11 LP-Based Approximation

15 July 2025

Prof. Dr. Sebastian Wild

CS627 (Summer 2025)
Philipps-Universität Marburg
version `2025-07-15 23:09`

Outline

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11.1 (Integer) Linear Optimization Recap

LPs in Standard Form

Definition 11.1 (LP)

A linear program (LP) in *standard form* with n *variables* and m *constraints* is characterized by a matrix $\boldsymbol{A} \in \mathbb{Z}^{m \times n}$, a vector $\boldsymbol{b} \in \mathbb{Z}^m$, and a vector $\boldsymbol{c} \in \mathbb{Z}^n$ and is written as

$$\begin{aligned} \min \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \end{aligned} \qquad\qquad \begin{aligned} \min \quad & \textstyle\sum_{j=1}^n c_j \cdot x_j \\ \text{s.t.} \quad & \textstyle\sum_{j=1}^n a_{ij} \cdot x_j \geq b_i \quad \text{for all } i \in [m] \\ & x_j \geq 0 \quad \text{for all } j \in [n] \end{aligned}$$

(Inequalities on vectors apply componentwise.)
Any vector $x \in \mathbb{R}^n$ with $Ax \geq b$ and $x \geq 0$ is called a *feasible solution* for the LP, and $c^T x$ is its objective value. An *optimal solution* is a feasible vector x^* with **min**imal objective value. ◀

Remark 11.2 (Rational coefficients)

We can in general allow $A \in \mathbb{Q}^{m \times x}$, $b \in \mathbb{Q}^m$ and $c \in \mathbb{Q}^n$; by multiplying constraints and scaling objective function with the common denominator we obtain an equivalent LP. ◀

Example LP

$$\begin{aligned} \min \quad & 7x_1 + x_2 + 5x_3 \\ \text{s.t.} \quad & x_1 - x_2 + 3x_3 \geq 10 \\ & 5x_1 + 2x_2 - x_3 \geq 6 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

$\rightsquigarrow$ Optimal solution $x^* = (1.75, 0, 2.75)$ with $c^T x^* = 26$.

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Extreme point: feasible point that is *not* a convex combination of two distinct feasible solutions.

Remark 11.3 (Facts on LPs)

1. More general versions of LP possible:
   $=$ constraints, unrestricted variables, max instead of min . . .
   $\rightsquigarrow$ can all be transformed into equivalent one in standard form.
2. LP can be *infeasible* (no solution), *unbounded* (no optimal solution) or *finite*.
3. If LP has optimal solution, there is an optimal extreme point $\rightsquigarrow$ finite problem!
4. Optimal solutions can be computed in polytime (ellipsoid method).

Integer Linear Program in Standard Form

Definition 11.4 (ILP)

An *integer linear program* in standard form is an LP with the additional integrality constraints $x_j \in \mathbb{N}_0$:

$$\begin{aligned} \min \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b} \\ & \boldsymbol{x} \in \mathbb{N}_0^n \end{aligned}$$

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Remark 11.5 (Facts on ILPs)

1. Generalized versions can again be transformed into standard form.
2. Decision version of the problem NP-complete.

11.2 LP Relaxations & Rounding

LP Relaxation Approximations

Since ILPs are **NP**-complete, any **NP** problem can be written as an ILP

well, for decision versions . . . but often very natural to write optimization problems as ILP

Hard part of approximation: Get a bound on OPT!

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$\rightsquigarrow$ A natural idea to obtain approximately optimal solutions for NPO problems:

1. Formulate problem as ILP (I)
2. Drop integrality constraints from (I) $\rightsquigarrow$ LP (P)
3. Obtain optimal fractional solution $\boldsymbol{x}^*$ for (P)
   Cost of $\boldsymbol{x}^*$ is bound for OPT!

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Note: Integrality gap of (I)LP is key barrier in this approach

Set Cover as ILP

$S = (S_1, \ldots, S_k)$

The Set Cover ILP

Idea $x_j = 1$ iff S_j in cover.

Notation: For $e \in U = [n]$ set $V(e) = \{j : e \in S_j\}$.

$$\begin{aligned} \min \quad & \sum_{j=1}^{k} c(S_j) \cdot x_j \\ \text{s. t.} \quad & \sum_{j \in V(e)} x_j \geq 1 \quad \forall e \in U \qquad \text{(I)} \\ & x \in \mathbb{N}_0^k \end{aligned}$$

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LP Relaxation: replace $x \in \mathbb{N}_0^k$ by $x \geq 0$.

