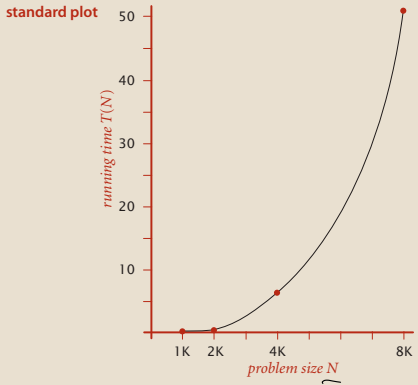
Run the program for various input sizes and measure running time.

N	time (seconds) †
250	0.0
500	0.0
1,000	0.1
2,000	0.8
4,000	6.4
8,000	51.1
16,000	?

$\frac{1}{6} n^3 \cdot 10^{-9}$

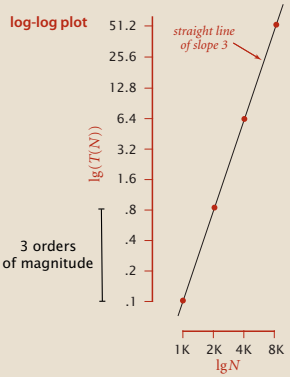
Data analysis

Standard plot. Plot running time $T(N)$ vs. input size N.



Data analysis

Log-log plot. Plot running time $T(N)$ vs. input size N using log-log scale.



$\lg(T(N)) = b \lg N + c$

$b = 2.999$

$c = -33.2103$

$T(N) = a N^b$, *where* $a = 2^c$

Regression. Fit straight line through data points: $a N^b$. (power law, slope)

Hypothesis. The running time is about $1.006 \times 10^{-10} \times N^{2.999}$ seconds.

Prediction and validation

Hypothesis. The running time is about $1.006 \times 10^{-10} \times N^{2.999}$ seconds.

"order of growth" of running time is about N^3 [stay tuned]

Predictions.

- 51.0 seconds for $N = 8{,}000$.
- 408.1 seconds for $N = 16{,}000$.

Observations.

N	time (seconds) †
8,000	51.1
8,000	51.0
8,000	51.1
16,000	410.8

validates hypothesis!

Doubling hypothesis

Doubling hypothesis. Quick way to estimate b in a power-law relationship.

Run program, doubling the size of the input.

N	time (seconds) †	ratio	lg ratio
250	0.0		–
500	0.0	4.8	2.3
1,000	0.1	6.9	2.8
2,000	0.8	7.7	2.9
4,000	6.4	8.0	3.0
8,000	51.1	8.0	3.0

lg (6.4 / 0.8) = 3.0

seems to converge to a constant b ≈ 3

$$\frac{T(2N)}{T(N)} = \frac{a(2N)^b}{aN^b} = 2^b$$

$T(N) \approx a \cdot N^3$

$\frac{T(2N)}{T(N)} = \frac{a(2N)^3}{aN^3}$

$= \frac{8N^3}{N^3} = 8$

Hypothesis. Running time is about $a N^b$ with b = lg ratio.

Caveat. Cannot identify logarithmic factors with doubling hypothesis.

Doubling hypothesis

Doubling hypothesis. Quick way to estimate b in a power-law relationship.

Q. How to estimate a (assuming we know b) ?

A. Run the program (for a sufficient large value of N) and solve for a.

N	time (seconds) †
8,000	51.1
8,000	51.0
8,000	51.1

$$51.1 = a \times 8000^3$$

$$\Rightarrow a = 0.998 \times 10^{-10}$$

Hypothesis. Running time is about $0.998 \times 10^{-10} \times N^3$ seconds.

almost identical hypothesis
to one obtained via linear regression

Experimental algorithmics

System independent effects.

- Algorithm.
- Input data.

determines exponent in power law

System dependent effects.

- Hardware: CPU, memory, cache, ...
- Software: compiler, interpreter, garbage collector, ...
- System: operating system, network, other apps, ...

determines constant in power law

Bad news. Difficult to get precise measurements.

Good news. Much easier and cheaper than other sciences.

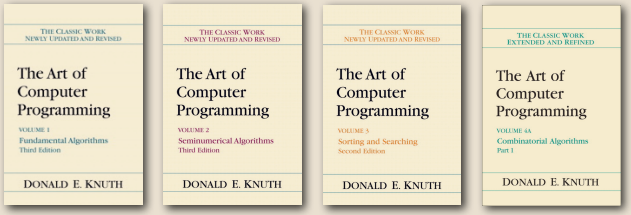
e.g., can run huge number of experiments

6.3 Mathematische Analyse von Algorithmen

Mathematical models for running time

Total running time: sum of cost × frequency for all operations.

