2 Complexity Theory Recap

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Prof. Dr. Sebastian Wild

CS627 (Summer 2025)
Philipps-Universität Marburg
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Outline

2 Complexity Theory Recap

2.1 P and NP Informally

Hard problems

- Some algorithmic problems are **"hard nuts" to crack.**
  - e. g., the *Traveling Salesperson Problem (TSP):*

    Given: n cities $S_1, \ldots, S_n$,
    all $n(n-1)$ pairwise
    distances $d(S_i, S_j) \in \mathbb{N} \quad (i \neq j)$

    Goal: Shortest round trip through all cities

  always exact, always correct polytime

  - no general, efficient algorithm known!
    (despite decades of intensive research . . .)

$\rightsquigarrow$ It *seems* as if there is no efficient algorithm for TSP!

  - **But:** can we *prove* that? 
  - Despite similarly intensive research: **No!** (not yet)

Doesn't sound like a shining example for theoretical computer science? . . . stay tuned!

United in incapacity

"I can't find an efficient algorithm, but neither can all these famous people."

Garey, Johnson 1979

Complexity Theory

- *Complexity theory* allows us to *compare* the *hardness* of algorithmic problems.

$\leq_p$

A: old problem
Consensus: hard

B: new problem
Status: unknown
(seems hard to *us* . . .)

Intuitive idea:

1. If A is a known hard nut, and
2. $\boxed{B \text{ is at least as hard as } A,}$

then B is a hard nut, too!

Formally:

efficient = polytime

1. A is NP-hard: probably $\nexists$ eff. alg. for A
2. $A \leq_p B$: $\exists$ eff. alg. for $B \implies \exists$ eff. alg. for A

$\leadsto$ B is NP-hart: probably $\nexists$ eff. alg. for B!

P and NP – Intuitive Synopsis

- P = class of problems for which there is an algorithm A and a polynomial p such that A **solves** every instance I in time $O(p(|I|))$.
  - P for "polynomial" – i. e., all problems where a solution can be *found* by a (deterministic) algorithm in polynomial time.

- NP = class of problems for which there is an algorithmus A and a polynomial p such that A can **verify** a given candidate solution $l(I)$ of a given instance I in time $O(p(|I|))$, i. e., check whether $l(I)$ solves I or not.
  - NP for "nondeterministically polynomial" – i. e., all problems where a solution can be *found* by a *nondeterministic* algorithm in polynomial time.
  - This is equivalent to the above characterization via verification.

- We know P $\subseteq$ NP. We *think* P $\subsetneq$ NP, i. e., P $\neq$ NP.

  The question "P = NP?" is one of the famous **millenium problems** and arguably the **most important open problem of theoretical computer science**.

2.2 Models of Computation

Mathematical Models of Computation

- complexity classes talk about sets of problems based upon whether they allow an algorithm of a certain cost
- in general, this depends on the allowable algorithms and their costs!

$\rightsquigarrow$ need to fix a machine model

A *machine model* decides

- what algorithms are possible
- how they are described (= programming language)
- what an execution *costs*

Goal: Machine models should be
detailed and powerful enough to reflect actual machines,
abstract enough to unify architectures,
simple enough to analyze.

Random Access Machines

Standard model for detailed complexity analysis:

Random access machine (RAM)

more detail in §2.2 of *Sequential and Parallel Algorithms and Data Structures* by Sanders, Mehlhorn, Dietzfelbinger, Dementiev

- unlimited *memory* $\mathtt{MEM}[0], \mathtt{MEM}[1], \mathtt{MEM}[2], \ldots$
- fixed number of *registers* $R_1, \ldots, R_r$ (say $r = 100$)
- memory cells $\mathtt{MEM}[i]$ and registers R_i store w-bit integers, i. e., numbers in $[0..2^w - 1]$
  w is the word width/size; typically $\boxed{w \propto \lg n}$ $\rightsquigarrow$ $2^w \approx n$
- Instructions:
  - load & store: $R_i := \mathtt{MEM}[R_j]$ $\mathtt{MEM}[R_j] := R_i$
  - operations on registers: $R_k := R_i + R_j$ (arithmetic is *modulo* 2^w!)
    also $R_i - R_j$, $R_i \cdot R_j$, $R_i \operatorname{div} R_j$, $R_i \bmod R_j$
    C-style operations (bitwise and/or/xor, left/right shift)
  - conditional and unconditional jumps
- time cost: number of executed instructions
- space cost: total number of touched memory cells

RAM-Program Example

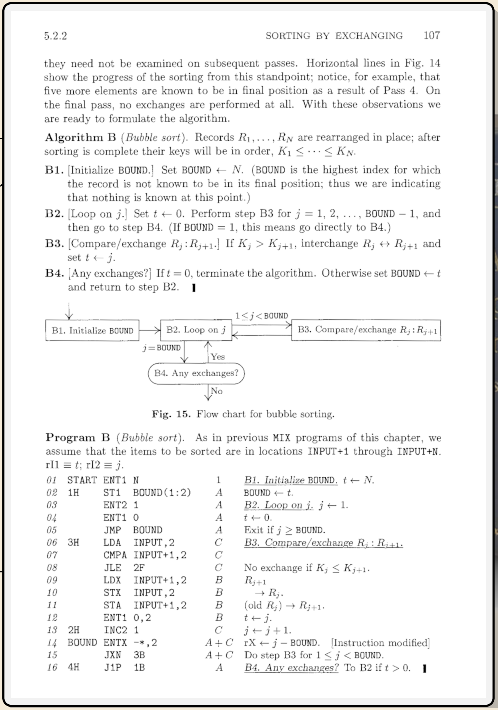
Example RAM program

```
1  // Assume: R_1 stores number N
2  // Assume: MEM[0..N) contains list of N numbers
3  R_2 := R_1;
4  R_3 := R_1 − 2;
5  R_4 := MEM[R_3];
6  R_5 := R_3 + 1;
7  R_6 := MEM[R_5];
8  if (R_4 ≤ R_6) goto line 11;
9  MEM[R_3] := R_6;
10 MEM[R_5] := R_4;
11 R_3 := R_3 − 1;
12 if (R_3 ≥ 0) goto line 5;
13 R_2 := R_2 − 1;
14 if (R_2 > 0) goto line 4;
15 // Done:
```

RAM-Program Example

Example RAM program

```
1  // Assume: R_1 stores number N
2  // Assume: MEM[0..N) contains list of N number
3  R_2 := R_1;
4  R_3 := R_1 − 2;
5  R_4 := MEM[R_3];
6  R_5 := R_3 + 1;
7  R_6 := MEM[R_5];
8  if (R_4 ≤ R_6) goto line 11;
9  MEM[R_3] := R_6;
10 MEM[R_5] := R_4;
11 R_3 := R_3 − 1;
12 if (R_3 ≥ 0) goto line 5;
13 R_2 := R_2 − 1;
14 if (R_2 > 0) goto line 4;
15 // Done: MEM[0..N) sorted
```



5.2.2 SORTING BY EXCHANGING 107

they need not be examined on subsequent passes. Horizontal lines in Fig. 14 show the progress of the sorting from this standpoint; notice, for example, that five more elements are known to be in final position as a result of Pass 4. On the final pass, no exchanges are performed at all. With these observations we are ready to formulate the algorithm.