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Write $OPT_{(I)}$ resp. $OPT_{(P)}$ for the optimal objective value $\rightsquigarrow$ $OPT_{(I)}$ ~~$\leq$~~ $\geq$ $OPT_{(P)}$

Simple Rounding

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procedure frequencyCutoffSetCover(n,S,c)
    f := global frequency of S
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    𝒞 := ∅
    for j := 1,...,k
        if x*_j ≥ 1/f then add j to 𝒞
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Theorem 11.6

frequencyCutoffSetCover is an f-approximation for SetCover.

Corollary 11.7

frequencyCutoffSetCover is a 2-approximation for WeightedVertexCover.

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Proof:

(1) $\mathcal{C}$ is a set cover

Let $e \in U$ be arbitrary. Since x^* is feasible, we have $\sum_{j \in V(e)} x_j^* \geq 1$.

$$\begin{aligned} \min \quad & \sum_{j=1}^{k} c(S_j) \cdot x_j \\ \text{s. t.} \quad & \sum_{j \in V(e)} x_j \geq 1 \quad \forall e \in U \qquad \text{(P)} \\ & x \geq 0 \end{aligned}$$

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$\rightsquigarrow \quad j \in \mathcal{C}$ and e is covered.

Simple Rounding [2]

Proof (cont.):

(2) f-approximation.

min-problem

x^* optimal for (P) $\rightsquigarrow$ $c^T x^* = OPT_{(P)} \leq OPT_{(I)}$. For every $j \in \mathcal{C}$, $x^*_j \geq 1/f$.

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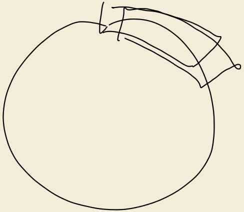
Consider the **"Fully Symmetric instance:"**

Suppose $f \mid n$
$U = [0..n)$ with $S_j = \{j, j+1, \ldots, j+f-1\} \bmod n$, for all $j \in [0..n)$
All sets of equal cost, $c(S_j) = 1$



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$\rightsquigarrow$ n/f sets suffice;
but $x^* = (\frac{1}{f}, \ldots, \frac{1}{f})$ is optimal for (P) $\rightsquigarrow$ frequencyCutoffSetCover outputs $\mathcal{C} = [0..n)$

11.3 Randomized Rounding

Fractions as probabilities

Another intuitive use of fractional solutions $x_j^* \in (0, 1)$: include S_j with probability x_j^* in $\mathcal{C}$.

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Intuition: If e occurs in f_e sets, we have

$$\mathbb{P}[e \text{ covered}] \;=\; 1 - \mathbb{P}\left[\bigcap_{j \in V(e)} S_j \notin \mathcal{C}\right] \;=\; 1 - \prod_{j \in V(e)} \left(1 - x_j^*\right)$$

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Assuming we keep trying and collect *all* sets ever chosen

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Curiously, H_n is also approx. ratio of greedy . . .

But randomized rounding is general & tweakable.

Randomized Rounding

```
procedure randomizedRoundingSet(n, S, c, r)
    x* := optimal solution of relaxed set cover LP.
    for i := 1, ..., r
        C_i := ∅
        for j := 1, ..., k
            b := coin flip with prob x*_j
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    $\mathcal{C}_i := \emptyset$
    for $j := 1, \ldots, k$
      b := coin flip with prob x_j^*
      if $b == 1$ **then** $\mathcal{C}_i := \mathcal{C}_i \cup \{j\}$
  return $\mathcal{C} := \bigcup_{i=1}^{r} \mathcal{C}_i$

safely above CC's H_n

For simplicity, always set $r = \lceil \ln(4n) \rceil$

Lemma 11.8
randomizedRoundingSet computes a feasible set-cover with probability $\geq \frac{3}{4}$. ◀

Proof:
Recall from calculation above that for $e \in U$ and a single iteration of the outer loop:

$$\mathbb{P}[e \text{ not covered by } \mathcal{C}_i] \leq \left(1 - \frac{1}{f_e}\right)^{f_e} \leq \frac{1}{e}$$

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$$\rightsquigarrow \quad \mathbb{P}[e \text{ not covered by } \mathcal{C}] = \prod_{i=1}^{r} \mathbb{P}[e \text{ not covered by } \mathcal{C}_i] \leq \left(\frac{1}{e}\right)^r$$

With the union bound over all n elements and $r = \ln(4n)$, we obtain
$\mathbb{P}[\mathcal{C} \text{ not a set cover}] \leq ne^{-r} = \frac{1}{4}$. ■

Randomized Rounding – Analysis

Lemma 11.9 (Expected quality)

Let $\mathcal{C}$ by computed by randomizedRoundingSet with r repetitions.
The *expected* cost are $\mathbb{E}[c(\mathcal{C})] \le r \cdot OPT_{(P)}$. ◀

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$\rightsquigarrow$ For $r = \ln(4n)$ we have by Markov's inequality: $\mathbb{P}\left[c(\mathcal{C}) \ge 4\ln(4n) \cdot OPT_{(P)}\right] \le \frac{1}{4}$

Randomized Rounding – Analysis

Lemma 11.9 (Expected quality)

Let $\mathcal{C}$ by computed by randomizedRoundingSet with r repetitions.
The *expected* cost are $\mathbb{E}[c(\mathcal{C})] \le r \cdot OPT_{(P)}$.

$\rightsquigarrow$ For $r = \ln(4n)$ we have by Markov's inequality: $\mathbb{P}\left[c(\mathcal{C}) \ge 4\ln(4n) \cdot OPT_{(P)}\right] \le \frac{1}{4}$

Proof:
We choose $\mathcal{C} = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_r$.
For the cost we get

$$\mathbb{E}[c(\mathcal{C})] \;\le\; \mathbb{E}\left[\sum_{i=1}^{r} c(\mathcal{C}_i)\right] \;=\; \sum_{i=1}^{r} \mathbb{E}[c(\mathcal{C}_i)] \;=\; r \cdot OPT_{(P)}$$

■

Randomized Rounding Approximation for Set Cover

So far, randomizedRoundingSet might return infeasible solution. ϟ

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```
procedure randomizedRoundingSetCover(n, S, c)
    𝒞 = randomizedRoundingSet(n, S, c, ⌈ln(4n)⌉)
    if 𝒞 is a set cover
        return 𝒞
    else
        return S
```