- Need to analyze program to determine set of operations.
- Cost depends on machine, compiler.
- Frequency depends on algorithm, input data.



Donald Knuth
1974 Turing Award

In principle, accurate mathematical models are available.

Cost of basic operations

Challenge. How to estimate constants.

operation	example	nanoseconds †
integer add	`a + b`	2.1
integer multiply	`a * b`	2.4
integer divide	`a / b`	5.4
floating-point add	`a + b`	4.6
floating-point multiply	`a * b`	4.2
floating-point divide	`a / b`	13.5
sine	`Math.sin(theta)`	91.3
arctangent	`Math.atan2(y, x)`	129.0
...	...	...

† Running OS X on Macbook Pro 2.2GHz with 2GB RAM

Cost of basic operations

Observation. Most primitive operations take constant time.

operation	example	nanoseconds †
variable declaration	`int a`	c_1
assignment statement	`a = b`	c_2
integer compare	`a < b`	c_3
array element access	`a[i]`	c_4
array length	`a.length`	c_5
1D array allocation	`new int[N]`	$c_6 N$
2D array allocation	`new int[N][N]`	$c_7 N^2$

Caveat. Non-primitive operations often take more than constant time.

novice mistake: abusive string concatenation

Beispiel: Knuthsche Analyse

Wir betrachten folgenden Code für das (wenig spannende) ***1-SUM-Problem***

```
1 public static int oneSum(int[] a) {
2     int count = 0, N = a.length;      1
3     for (int i = 0; i < N; ++i)       N+1
4         if (a[i] == 0)                N
5             count++;                  [0..N]   worst case N
6     return count;                     1
7 }
```

oneSum(A[0..n))

Wie viele Instruktionen
als Funktion in N?

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```

Wie viele Instruktionen
als Funktion in N?

Zeile Z	Frequenz $f_Z(N)$
2	1
3	$N+1$
4	N
5	$[0..N]$
6	1

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Analyse aller Operationen

Operation	Frequenz	Kosten
Variablendeklaration	3	c_1
Zuweisung	3	c_2
Vergleich $<$	$N+1$	c_3
Vergleich $==$	N	c_4
Array-Zugriff	N	c_5
Inkrement	$[N..2N]$	c_6

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$\rightsquigarrow$ Gesamtkosten