Algorithm B (*Bubble sort*). Records $R_1, \ldots, R_N$ are rearranged in place; after sorting is complete their keys will be in order, $K_1 \le \cdots \le K_N$.

B1. [Initialize BOUND.] Set BOUND $\leftarrow N$. (BOUND is the highest index for which the record is not known to be in its final position; thus we are indicating that nothing is known at this point.)

B2. [Loop on j.] Set $t \leftarrow 0$. Perform step B3 for $j = 1, 2, \ldots,$ BOUND $- 1$, and then go to step B4. (If BOUND $= 1$, this means go directly to B4.)

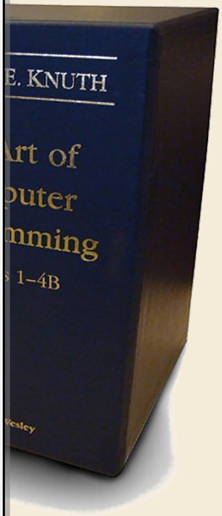
B3. [Compare/exchange $R_j : R_{j+1}$.] If $K_j > K_{j+1}$, interchange $R_j \leftrightarrow R_{j+1}$ and set $t \leftarrow j$.

B4. [Any exchanges?] If $t = 0$, terminate the algorithm. Otherwise set BOUND $\leftarrow t$ and return to step B2. ▮

Fig. 15. Flow chart for bubble sorting.

Program B (*Bubble sort*). As in previous MIX programs of this chapter, we assume that the items to be sorted are in locations INPUT+1 through INPUT+N. rI1 $\equiv t$; rI2 $\equiv j$.

```
01 START ENT1 N            1     B1. Initialize BOUND. t ← N.
02 1H    ST1  BOUND(1:2)   A     BOUND ← t.
03       ENT2 1            A     B2. Loop on j. j ← 1.
04       ENT1 0            A     t ← 0.
05       JMP  BOUND        A     Exit if j ≥ BOUND.
06 3H    LDA  INPUT,2      C     B3. Compare/exchange R_j : R_{j+1}.
07       CMPA INPUT+1,2    C
08       JLE  2F           C     No exchange if K_j ≤ K_{j+1}.
09       LDX  INPUT+1,2    B     R_{j+1}
10       STX  INPUT,2      B       → R_j.
11       STA  INPUT+1,2    B     (old R_j) → R_{j+1}.
12       ENT1 0,2          B     t ← j.
13 2H    INC2 1            C     j ← j + 1.
14 BOUND ENTX -*,2         A+C   rX ← j − BOUND. [Instruction modified]
15       JXN  3B           A+C   Do step B3 for 1 ≤ j < BOUND.
16 4H    J1P  1B           A     B4. Any exchanges? To B2 if t > 0. ▮
```



2.3 Turing Machines

Keep it Simple, Stupid

- word-RAM (rather) realistic, but complicated
  - note that the machine has to grow with the inputs(!)
- for a coarse distinction of running time complexity, simpler models suffice
  - useful to reason about "all algorithms"
  - machine is fixed for all inputs sizes apart from storage for input

Many models of computation . . .

- μ-recursive function
- Turing machines (TM)
- counter machines
- λ-calculus
- While-programs
- any Turing-complete language
- quantum computers
- . . .

. . . with strong equivalences:

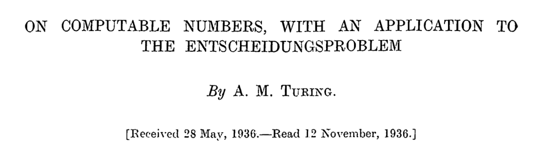
1. all proven to lead to the *same* set of computable functions
2. **Church-Turing thesis**:
   any formalization of "effectively computable" is equivalent in this sense
3. ***Extended Church-Turing Thesis:***
   . . . and can be simulated with *polynomial overhead* on a TM
   - true for all on left . . .
   - except theoretical quantum computers!
   - ignore them for now

wake me when they exist

Turing Machines

- invented by ***Alan Turing*** in 1936 as formalization for “computable by hand”

  In same paper, Turing proved undecidability of halting problem!

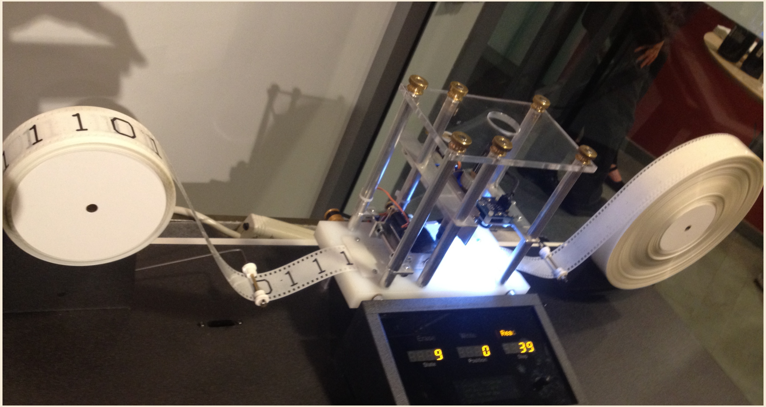


ON COMPUTABLE NUMBERS, WITH AN APPLICATION TO THE ENTSCHEIDUNGSPROBLEM

By A. M. Turing.

[Received 28 May, 1936.—Read 12 November, 1936.]