Theorem 11.10 (randomizedRoundingSetCover randomized approx)

randomizedRoundingSetCover is a randomized $4\ln(4n)$-approximation for SetCover. ◀

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Proof:

$$\mathbb{P}[\mathcal{C} \text{ not SC} \;\vee\; c(\mathcal{C}) > 4\ln(4n)\cdot OPT_{(P)}] \;\le\; \mathbb{P}[\mathcal{C} \text{ not SC}] + \mathbb{P}[c(\mathcal{C}) > 4\ln(4n)\cdot OPT_{(P)}]$$

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Proof:

$$\begin{aligned}
\mathbb{P}[\mathcal{C} \text{ not SC} \ \vee\ c(\mathcal{C}) > 4\ln(4n)\cdot OPT_{(P)}] &\le \mathbb{P}[\mathcal{C} \text{ not SC}] + \mathbb{P}[c(\mathcal{C}) > 4\ln(4n)\cdot OPT_{(P)}] \\
&\underset{\text{Lemma 11.8, Lemma 11.9}}{\le} \frac{1}{4} + \frac{1}{4} \\
&= \frac{1}{2}.
\end{aligned}$$

■

11.4 LP Duality

LPs for Approximation

Suppose we consider a minimization NPO problem.

Recall: Key use of LP relaxation for approximation: Get lower bound for OPT.

LPs for Approximation

Suppose we consider a minimization NPO problem.

Recall: Key use of LP relaxation for approximation: Get lower bound for *OPT*.

There's another powerful technique from linear optimization that can do that: the *dual problem*!

Bounding optimal values of LPs

Starting with an original ("primal") LP, how can we bound on its optimal objective value?

$$
\begin{aligned}
\min \quad & 7x_1 + x_2 + 5x_3 \\
\text{s.t.} \quad & y_1 \cdot (x_1 - x_2 + 3x_3) \geq 10\, y_1 \quad (a) \\
& y_2 \cdot (5x_1 + 2x_2 - x_3) \geq 6\, y_2 \quad (b) \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

goal: lower bound $7x_1 + x_2 + 5x_3$

$7x_1 + x_2 + 5x_3 \underset{x \geq 0}{\geq} x_1 - x_2 + 3x_3 \underset{\text{feasible}}{\geq} 10$

$7x_1 + x_2 + 5x_3 \not\geq 5x_1 + 2x_2 - x_3 \geq 6$

$7x_1 + x_2 + 5x_3 \underset{a+b}{\geq} 6x_1 + x_2 + 2x_3 \geq 16$

Dual

$$
\begin{aligned}
\max \quad & 10y_1 + 6y_2 \\
& y_1 + 5y_2 \leq 7 \quad // x_1 \\
& -y_1 + 2y_2 \leq 1 \quad // x_2 \\
& 3y_1 - y_2 \leq 5 \quad // x_3 \\
& y_1, y_2 \geq 0
\end{aligned}
$$

Optimal solution:
$x^* = (1.75, 0, 2.75)$ with $c^T x^* = 26$.