$$T(N) \leq (c_3 + c_4 + c_5 + 2c_6)N + 2(c_1 + c_2) + c_3$$

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4         if (a[i] == 0)
5             count++;
6     return count;
7 }
```

Wie viele Instruktionen als Funktion in N?

Zeile Z	Frequenz $f_Z(N)$
2	1
3	$N+1$
4	N
5	$[0..N]$
6	1

Analyse aller Operationen

Operation	Frequenz	Kosten
Variablendeklaration	3	c_1
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$\rightsquigarrow$ Wählen *abstraktes Kostenmaß*: **# Arrayzugriffe**

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alle Instruktionen zählen i.d.R. zu aufwändig

$\rightsquigarrow$ Wählen *abstraktes Kostenmaß*: **# Arrayzugriffe**

$\rightsquigarrow$ Kosten von 1-SUM: $\sim N$

Simplifying the calculations

> *" It is convenient to have a measure of the amount of work involved in a computing process, even though it be a very crude one. We may count up the number of times that various elementary operations are applied in the whole process and then given them various weights. We might, for instance, count the number of additions, subtractions, multiplications, divisions, recording of numbers, and extractions of figures from tables. In the case of computing with matrices most of the work consists of multiplications and writing down numbers, and we shall therefore only attempt to count the number of multiplications and recordings. "* — *Alan Turing*

ROUNDING-OFF ERRORS IN MATRIX PROCESSES

By A. M. TURING

(*National Physical Laboratory, Teddington, Middlesex*)

[Received 4 November 1947]

SUMMARY

A number of methods of solving sets of linear equations and inverting matrices are discussed. The theory of the rounding-off errors involved is investigated for some of the methods. In all cases examined, including the well-known 'Gauss elimination process', it is found that the errors are normally quite moderate: no exponential build-up need occur.

Von Code zu Analyse

Achtung: Im Allgemeinen Ausführungsfrequenz von Codestelle vielleicht unmöglich zu analysieren.

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Oft geht es aber, z. B. nach einer der folgenden Regeln

Code	**Laufzeit** T
`x = 0`, `a[i]` `x + y`, `x * y`, `x++` etc.	1 (falls Teil des abstrakte Kostenmaßes, sonst 0)

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`for (int j = a; j <= b; ++j) {`
`  B(j) // weiterer Code`
`}`
	$\sum_{j=a}^{b} T_{\mathsf{B}}(j)$ für $T_{\mathsf{B}}(j)$ Laufzeit des Schleifenrumpfs mit Wert j

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`  B(j) // weiterer Code`
`}`
	$\sum_{j=a}^{b} T_B(j)$ für $T_B(j)$ Laufzeit des Schleifenrumpfs mit Wert j

`if (...) { A } else { B }`	$\leq \max\{T_A, T_B\}$

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`if (...) { A } else { B }`	$\leq \max\{T_A, T_B\}$

`while (n > c) {`
`  A(n) // weiterer Code`
`  n = n/2;`
`  B(n) // weiterer Code`
`}`	

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`x = 0`, `a[i]` `x + y`, `x * y`, `x++` etc.	1 (falls Teil des abstrakte Kostenmaßes, sonst 0)

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`}`
	$\sum_{j=a}^{b} T_{\mathsf{B}}(j)$ für $T_{\mathsf{B}}(j)$ Laufzeit des Schleifenrumpfs mit Wert j

`if (...) { A } else { B }`	$\leq \max\{T_{\mathsf{A}}, T_{\mathsf{B}}\}$

`while (n > c) {`
`  A(n) // weiterer Code`
`  n = n/2;`
`  B(n) // weiterer Code`
`}`	

`void m(n) {`
`  if (n <= c) return;`
`  A(n) // weiterer Code`
`  m(n/2); // Rekursion`
`  B(n/2) // weiterer Code`
`}`	

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Oft geht es aber, z. B. nach einer der folgenden Regeln

Code	Laufzeit T
`x = 0`, `a[i]` `x + y`, `x * y`, `x++` etc.	1 (falls Teil des abstrakte Kostenmaßes, sonst 0)

`for (int j = a; j <= b; ++j) {`
`  B(j) // weiterer Code`
`}`
	$\sum_{j=a}^{b} T_{\mathsf{B}}(j)$ für $T_{\mathsf{B}}(j)$ Laufzeit des Schleifenrumpfs mit Wert j

`if (...) { A } else { B }`	$\leq \max\{T_{\mathsf{A}}, T_{\mathsf{B}}\}$

`while (n > c) {`
`  A(n) // weiterer Code`
`  n = n/2;`
`  B(n) // weiterer Code`
`}`
	Für allgemeine While-Schleifen und rekursive Methoden

`void m(n) {`
`  if (n <= c) return;`
`  A(n) // weiterer Code`
`  m(n/2); // Rekursion`
`  B(n/2) // weiterer Code`
`}`
	Rekursionsgleichung
	$T(n) = T(n/2) + T_{\mathsf{A}}(n) + T_{\mathsf{B}}(n/2) \quad (n > c)$
	$T(n) = 1 \quad (n \leq c)$
	$\rightsquigarrow$ Master-Theorem

Example: 2-SUM

Q. Approximately how many array accesses as a function of input size N?

```
int count = 0;
for (int i = 0; i < N; i++)
   for (int j = i+1; j < N; j++)
      if (a[i] + a[j] == 0)
         count++;
```

2

$T_B(i,j) = 2$

"inner loop"

$$0 + 1 + 2 + \ldots + (N-1) = \frac{1}{2} N (N-1)$$
$$= \binom{N}{2}$$

$$T_C(i) = \sum_{j=i+1}^{N-1} T_B(i,j) \quad \backslash\backslash \; 2$$

A. $\sim N^2$ array accesses.

$$\sum_{i=0}^{N-1} T_C(i)$$

Bottom line. Use cost model and tilde notation to simplify counts.

Example: 3-SUM

Q. Approximately how many array accesses as a function of input size N?

```
int count = 0;
for (int i = 0; i < N; i++)
   for (int j = i+1; j < N; j++)
      for (int k = j+1; k < N; k++)
         if (a[i] + a[j] + a[k] == 0)
            count++;
```

"inner loop"

$$\binom{N}{3} = \frac{N(N-1)(N-2)}{3!} \sim \frac{1}{6}N^3$$

A. $\sim \frac{1}{6} N^3$ array accesses.

Bottom line. Use cost model and tilde notation to simplify counts.

6.4 Typische Wachstumsraten

Common order-of-growth classifications

Θ - Klasse des Laufzeit

Definition. If $f(N) \sim c\, g(N)$ for some constant $c > 0$, then the order of growth of $f(N)$ is $g(N)$.

- Ignores leading coefficient.
- Ignores lower-order terms.

Ex. The order of growth of the running time of this code is N^3.