- *minimalistic* model of universal computer, but can be built:

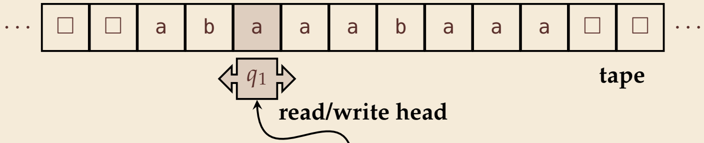


Turingmaschinen in Circulation

Turing Machines – Informal Recap

A *Turing machine* has

- a **finite control** via states
- an **input/output-tape**
  - unbounded length
  - initially contains input
  - all other cells contain "□"
- a **read/write head**
  - reads the current symbol
  - overwrites it with a new symbol
  - initially placed on beginning of input



State	Tape Symbol	→	New Tape Symbol	Head Movement	New State
q_0	a	→	a	right	q_0
q_0	b	→	b	right	q_1
q_0	□	→	□	none	q_e
q_1	a	→	b	right	q_1
q_1	b	→	b	right	q_0
q_1	□	→	□	none	q_1
q_e	–	→	— terminate the computation —		

finite control

Turing Machines – Formal Syntax

Definition 2.1 (Turing Machine (TM))

A *Turing machine* is a 7-tuple $M = (Q, \Sigma, \Gamma, \delta, q_0, \square, q_{\text{halt}})$ with

- a finite set of *states* Q,
- an *input alphabet* Σ,
- a *tape alphabet* $\Gamma \supset \Sigma$,
- for deterministic TMs a *transition function* $\delta : (Q \setminus \{q_{\text{halt}}\}) \times \Gamma \rightarrow Q \times \Gamma \times \{L, R, N\}$
for **non**deterministic TMs a *transition relation* $\delta : (Q \setminus \{q_{\text{halt}}\}) \times \Gamma \rightarrow 2^{Q \times \Gamma \times \{L,R,N\}}$
- an *initial state* $q_0 \in Q$,
- a *blank symbol* $\square \in \Gamma \setminus \Sigma$, and
- a halting state $q_{\text{halt}} \in Q$

Turing Machine – Computation Step

- Each step of a computation of TM M has the form

  $\delta(q, a) = (q', b, d)$ resp. $\delta(q, a) \ni (q', b, d)$,

  with the semantics that

  - M is in state $q \neq q_{\text{halt}}$
  - the cell below the read/write head currently contains symbol a
  - M now changes (based on its finite control)
    - into state q',
    - writes b into the cell under the read/write head
    - and finally moves the read/write head in direction $d \in \{L, R, N\}$.
      (L = <u>l</u>eft, R = <u>r</u>ight, N = <u>n</u>one (stay))

- for deterministic TM M, q and a uniquely determine this action;
  for nondeterministic TM, we may have several possible actions.

- to formally define an entire computation, we have to encode the tape contents as well

Turing Machines – Configurations

Definition 2.2 (TM Configuration)

A *configuration (config)* of a TM M is a string $C \in \Gamma^\star Q\, \Gamma^\star$.

The semantics of a config $C = \alpha q \gamma$, $q \in Q$, is tape content $\alpha\beta$ and head at first symbol of β.

Definition 2.3 (TM Computation Relation)

The *computation relation* $\vdash$ is defined on the set of configurations of a TM M as follows.

$$a_1 \dots a_m\, q\, b_1 \dots b_n \;\vdash\; \begin{cases} a_1 \dots a_m \quad q'\, c\, b_2 \dots b_n, & \delta(q, b_1) = (q', c, N),\ m \geq 0,\ n \geq 1, \\ a_1 \dots a_m\, c \;\; q'\, b_2 \dots b_n, & \delta(q, b_1) = (q', c, R),\ m \geq 0,\ n \geq 2, \\ a_1 \dots a_{m-1}\, q'\, a_m\, c\, b_2 \dots b_n, & \delta(q, b_1) = (q', c, L),\ m \geq 1,\ n \geq 1. \end{cases}$$

For the boundary case $n = 1$ and direction right, we set

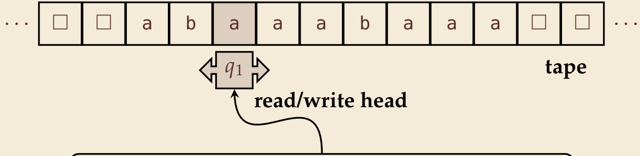
$$a_1 \dots a_m\, q\, b_1 \;\vdash\; a_1 \dots a_m\, c\, q' \,\square \qquad \text{if } \delta(q, b_1) = (q', c, R),$$

For $m = 0$ and direction left, we similarly have set

$$q\, b_1 \dots b_n \;\vdash\; q' \,\square c\, b_2 \dots b_n \qquad \text{if } \delta(q, b_1) = (q', c, L).$$

Turing Machines – Configuration Example

Example:
For the shown TM M,
the current configuration is:
$C = \mathtt{ab}\, q_1\, \mathtt{aaabaaa}$



State	Tape Symbol	$\to$	New Tape Symbol	Head Movement	New State
q_0	a	$\to$	a	right	q_0
q_0	b	$\to$	b	right	q_1
q_0	□	$\to$	□	none	q_e
q_1	a	$\to$	b	right	q_1
q_1	b	$\to$	b	right	q_0
q_1	□	$\to$	□	none	q_1
q_e	–	$\to$	— terminate the computation —		

- TM Config $C = \alpha q \beta$ completely describes current state of computation
  - $\alpha\beta$ is the (non-blank) tape content
  - q is the current state of the TM
  - the read/write head is on the first symbol of β

Turing Machines – Computed Function

- With this setup, we can now formally define what a Turing machine computes.

Definition 2.4 (Function computed by a TM)

Let $M = (Q, \Sigma, \Gamma, \delta, q_0, \square, q_{\text{halt}})$ by a TM. The function computed by M on input $x \in \Sigma^\star$, written $M(x)$, is defined as
$M(x) = \{y : q_0 x \vdash^\star q_{\text{halt}} y\}$.
For deterministic TMs, we will also have $|M(x)| \leq 1$ and we write $M(x) = y$ for $M(x) = \{y\}$. ◀

Definition 2.5 (Time and Space cost)

For a TM M and input $x \in \Sigma^*$, we define
$time_M(x) = \inf\{t : q_0 x \vdash^t q_{\text{halt}} y\} \cup \{0\}$.
We define $space_M(x) = \inf\Big\{|\alpha\beta| : q_0 x \vdash^\star \alpha q \beta \vdash^\star q_{\text{halt}} y, |\alpha\beta|_\square \leq 2\Big\} \cup \{0\}$. ◀

$\square$ in $\alpha\beta$

- Note: *time* and *space* can be ∞ or 0 for nondeterministic TMs.

Turing Machines – Accepted Language

- Often convenient to use language acceptance instead of function computation.
- for deterministic TM, compute *characteristic function* of L: $\mathbb{1}_L(x) = \begin{cases} 1 & \text{if } x \in L \\ 0 & \text{otherwise} \end{cases}$
- care needed for nondeterministic TM

Definition 2.6 (Language of TM)

The language $\mathcal{L}(M)$ accepted by a TM M is defined as

$$\mathcal{L}(M) \quad = \quad \{w \in \Sigma^\star : 1 \in M(w)\}.$$

◀

$\rightsquigarrow$ $\boxed{\text{nondeterministic TM accepts } w \text{ iff some computation accepts } w}$

Turing Machines – Several tapes

Remark 2.7 (k-tape TMs)

We only consider one-tap TMs here. In general, k-tape TMs can be faster.
However, any language accepted by a k-tape TM in time $f(n)$
is also be accepted by a 1-tape TM with running time $O(f^2(n))$.
The models are thus *polynomially equivalent*. ◄

Turing Machines – Totality

- In complexity theory, we will restrict ourselves to TMs that always halt.

Definition 2.8 (Terminating TM)

A TM M is *always terminating / total* if there is a function $T : \mathbb{N} \to \mathbb{N}$ such that $q_0 x = C_0 \vdash C_1 \vdash \cdots \vdash C_t$ implies $t \leq T(|x|)$, and there is a y such that $q_0 x \vdash^\star q_{\text{halt}} y$. ◀

- Note that in a terminating TM, we always have $1 \leq time_M(x) \leq T(|x|)$.

Lemma 2.9 (Time-constrained TM)

Given a (potentially nonterminiating) TM M and a function $T : \mathbb{N} \to \mathbb{N}$ **computable** in time $T(n)$, we can construct an *always terminating* TM M' that simulates M for $T(|x|)$ on x and outputs `TIMEOUT` if M has not terminated yet. Moreover, $time_{M'}(x) = O(T^2(|x|))$ for all $x \in \Sigma^\star$. ◀

⇝ $\boxed{\text{If we are only interested in the (non)existence of a polynomial-time TM, we can restriction ourselves to total TMs that will never TIMEOUT.}}$

Models of Computation – Summary

- Concrete model such as TMs useful for some proofs
- Often, details do not matter as long as models are polynomially equivalent
  - Note: TM always means we are in the logarithmic cost model for arithmetic operations
- In the following, discuss more abstract notion of "algorithm"
  (fine to substitute by TM in each case)

2.4 The Classes P und NP

Worst Case Complexity

Definition 2.10 (Time and Space Complexity – Generic)

Let Σ_I and Σ_O two alphabets and A an algorithm implementing a **total** mapping $\Sigma_I^\star \to \Sigma_O^\star$. Then for each $x \in \Sigma_I^\star$ we denote by $time_A(x)$ (resp. $space_A(x)$) the logarithmic time complexity (resp. logarithmic space complexity) for A on x. ◀

Where needed, we can unpack this in full detail for Turing machines!

Definition 2.11 (Worst-Case Complexity)

Let Σ_I and Σ_O be two alphabets and A an algorithm implementing a **total** mapping $\Sigma_I^\star \to \Sigma_O^\star$. The *worst case time complexity of A* is the function $Time_A : \mathbb{N} \to \mathbb{N}$ with

$$Time_A(n) = \max\{time_A(x) : x \in \Sigma_I^n\},$$

for each $n \in \mathbb{N}$. The *worst case space complexity of A* is given by function $Space_A : \mathbb{N} \to \mathbb{N}$ with

$$Space_A(n) = \max\{space_A(x) : x \in \Sigma_I^n\}.$$

◀

Decision Problems = Languages

Definition 2.12 (Decision Problem and Algorithms)

A *decision problem* is given by $P = (L, U, \Sigma)$ for Σ an alphabet and $L \subseteq U \subseteq \Sigma^\star$. An algorithm *$A$ solves (decides)* decision problem P, if for all $x \in U$

1. $A(x) = 1$ for $x \in L$, and
2. $A(x) = 0$ for $x \in U \setminus L$ (i. e., $x \notin L$)

holds. Here $A(x)$ denotes the output of A on input x.
If $U = \Sigma^\star$ holds we denote P briefly by (L, Σ).

$\rightsquigarrow$ A computes a total function $A : U \to \{0, 1\}$,
the *characteristic function* $\mathbb{1}_L : U \to \{0, 1\}$ of language L.
We then write $L = \mathcal{L}(A)$, the language accepted by A.

We restrict our attention to ***decision problems***:
Given: $w \in \Sigma^\star$.
Goal: Is $w \in L$?

Example:
w is an encoding of an instance of the traveling salesperson problem **and** a threshold D

$L = \{w : w \text{ encodes instance w/ opt. round trip length} \leq D\}$

Optimal Algorithms

Definition 2.13 (Upper/Lower Bounds, Optimal Algorithms)

Let U be an algorithmic problem and f, g functions $\mathbb{N}_0 \to \mathbb{R}^+$.

- We call $O(g(n))$ an *upper bound for time complexity of U* if there is an algorithm A that solves U in time $Time_A(n) \in O(g(n))$.
- We say $\Omega(f(n))$ is a *lower bound for time complexity of U* if every algorithm A that solves U needs time $Time_A(n) \in \Omega(f(n))$.
- An algorithm A is called *optimal for U* if $Time_A(n) \in O(g(n))$ and $\Omega(g(n))$ is a lower bound for the time complexity of U.

Running Time

Definition 2.14 (time classes)

For function $f : \mathbb{N} \to \mathbb{N}$, the class $\mathit{TIME}(f(n))$ is the set of all languages A, for which there is a *deterministic* Turing machine M with $\mathcal{L}(M) = A$ and $\mathit{time}_M(w) \le f(|w|)$ für alle $w \in \Sigma^\star$. ◀

Definition 2.15 (P, tractable)

We define the class of languages P decidable in polynomial time by

$$\mathsf{P} \;:=\; \bigcup_{p \text{ polynomial}} \mathit{TIME}(p(n)).$$

A language (a decision problem) $L \in \mathsf{P}$ is called *tractable / efficiently decidable*. ◀

Nondeterministic Running Time

Recall:

- A nondeterministic Turing machine / algorithm M accepts L ($\mathcal{L}(M) = L$) if for all $x \in L$ there is at least one computation of M which accepts x and for all $y \notin L$ every computation of M rejects y.
- We only consider always-terminating Turing machines.
- The running time $time_M(x)$ of M on x is given by the longest computation of M on x.