```
int count = 0;
for (int i = 0; i < N; i++)
   for (int j = i+1; j < N; j++)
      for (int k = j+1; k < N; k++)
         if (a[i] + a[j] + a[k] == 0)
            count++;
```

Typical usage. With running times.

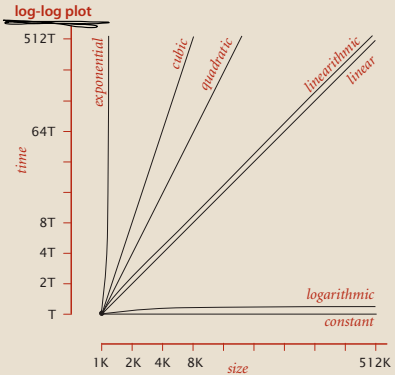
where leading coefficient depends on machine, compiler, JVM, ...

Common order-of-growth classifications

Good news. The set of functions

$$1,\ \log N,\ N,\ N \log N,\ N^2,\ N^3, \text{ and } 2^N$$

suffices to describe the order of growth of most common algorithms.



Typical orders of growth

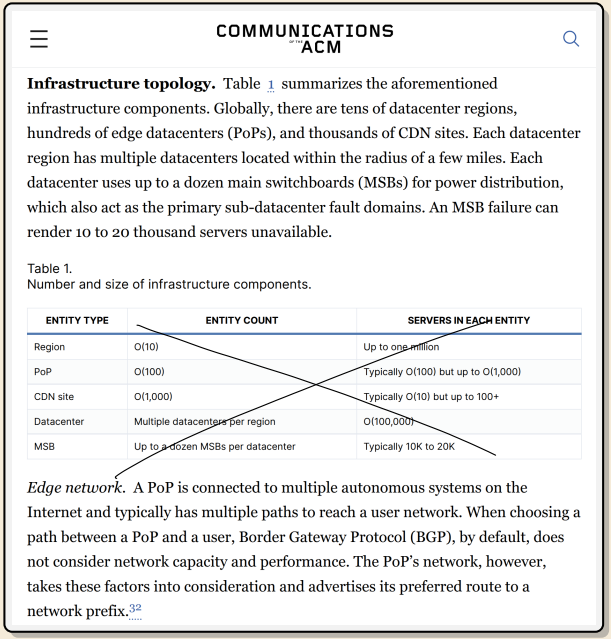
Common order-of-growth classifications

order of growth	name	typical code framework	description	example	$T(2N) / T(N)$
$\Theta(1)$	**constant**	`a = b + c;`	statement	add two numbers	1
$\Theta(\log N)$	**logarithmic**	`while (N > 1) { N = N / 2; ... }`	divide in half	binary search	~ 1
$\Theta(N)$	**linear**	`for (int i = 0; i < N; i++) { ... }`	loop	find the maximum	2
$\Theta(N \log N)$	**linearithmic**	[see mergesort lecture]	divide and conquer	mergesort	~ 2
$\Theta(N^2)$	**quadratic**	`for (int i = 0; i < N; i++) for (int j = 0; j < N; j++) { ... }`	double loop	check all pairs	4
$\Theta(N^3)$	**cubic**	`for (int i = 0; i < N; i++) for (int j = 0; j < N; j++) for (int k = 0; k < N; k++) { ... }`	triple loop	check all triples	8
$\Theta(2^N)$	**exponential**	[see combinatorial search lecture]	exhaustive search	check all subsets	$T(N)$

aus Daten nicht erkennbar

Move Fast & Constantly

Meta's Hyperscale Infrastructure: Overview and Insights



COMMUNICATIONS of the ACM

Infrastructure topology. Table 1 summarizes the aforementioned infrastructure components. Globally, there are tens of datacenter regions, hundreds of edge datacenters (PoPs), and thousands of CDN sites. Each datacenter region has multiple datacenters located within the radius of a few miles. Each datacenter uses up to a dozen main switchboards (MSBs) for power distribution, which also act as the primary sub-datacenter fault domains. An MSB failure can render 10 to 20 thousand servers unavailable.

Table 1.
Number and size of infrastructure components.

ENTITY TYPE	ENTITY COUNT	SERVERS IN EACH ENTITY
Region	O(10)	Up to one million
PoP	O(100)	Typically O(100) but up to O(1,000)
CDN site	O(1,000)	Typically O(10) but up to 100+
Datacenter	Multiple datacenters per region	O(100,000)
MSB	Up to a dozen MSBs per datacenter	Typically 10K to 20K

Edge network. A PoP is connected to multiple autonomous systems on the Internet and typically has multiple paths to reach a user network. When choosing a path between a PoP and a user, Border Gateway Protocol (BGP), by default, does not consider network capacity and performance. The PoP's network, however, takes these factors into consideration and advertises its preferred route to a network prefix.[32]

https://cacm.acm.org/research/metas-hyperscale-infrastructure-overview-and-insights/

6.5 Beispiel: Suchen in einem Array

Clicker Question

Was ist die (Worst-Case)-Laufzeit um mittels binärer Suche nach einem Element in einem sortierten Array $A[0..n)$ zu suchen?

A binäre Was?

B $\Theta(1)$

C $\Theta(10000)$

D $\Theta(\log n)$

E $\Theta(\log^2 n)$

F $\Theta(\sqrt[3]{n})$

G $\Theta(\sqrt{n})$

H $\Theta(n)$

I $\Theta(n^2)$

J $\Theta(n^3)$

→ *sli.do/cs210*

Clicker Question

Was ist die (Worst-Case)-Laufzeit um mittels binärer Suche nach einem Element in einem sortierten Array $A[0..n)$ zu suchen?

- **A** ~~binäre Was?~~
- **B** ~~$\Theta(1)$~~
- **C** ~~$\Theta(10000)$~~
- **D** $\Theta(\log n)$ ✓
- **E** ~~$\Theta(\log^2 n)$~~
- **F** ~~$\Theta(\sqrt[3]{n})$~~
- **G** ~~$\Theta(\sqrt{n})$~~
- **H** ~~$\Theta(n)$~~
- **I** ~~$\Theta(n^2)$~~
- **J** ~~$\Theta(n^3)$~~

→ *sli.do/cs210*

Lineare Suche

Gegeben: Array $A[0..n)$ von Zahlen, Zahl x

Ziel: Kleinster Index $i \in [0..n)$ mit $A[i] == x$, oder $i = n$ falls x nicht in A vorkommt

Falls wir sonst nichts über A annehmen können, müssen wir alle Indices durchprobieren.

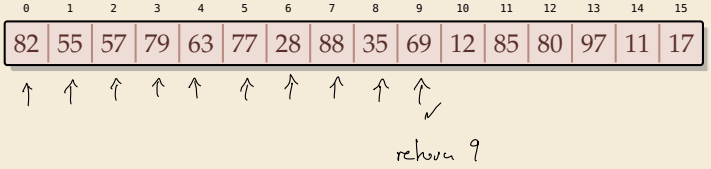
Lineare Suche

Gegeben: Array $A[0..n)$ von Zahlen, Zahl x

Ziel: Kleinster Index $i \in [0..n)$ mit $A[i] == x$, oder $i = n$ falls x nicht in A vorkommt

Falls wir sonst nichts über A annehmen können, müssen wir alle Indices durchprobieren.

Beispiel: Suche nach 69



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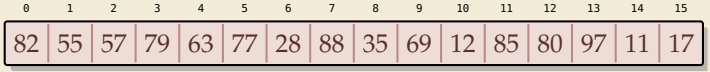
Beispiel: Suche nach 69