Definition 2.16 (NTIME, NP)

For function $f : \mathbb{N} \to \mathbb{N}$, the class $NTIME(f(n))$ is the set of all languages A, for which there is a ***nondeterministic*** Turing machine M with $\mathcal{L}(M) = A$ and $time_M(w) \le f(|w|)$ für alle $w \in \Sigma^\star$.
The class of languages NP is defined by

$$\mathsf{NP} \;:=\; \bigcup_{p \text{ polynomial}} NTIME(p(n)).$$

◀

2.5 Nondeterminism = Verification

Nondeterminism?

- The original definition of NP via nondeterministic Turing machines is not very intuitive.
- There is an equivalent characterization that is usually more convenient to use: Certificates and verifiers.

Polynomially verifiable

Definition 2.17 (Certificates, Verifier, VP)

Let $L \subseteq \Sigma^\star$ be a language.

- An algorithm A acting on inputs from $\Sigma^\star \times \{0,1\}^\star$ is called *verifier for L* (notation $L = \mathcal{V}(A)$), if

$$L \quad = \quad \{w \in \Sigma^\star : \exists c \in \{0,1\}^\star \; A(w,c) = 1\}.$$

  If A accepts input (w,c) we say c is *proof* or *certificate* for $w \in L$.

- A verifier A for L is a *polynomial-time verifier* if there is a $d \in \mathbb{N}$ such that for all $w \in L$, there is a proof c (for $w \in L$) with $time_A(w,c) \in O(|w|^d)$.

- We define the class of polynomially verifiable languages VP by

$$\mathsf{VP} \quad = \quad \{\mathcal{V}(A) : A \text{ is polynomial time verifier}\}.$$

◀

Nondeterminism ↔ certificate

Theorem 2.18
NP = VP.

2.6 Karp-Reductions und NP-Completeness

Recap

- We restrict our attention to ***decision problems***:
  Given: $w \in \Sigma^\star$.
  Goal: Is $w \in L$?

Example:
w is an encoding of an instance of the traveling salesperson problem **and** a threshold D

$L = \{w : w \text{ encodes instance w/ opt. round trip length} \le D\}$

$\rightsquigarrow$ problems = (formal) languages $L \subseteq \Sigma^\star$

- problem instance = word $w \in \Sigma^\star$
- $w \in \Sigma^\star$ is a *Yes instance* if $w \in L$, otherwise a *No instance*

$\rightsquigarrow$ For problems on structures, e. g., graphs, we need an *encoding* of the instance as a string.
(often simple; standard data structures do the trick)

$\rightsquigarrow$ input size n of instance = length of the *encoding* of instance

$\rightsquigarrow$ all running times are worst case over instances of encoding length n

Karp Reductions

$\leq_p$

A: old problem
Consensus: hard

B: new problem
Status: unknown
(seems hard to *us* ...)

- **Goal:** Show that B is at least as hard as A.

short for: deterministic TM M w/ $Time_M(n) = O(n^k)$ for constant k.

Assume there *was* a polytime algorithm M für B.
Solve A using M (in polytime).

$\rightsquigarrow$ polytime algo for B implies polytime algo for A

$\rightsquigarrow$ B at least as hard as A

Formally: (strong notion than intuition above!)

Definition 2.19 (polytime reduction, $\leq_p$)

Let $A \subseteq \Sigma^\star$ and $B \subseteq \Gamma^\star$ be languages (decision problems).
A is *polytime reducible to* B – written $A \leq_p B$ – if there is a total function $g : \Sigma^\star \to \Gamma^\star$, computable in polynomial time, with

$$\forall w \in \Sigma^\star \ : \ w \in A \Leftrightarrow g(w) \in B.$$

◀

- This type of reduction is called a *Karp-reduction*
- It is more restrictive than our intuitive version would need, but allows finer complexity classification (NP and co-NP)

Implication of Reductions

Lemma 2.20 (Membership reduction)

If $A \leq_p B$ and $B \in \mathsf{P}$ (resp. $B \in \mathsf{NP}$), then $A \in \mathsf{P}$ (resp. $A \in \mathsf{NP}$).

Proof:

Since $B \in \mathsf{P}$, there is polytime TM M with $\mathcal{L}(M) = B$.
Since $A \leq_p B$, there further is polytime TM g mit $\forall w : w \in A \Leftrightarrow g(w) \in B \quad (*)$.
We construct TM M' for A:
first simulate g on input w, then simulate M on $g(w)$.

Since M and g are polytime TMs, so is M', and $\mathcal{L}(M') = A$ (since $(*)$).
$\rightsquigarrow \quad A \in \mathsf{P}$.

(The version with $B \in \mathsf{NP}$ is similar, just using nondeterministic polytime). ■

NP Completeness

Definition 2.21 (NP-hard, NP-complete)

A language A is called NP-*hard*, if we have for **all** languages $L \in \mathsf{NP}$ that $L \le_p A$.
A language A is called NP-*complete*, if A is NP-hard and $A \in \mathsf{NP}$.

Theorem 2.22 (One for all and all for one)

For an NP-complete language A holds: $A \in \mathsf{P} \iff \mathsf{P} = \mathsf{NP}$.

$\rightsquigarrow$ Under the *consensus hypothesis* that $\mathsf{P} \subsetneq \mathsf{NP}$, this means that
for an NP-complete problem A, we should expect **no efficient solution** for A.

Proof:

$\Rightarrow$ Let $L \in \mathsf{NP}$ be arbitrary.
Since A is (by assumption) NP-hard, we have $L \le_p A$.
Since $A \in \mathsf{P}$, by membership reduction (Lemma **??**) also $L \in \mathsf{P}$.
Since L was an *arbitrary* language from NP, $\mathsf{P} = \mathsf{NP}$ follows.

$\Leftarrow$ Let conversely $\mathsf{P} = \mathsf{NP}$.
Then, by assumption, $A \in \mathsf{NP} = \mathsf{P}$. ■

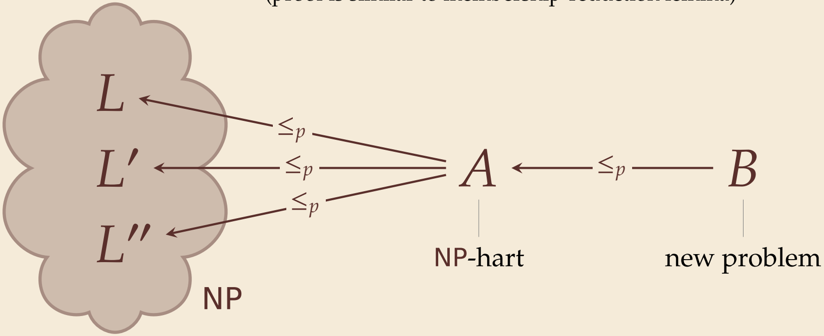
Implications of NP-Completeness

- NP-completeness tells us a lot about a problem!

But shall we possible prove $L \leq_p A$ *for all* possible $L \in \mathsf{NP}$?

- One can show: $\leq_p$ is a *transitive* relation on languages.

  (proof is similar to membership-reduction lemma)



A NP-hard and $A \leq_p B$

$\rightsquigarrow$ B NP-hard

The Mother of All Problems

- It remains to identify a first NP-complete problem! Are there any *at all?*

Theorem 2.23 (Cook-Levin)

SAT is NP-complete.

- SAT is the *satisfiability problem* of propositional logic:
  Given: Boolean (propositional logic) formula φ over variables $x_1, \ldots, x_n$.
  Goal: Is there a variable assignment $V : \{x_1, \ldots, x_n\} \to \{\mathit{true}, \mathit{false}\}$,
  so that φ evaluates under V to true?

Proof (Theorem ??):

Idea: Given any nondeterministic TM M for an arbitrary language $L \in$ NP, construct from a word w a formula $\varphi(w)$, which exactly encodes all valid computations of M.

Variables $x_{q,t}$: Is M be in state q at time t?
$y_{c,i,t}$: Does tape cell i contain char c at time t?
$z_{i,t}$ Does read/write head stand at position i at time t?

$\rightsquigarrow$ $\varphi(w)$ satisfiable iff M accepts w.

(for details, see, e. g., §2.3 of S. Arora and B. Barak: *Computational Complexity: A Modern Approach*) ■

2.7 Example of an NP-completeness proof

3SAT

- Let's do a more typical full example.
- Need one more NP-complete problem first

Definition 2.24 (3SAT)

Given: A Boolean formula φ in *3-CNF*:
conjunctive normal form with *at most* 3 literals per clause

Goal: Is there an assignment V of the variables in φ, so that φ evaluates to *true*?
(a. k. a. Is φ *satisfiable*?)

Example:

$$(x_1 \vee \neg x_3 \vee x_2) \;\wedge\; (\neg x_3 \vee x_4 \vee \neg x_5) \;\wedge\; (x_5 \vee \neg x_5 \vee \neg x_1) \;\wedge\; (x_1 \vee x_3 \vee x_5)$$

satisfiable, e. g., via $x_1 \mapsto \mathit{true}$, $x_5 \mapsto \mathit{false}$ (other variables arbitrary)

Theorem 2.25 (3SAT)

SAT $\le_p$ 3SAT and 3SAT $\in$ NP.

Corollary 2.26 (3SAT)

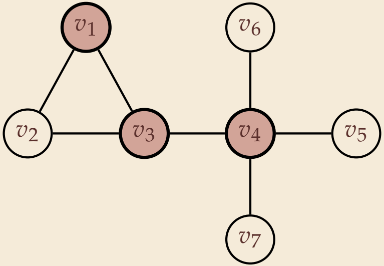
3SAT is NP-complete.

Vertex Cover

Definition 2.27 (VertexCover)

Given: A (simple, undirected) graph $G = (V, E)$, $E \subseteq \binom{V}{2}$, threshold k.

Goal: $\exists S \subseteq V$ with $|S| \le k$, such that $\forall e \in E : S \cap e \ne \emptyset$?



Intuitively: a small subset S of vertices of a graph, such that every edge is *covered* by S

Theorem 2.28 (VertexCover hard)

VertexCover is NP-complete.

Proof:

We will prove (i) VertexCover $\in$ VP and (ii) 3SAT $\le_p$ VertexCover.

$\rightsquigarrow$ Theorem **??** (since 3SAT is NP-complete and VP = NP).

(i) certificate = S; verifier whether all edges $e \in E$ are covered and whether $|S| \le k$

$\rightsquigarrow$ clearly doable in polytime $\rightsquigarrow$ VertexCover $\in$ VP.

(ii) 3SAT $\leq_p$ VertexCover

Proof:

- Intuition: Express 3SAT instance as a VertexCover instance.
- So, let φ be an arbitrary formula in 3-CNF over variables $x_1, \ldots, x_m$

$\rightsquigarrow$ φ has the form $\underbrace{(l_{1,1} \vee l_{1,2} \vee l_{1,3})}_{C_1} \wedge \underbrace{(l_{2,1} \vee l_{2,2} \vee l_{2,3})}_{C_2} \wedge \cdots \wedge \underbrace{(l_{n,1} \vee l_{n,2} \vee l_{n,3})}_{C_n}$,
with $l_{i,j} \in \{x_1, \neg x_1, \ldots, x_m, \neg x_m\}$ for $i = 1, \ldots, n$ and $j = 1, 2, 3$.

- Define a graph $G = (V, E)$ via

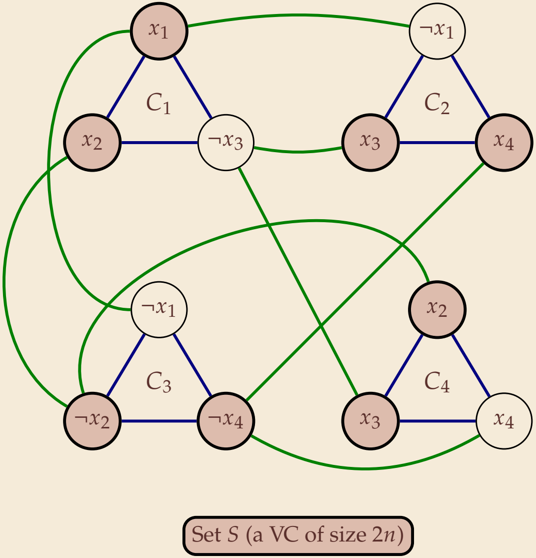
$$V = \{L_{i,j} : i = 1, \ldots, n;\ j = 1, 2, 3\}$$
$$E = \Big\{\{L_{i,j}, L_{p,q}\} : l_{i,j} \equiv \neg l_{p,q}\Big\} \cup \Big\{\{L_{i,1}, L_{i,2}\}, \{L_{i,2}, L_{i,3}\}, \{L_{i,3}, L_{i,1}\} : i = 1, \ldots, n\Big\}$$

We "draw" a vertex for every literal of a clause. We connect them if
(a) they are literals in the same clause or (b) they are negations of each other

$\rightsquigarrow$ **Claim:** φ satisfiable $\iff$ G has vertex cover of size $\leq 2n$.