```
static int linearSearch(int[] A, int x) {
    int n = A.length;
    for (int i = 0; i < n; ++i)
        if (A[i] == x) return i;
    return n; // i == n
}
```

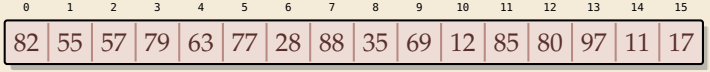
Lineare Suche

Gegeben: Array $A[0..n)$ von Zahlen, Zahl x

Ziel: Kleinster Index $i \in [0..n)$ mit $A[i] == x$, oder $i = n$ falls x nicht in A vorkommt

Falls wir sonst nichts über A annehmen können, müssen wir alle Indices durchprobieren.

Beispiel: Suche nach 69

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1 static int linearSearch(int[] A, int x) {
2     int n = A.length;
3     for (int i = 0; i < n; ++i)
4         if (A[i] == x) return i;
5     return n;
6 }
```

- Invariante: $A[0..i)$ enthält x nicht

Lineare Suche

Gegeben: Array $A[0..n)$ von Zahlen, Zahl x

Ziel: Kleinster Index $i \in [0..n)$ mit $A[i] == x$, oder $i = n$ falls x nicht in A vorkommt

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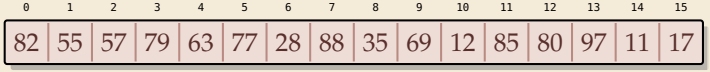
Beispiel: Suche nach 69




```java
static int linearSearch(int[] A, int x) {
    int n = A.length;
    for (int i = 0; i < n; ++i)
        if (A[i] == x) return i;
    return n;
}
```

- Invariante: $A[0..i)$ enthält x nicht
- i #Arrayzugriffe für Rückgabewert i
  - ⇝ Worst-Case n, Best-Case 1, Average-Case $\sim \frac{n}{2}$

Binäre Suche

Gegeben: Array $A[0..n)$ von Zahlen, Zahl x, $\boxed{A[0] \leq A[1] \leq \cdots \leq A[n-1]}$

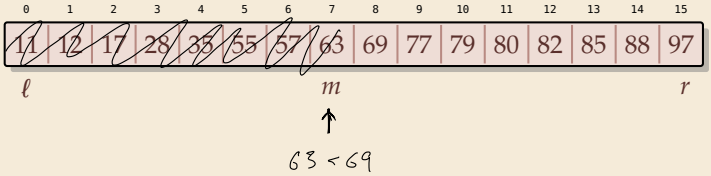
Ziel: Kleinster Index $i \in [0..n)$ mit $A[i] == x$, oder $i = n$ falls x nicht in A vorkommt

Binäre Suche

Gegeben: Array $A[0..n)$ von Zahlen, Zahl x, $\boxed{A[0] \leq A[1] \leq \cdots \leq A[n-1]}$

Ziel: Kleinster Index $i \in [0..n)$ mit $A[i] == x$, oder $i = n$ falls x nicht in A vorkommt

*Es hilft sehr, wenn das Array **sortiert** ist!* **Beispiel:** Suche nach 69

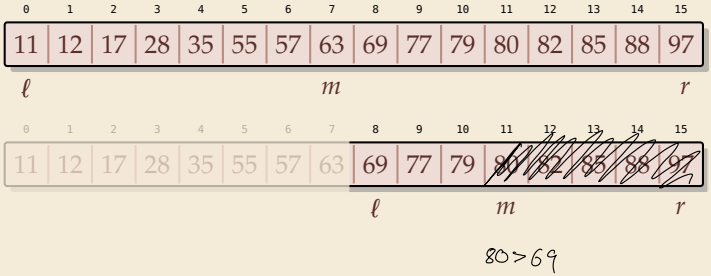


Binäre Suche

Gegeben: Array $A[0..n)$ von Zahlen, Zahl x, $A[0] \leq A[1] \leq \cdots \leq A[n-1]$

Ziel: Kleinster Index $i \in [0..n)$ mit $A[i] == x$, oder $i = n$ falls x nicht in A vorkommt

*Es hilft sehr, wenn das Array **sortiert** ist!* **Beispiel:** Suche nach 69



Binäre Suche

Gegeben: Array $A[0..n)$ von Zahlen, Zahl x, $\boxed{A[0] \leq A[1] \leq \cdots \leq A[n-1]}$

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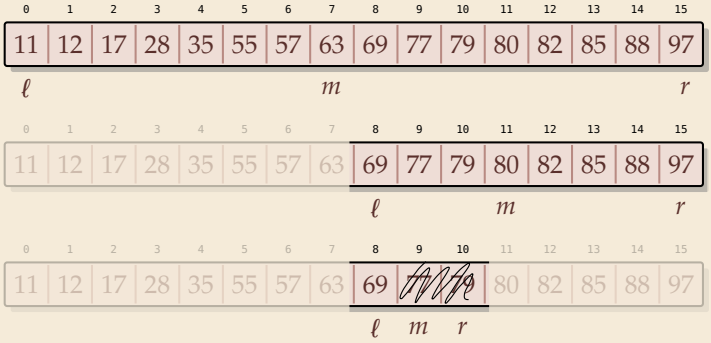
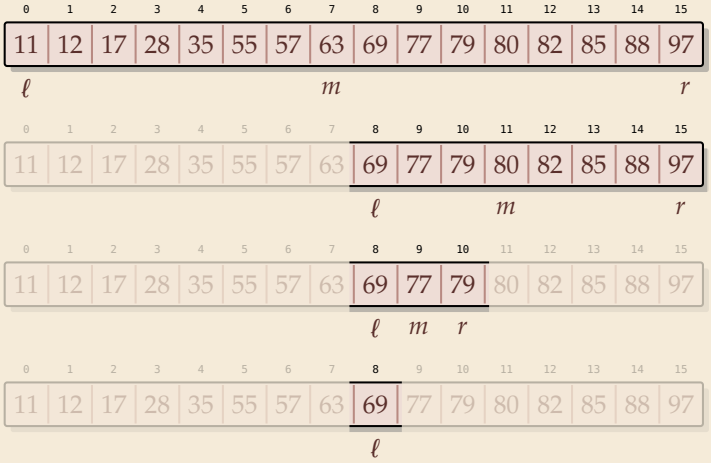


Binäre Suche

Gegeben: Array $A[0..n)$ von Zahlen, Zahl x, $\boxed{A[0] \leq A[1] \leq \cdots \leq A[n-1]}$

Ziel: Kleinster Index $i \in [0..n)$ mit $A[i] == x$, oder $i = n$ falls x nicht in A vorkommt

*Es hilft sehr, wenn das Array **sortiert** ist!* **Beispiel:** Suche nach 69



Binäre Suche – Java Implementierung

trivial zu implementieren?