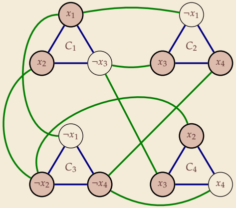
(ii) 3SAT $\leq_p$ VERTEXCOVER – Example

$\varphi = (x_1 \lor x_2 \lor \neg x_3) \land (\neg x_1 \lor x_3 \lor x_4) \land (\neg x_1 \lor \neg x_2 \lor \neg x_4) \land (x_2 \lor x_3 \lor x_4)$



- **Idea:** Vertices ***not in*** vertex cover S define a variable assignment.
  - Cannot be contradictory, otherwise "negation"-edge not covered.
  - Must take ≥ 2 vertices per clause into S (otherwise triangle not covered)
  - $\rightsquigarrow$ $|S| \geq 2n$ for every vertex cover.
- In the example:
  - Fat vertices form a vertex cover for G
  - corresponding assignment: $V = \{x_1 \mapsto 0, x_2 \mapsto 0, x_3 \mapsto 0, x_4 \mapsto 1\}$ ($0 \mathrel{\widehat{=}} \mathit{false}, 1 \mathrel{\widehat{=}} \mathit{true}$)
  - $\rightsquigarrow$ φ satisfiable

(ii) 3SAT $\le_p$ VertexCover – Correctness Proof



Claim: φ satisfiable $\iff$ G has VC S of size $|S| \le 2n$.

$\Rightarrow$ Let φ erfüllbar; $\rightsquigarrow$ every C_i has a satisfied literal $l_{i,j}$.
Add the *other two* vertices of the clause to S.
$\rightsquigarrow$ $|S| = 2n$ and S covers all clause triangle edges.
Remaining edges have the form $\{x, \neg x\}$.
If such an edge remained uncovered by S, we would have that both x and $\neg x$ are satisfied literals $\lightning$
$\rightsquigarrow$ G has VC of size $2n$.

$\Leftarrow$ Given a VC S of G with $|S| \le 2n$.
S must contain 2 vertices per clause triangle $\rightsquigarrow$ $|S| = 2n$ and S is a *minimal* VC.
Define assignment V so that all literals *not* in S are satisfied.
(Variables which are not assigned a value via this procedure can be assigned an arbitrary value.)
V is well defined, since $\{x, \neg x\}$-edges must be covered.
Moreover, V makes φ *true*:
from every clause, at least one literal is satisfied since $|S| = 2n$.
$\rightsquigarrow$ φ satisfiable. ∎

(ii) 3SAT $\leq_p$ VERTEXCOVER – Running Time

- Construction of G upon input φ can easily be done in polytime
  - $|V| = O(n)$, $|E| = O(n^2)$
  - Construction of E in time $O(n^2)$ easy to do, e. g., on RAM $\rightsquigarrow$ $\exists$ polytime TM.

$\rightsquigarrow$ 3SAT $\leq_p$ VERTEXCOVER. $\blacksquare$

2.8 Important NP-Complete Problems

Further NP-complete problems [1]

Apart from SAT, 3SAT, and VertexCover, here are some of the most useful NP-complete problems.

Definition 2.29 (Dominating Set)

Given: graph $G = (V, E)$ and $k \in \mathbb{N}$

Question: $\exists V' \subset V \quad : \quad |V'| \leq k \ \wedge \ \forall v \in V : \big(v \in V' \ \vee \ \exists u \in N(v) : u \in V'\big)$ ◀

Definition 2.30 (Hamiltonian Cycle)

Given: graph $G = (V, E)$ (directed and undirected version)

Question: Is there a vertex-simple cycle in G of length $|V|$? ◀

Definition 2.31 (Clique)

Given: graph $G = (V, E)$ and $k \in \mathbb{N}$

Question: $\exists V' \subset V \quad : \quad |V'| \geq k \ \wedge \ \forall u, v \in V' : \{u, v\} \in E$ ◀

Definition 2.32 (Independent Set)

Given: graph $G = (V, E)$ and $k \in \mathbb{N}$

Question: $\exists V' \subset V \quad : \quad |V'| \geq k \ \wedge \ \forall u, v \in V' : \{u, v\} \notin E$ ◀

Further NP-complete problems [2]

Definition 2.33 (Traveling Salesperson (TSP))

Given: distance matrix $D \in \mathbb{N}^{n \times n}$ and $k \in \mathbb{N}$

Question: Is there a permutation $\pi : [n] \to [n]$ with $\sum_{i=1}^{n-1} D_{\pi(i),\pi(i+1)} + D_{\pi(n),\pi(1)} \leq k$? ◀

Definition 2.34 (Graph Coloring)

Given: graph $G = (V, E)$ and $k \in \mathbb{N}$

Question: $\exists c : V \to [k] \;:\; \forall \{u, v\} \in E : c(u) \neq c(v)$? ◀

Definition 2.35 (Set Cover)

Given: $n \in \mathbb{N}$, sets $S_1, \ldots, S_m \subseteq [n]$ and $k \in \mathbb{N}$

Question: $\exists I \subseteq [m] \;:\; \bigcup_{i \in I} S_i = [n] \;\wedge\; |I| \leq k$? ◀

Definition 2.36 (Weighted Set Cover)

Given: $n \in \mathbb{N}$, sets $S_1, \ldots, S_m \subseteq [n]$, costs $c_1, \ldots, c_m \in \mathbb{N}_0$ and $k \in \mathbb{N}$

Question: $\exists I \subseteq [m] \;:\; \bigcup_{i \in I} S_i = [n] \;\wedge\; \sum_{i \in I} c_i \leq k$? ◀

Further hard problems [3]

Definition 2.37 (Closest String)

Given: $s_1, \ldots, s_n \in \Sigma^m$ and $k \in \mathbb{N}$
Question: $\exists s \in \Sigma^m \quad : \quad \forall i \in [n] : d_H(s, s_i) \leq k$? $\qquad$ (d_H Hamming-distance) ◀

Definition 2.38 (Max Cut)