- erster Code 1946 veröffentlicht
- erster Bug-freier Code 1962
- subtiler Bug in Java's `Arrays.binarySearch()` 2006(!) entdeckt

  https://research.google/blog/extra-extra-read-all-about-it-nearly-all-binary-searches-and-mergesorts-are-broken/

Binäre Suche – Java Implementierung

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```
1  static int binarySearch(int[] A, int x) {
2      int n = A.length;
3      int l = 0, r = n-1;
4      while (l <= r) {
5          int m = l + (r - l) / 2;
6          if (x < A[m]) r = m - 1;
7          if (A[m] < x) l = m + 1;
8          else return m;
9      }
10     return l;
11 }
```

◎ **Invariante**: Falls x in $A[0..n)$ vorkommt, $A[i] = x$, dann gilt stets $A[l] \leq x \leq A[r]$

- Da $A[0..n)$ sortiert ist, gilt auch $l \leq i \leq r$

$\rightsquigarrow$ irgendwann $m = i$ $\rightsquigarrow$ geben m zurück

Binäre Suche – Java Implementierung

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```
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```

◎ **Invariante**: Falls `x` in $A[0..n)$ vorkommt, $A[i] = x$, dann gilt stets $A[l] \leq x \leq A[r]$

- Da $A[0..n)$ sortiert ist, gilt auch $l \leq i \leq r$

$\rightsquigarrow$ irgendwann $m = i$ $\rightsquigarrow$ geben m zurück

Falls `x` nicht in $A[0..n)$ vorkommt

- geben $i \in [0..n]$ zurück sodass $A[0..i), x, A[i..n)$ sortiert wäre
  d.h. $A[i-1] < x$ (falls $i > 0$) und $A[i] > x$ (falls $i < n$)

```java
static int binarySearch(int[] A, int x) {
    int n = A.length;
    int l = 0, r = n-1;
    while (l <= r) {
        int m = l + (r - l) / 2;
        if (x < A[m]) r = m - 1;
        if (A[m] < x) l = m + 1;
        else return m;
    }
    return l;
}
```

1 Vergleich

$\Theta(1)$

abstraktes Kostenmaß # Vergleiche (3-way)

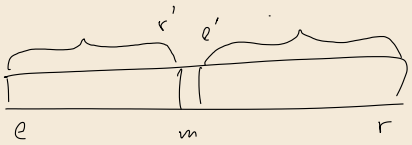
Iterationen der while-Schleife

while ($\ell \le r$) | while ($n \ge 1$)

$A(\ell, r)$ | $A(n)$

$n = r - \ell + 1$

$n := n' \le \frac{n}{2}$



Obere Schranke für # Vergleiche

$$T(n) = T_A(n) + T\left(\frac{n}{2}\right)$$

$T_A(n) = 1$, $T\left(\frac{n}{2}\right) \cdot 1$

$T(1) = 0\,1$ *Wir machen für n=1 noch einen Vergleich ...*

$c = \log_2(1) = 0$ $\quad f(n)$ vs. $\Theta(n^c) = \Theta(1)$

$\Rightarrow$ MT-b) $\quad T(n) = \Theta(\log n)$

Binary search: mathematical analysis

Proposition. Binary search uses at most $1 + \lg N$ key compares to search in a sorted array of size N.

Def. $T(N)$ = # key compares to binary search a sorted subarray of size $\leq N$.

Binary search recurrence. $T(N) \leq T(N/2) + 1$ for $N > 1$, with $T(1) = 1$.

- $T(N/2)$: left or right half (floored division)
- $+ 1$: possible to implement with one 2-way compare (instead of 3-way)

Pf sketch. [assume N is a power of 2]

$$
\begin{aligned}
T(N) &\leq T(N/2) + 1 && \text{[given]} \\
&\leq T(N/4) + 1 + 1 && \text{[apply recurrence to first term]} \\
&\leq T(N/8) + 1 + 1 + 1 && \text{[apply recurrence to first term]} \\
&\vdots \\
&\leq T(N/N) + 1 + 1 + \ldots + 1 && \text{[stop applying, } T(1) = 1 \text{]} \\
&= 1 + \lg N
\end{aligned}
$$

6.6 Speicherbedarf

Clicker Question

Wie viel Speicherplatz benötigt ein Objekt dieser Klasse?

```
class Date {
    int day, month, year;
}
```

A	0 Bit	H	8 Byte	O	32 Byte
B	1 Bit	I	10 Byte	P	48 Byte
C	8 Bit	J	12 Byte	Q	64 Byte
D	1 Byte	K	16 Byte	R	128 Byte
E	2 Byte	L	20 Byte	S	keine Ahnung
F	4 Byte	M	24 Byte		
G	6 Byte	N	30 Byte		

→ *sli.do/cs210*

Basics

Bit. 0 or 1.

Byte. 8 bits.

Megabyte (MB). 1 million or 2^{20} bytes.

Gigabyte (GB). 1 billion or 2^{30} bytes.

NIST

most computer scientists

MB MiB

64-bit machine. We assume a 64-bit machine with 8-byte pointers.

- Can address more memory.
- Pointers use more space.

some JVMs "compress" ordinary object pointers to 4 bytes to avoid this cost

Typical memory usage for primitive types and arrays

type	bytes
boolean	1
byte	1
char	2
int	4
float	4
long	8
double	8

primitive types

type	bytes
char[]	$2N + 24$
int[]	$4N + 24$
double[]	$8N + 24$

one-dimensional arrays

type	bytes
char[][]	$\sim 2MN$
int[][]	$\sim 4MN$
double[][]	$\sim 8MN$

two-dimensional arrays

Typical memory usage for objects in Java

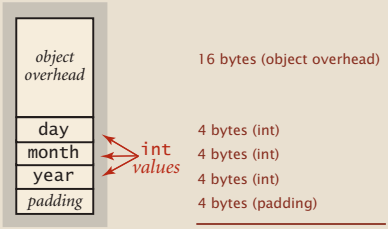
Object overhead. 16 bytes.

Reference. 8 bytes.

Padding. Each object uses a multiple of 8 bytes.

Ex 1. A `Date` object uses 32 bytes of memory.

```
public class Date
{
   private int day;
   private int month;
   private int year;
...
}
```



Clicker Question

Wie viel Speicherplatz benötigt ein Objekt dieser Klasse?

```
class Date {
    int day, month, year;
}
```

- A ~~0 Bit~~
- B ~~1 Bit~~
- C ~~8 Bit~~
- D ~~1 Byte~~
- E ~~2 Byte~~
- F ~~4 Byte~~
- G ~~6 Byte~~
- H ~~8 Byte~~
- I ~~10 Byte~~
- J ~~12 Byte~~
- K ~~16 Byte~~
- L ~~20 Byte~~
- M ~~24 Byte~~
- N ~~30 Byte~~
- O 32 Byte ✓
- P ~~48 Byte~~
- Q ~~64 Byte~~
- R ~~128 Byte~~
- S keine Ahnung ✓

→ *sli.do/cs210*

Typical memory usage summary

Total memory usage for a data type value:

- Primitive type: 4 bytes for `int`, 8 bytes for `double`, ...
- Object reference: 8 bytes.
- Array: 24 bytes + memory for each array entry.
- Object: 16 bytes + memory for each instance variable.
- Padding: round up to multiple of 8 bytes.

+ 8 extra bytes per inner class object
(for reference to enclosing class)

Shallow memory usage: Don't count referenced objects.

Deep memory usage: If array entry or instance variable is a reference, count memory (recursively) for referenced object.

Example

Q. How much memory does `WeightedQuickUnionUF` use as a function of N? Use tilde notation to simplify your answer.

```
public class WeightedQuickUnionUF
{
   private int[] id;
   private int[] sz;
   private int count;

   public WeightedQuickUnionUF(int N)
   {
      id = new int[N];
      sz = new int[N];
      for (int i = 0; i < N; i++) id[i] = i;
      for (int i = 0; i < N; i++) sz[i] = 1;
   }
   ...
}
```

16 bytes (object overhead)

8 + (4N + 24) bytes each (reference + `int[]` array)

4 bytes (`int`)

4 bytes (padding)

8N + 88 bytes

A. $8N + 88 \sim 8N$ bytes.

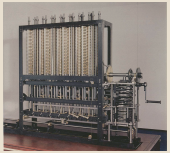
Turning the crank: summary

Empirical analysis.

- Execute program to perform experiments.
- Assume power law and formulate a hypothesis for running time.
- Model enables us to make predictions.

Mathematical analysis.

- Analyze algorithm to count frequency of operations.
- Use tilde notation to simplify analysis.
- Model enables us to explain behavior.



Scientific method.

- Mathematical model is independent of a particular system; applies to machines not yet built.
- Empirical analysis is necessary to validate mathematical models and to make predictions.