Given: graph $G = (V, E)$ and $k \in \mathbb{N}$
Question: $\exists C \subset V \quad : \quad \big|E \cap \{\{u, v\} \mid u \in C, v \notin C\}\big| \geq k$? ◀

Definition 2.39 (Exact Cover)

Given: $n \in \mathbb{N}$, sets $S_1, \ldots, S_m \subseteq [n]$
Question: $\exists I \subseteq [m] \quad : \quad \bigcup_{i \in I} S_i \ = \ [n] \ \wedge \ \sum_{i \in I} |S_i| = n$? ◀

Further hard problems [4]

Definition 2.40 (Subset Sum)

Given: $x_1, \dots, x_n \in \mathbb{Z}$

Question: $\exists I \subseteq [n] \ : \ I \neq \emptyset \wedge \sum_{i \in I} x_i = 0$? ◀

Definition 2.41 ((0/1) Knapsack)

Given: $w_1, \dots, w_n \in \mathbb{N}$, $v_1, \dots, v_n \in \mathbb{N}$ and $b, k \in \mathbb{N}$

Question: $\exists I \subseteq [n] \ : \ \sum_{i \in I} w_i \le b \ \wedge \ \sum_{i \in I} v_i \ge k$? ◀

Definition 2.42 (Bin Packing)

Given: $w_1, \dots, w_n \in \mathbb{N}$, $b \in \mathbb{N}$, $k \in \mathbb{N}$

Question: $\exists a : [n] \to [k] \ : \ \forall j \in [k] : \sum_{\substack{i=1,\dots,n \\ a[i]=j}} w_i \ \le \ b$? ◀

Definition 2.43 (0/1 Integer Programming)

Given: integer linear program (ILP) $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$ and $c \in \mathbb{Z}^n$ and $k \in \mathbb{Z}$

Question: Is there $x \in \{0,1\}^n$ with $Ax \le b$ and $c^T x \ge k$? ◀

2.9 Optimization Problems

Optimization Problems

Definition 2.44 (Optimization Problem)

An *optimization problem* is given by 7-tuple $U = (\Sigma_I, \Sigma_O, L, L_I, M, cost, goal)$ with

1. Σ_I an alphabet (called input alphabet),
2. Σ_O an alphabet (called output alphabet),
3. $L \subseteq \Sigma_I^\star$ the language of allowable problem instances (for which U is well-defined),
4. $L_I \subseteq L$ the language of actual problem instances for U (for those we want to determine U's complexity),
5. $M : L \to 2^{\Sigma_O^\star}$ and with $x \in L$, $M(x)$ is the set of all feasible solutions for x.
6. $cost$ is a cost function, which assigns for $x \in L$ each pair (u, x) with $u \in M(x)$ a positive real number,
7. $goal \in \{\min, \max\}$.

◀

Optimal Solutions

Definition 2.45 (Optimal Solutions, Solution Algorithms)

Let $U = (\Sigma_I, \Sigma_O, L, L_I, M, cost, goal)$ an optimization problem. For each $x \in L_I$ a feasible solution $y \in M(x)$ is called *optimal for x and U*, if

$$cost(y, x) \quad = \quad goal\{cost(z, x) \mid z \in M(x)\}.$$

An algorithm A is *consistent with* U if $A(x) \in M(x)$ for all $x \in L_I$.
We say *algorithm B solves* U, if

1. B is consistent with U and
2. for all $x \in L_I$, $B(x)$ is optimal for x and U.

Optimization Problems – Examples

Natural examples: Problems above with an input parameter k.

Less immediate example:

Definition 2.46 (Max-SAT)

Given: CNF-Formula $\phi = C_1 \wedge \cdots \wedge C_m$ over variables $x_1, \ldots, x_n$
Allowable (=Actual) Instances: encodings of ϕ
$M(\phi) = \{0,1\}^n$ (variable assignments)
$cost(u, x)$: # of satisfied clauses in u under given assignment x
$goal = \max$

Classes of Optimization Problems

Definition 2.47 (NPO)

$\mathcal{NPO}$ is the class if optimization problems $U = (\Sigma_I, \Sigma_O, L, L_I, M, cost, goal)$ with

1. $L_I \in \mathsf{P}$,
2. there is a polynomial p_U with
   a) $\forall x \in L_I \ \forall y \in M(x) : |y| \leq p_U(|x|)$ and
   b) there is a polynomial time algorithm which for all $y \in \Sigma_O^\star$, $x \in L_I$ with $|y| \leq p_U(|x|)$ decides whether $y \in M(x)$ holds, and
3. function $cost$ can be computed in polynomial time. ◀

Definition 2.48 (PO)

PO is the class of optimization problems $U = (\Sigma_I, \Sigma_O, L, L_I, M, cost, goal)$ with

1. $U \in \mathcal{NPO}$, and
2. there is an algorithm of polynomial time complexity which for all $x \in L_I$ computes an optimal solution for x and U. ◀

From Optimization to Decision

Definition 2.49 (Threshold Languages)

Let $U = (\Sigma_I, \Sigma_O, L, L_I, M, cost, goal)$ an optimization problem, $U \in \mathcal{NPO}$.
For $Opt_U(x)$ the cost of an optimal solutions for x and U we define the *threshold language for U* as

$$Lang_U = \begin{cases} \left\{(x,k) \in L_I \times \{0,1\}^\star \;\middle|\; Opt_U(x) \leq k_2\right\}, & \text{if } goal = \min, \\ \left\{(x,k) \in L_I \times \{0,1\}^\star \;\middle|\; Opt_U(x) \geq k_2\right\}, & \text{if } goal = \max. \end{cases}$$

We say U *is* NP-*hard*, if $Lang_U$ is $\mathcal{NP}$-hard. ◀

Corollary 2.50 (Optimization is harder than Threshold)

Let U an optimization problem.
If $Lang_U$ is NP-hard and if P $\neq$ NP holds, we have $U \notin$ PO. ◀

Max-SAT is hard

Corollary 2.51 (Max-SAT is hard)

MAX-SAT is NP-hard. ◀

Summary

- We have formalized the classic notion of intractable problems.
  - What is running time, what is "polytime"?
  - Decision problems $\leftrightarrow$ (formal) languages
  - P, NP via Turing machines $\leftrightarrow$ certificates and verifiers
  - For the typical case of optimization problems, there are different versions of the problem, but (in)tractability typically carries over.

$\rightsquigarrow$ We can mathematically prove a problem is intractable (NP-hard).

… but how can we tackle hard problems anyway?