Dual LPs

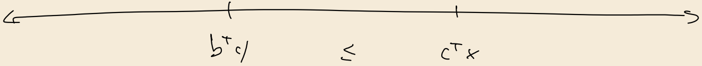
$$\begin{aligned} \min\ & c^T x \\ \text{s.t.}\ & Ax \geq b \qquad \text{(P)} \\ & x \geq 0 \end{aligned} \qquad\qquad \begin{aligned} \max\ & b^T y \\ \text{s.t.}\ & A^T y \leq c \qquad \text{(D)} \\ & y \geq 0 \end{aligned}$$

Generalizations:

- ith constraint in primal with '$\geq$' $\leftrightsquigarrow$ $y_i \geq 0$
- ith constraint in primal with '$=$' $\leftrightsquigarrow$ y_i unconstrained

Lemma 11.11 (Weak Duality)

If x and y are *feasible* solutions for the primal resp. dual LP, it holds that $\boxed{c^T x \geq b^T y.}$



Dual LPs

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Lemma 11.11 (Weak Duality)

If x and y are *feasible* solutions for the primal resp. dual LP, it holds that $\boxed{c^T x \geq b^T y.}$

Proof:

Dual constraint $A^T y \leq c$ implies $c^T \geq (A^T y)^T = y^T A$.

$$\rightsquigarrow \quad c^T x \;\geq\; (y^T A)x \;=\; y^T (Ax) \underset{\text{prim. constr.}}{\geq} y^T b \;=\; b^T y$$

∎

Duality Theory

Indeed, one can show by a closer study that the optimal objective values *always coincide*.

Theorem 11.12 (Strong duality)

The primal LP has a finite optimal objective if and only if the dual has. If x^* resp. y^* are two optimal solutions to the primal resp. dual LP then $c^T x^* = b^T y^*$ holds. ◀

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Theorem 11.13 (Complementary Slackness Conditions (CSC))

Let x and y be feasible solutions to the primal and dual LP.
The pair (x, y) is optimal *if and only if*

1. $\forall j \in [n] \,:\, x_j = 0 \vee \sum_{1 \le i \le m} a_{i,j} \cdot y_i = c_j$ and
2. $\forall i \in [m] \,:\, y_i = 0 \vee \sum_{1 \le j \le n} a_{i,j} \cdot x_j = b_i$. ◀

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2. $\forall i \in [m] \;:\; y_i = 0 \;\vee\; \sum_{1 \le j \le n} a_{i,j} \cdot x_j = b_i$. ◀

Remark 11.14

1. Strong duality implies that the LP threshold decision problem is in NP ∩ co-NP:
   Yes-certificate: feasible solution; No-certificate: feasible solution *for the dual*.
   (We know it actually lies in P)
2. For ILPs, we only get weak duality. ◀

11.5 Vertex Cover and Matching Revisited

Vertex Cover & Maximum Matching

Vertex Cover

$$\min \sum_{v \in V} x_v$$

$$\text{s.t. } x_v + x_w \geq 1 \quad \forall vw \in E$$

$$x_v \in \{0,1\} \quad \forall v \in V$$

Maximum Matching

$$\max \sum_{e \in E} y_e$$

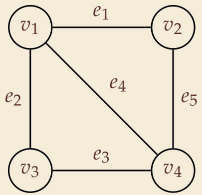
$$\text{s.t. } \sum_{vw \in E} y_{vw} \leq 1 \quad \forall v \in V$$

$$y_e \in \{0,1\} \quad \forall e \in E$$

⇝ Consider the LP relaxations

Vertex Cover & Maximum Matching – Example

Graph G



Minimum Vertex Cover

$$
\begin{array}{lrcl}
\min & x_1 + x_2 + x_3 + x_4 & & \\
\text{s.t.} & x_1 + x_2 & \geq & 1 \\
& x_1 + x_3 & \geq & 1 \\
& x_3 + x_4 & \geq & 1 \\
& x_1 + x_4 & \geq & 1 \\
& x_2 + x_4 & \geq & 1 \\
& x_1,\ x_2,\ x_3,\ x_4 & \geq & 0
\end{array}
$$

Maximum Matching

$$
\begin{array}{lrcl}
\max & y_1 + y_2 + y_3 + y_4 + y_5 & & \\
\text{s.t.} & y_1 + y_2 + y_4 & \leq & 1 \\
& y_1 + y_5 & \leq & 1 \\
& y_2 + y_3 & \leq & 1 \\
& y_3 + y_4 + y_5 & \leq & 1 \\
& y_1,\ y_2,\ y_3,\ y_4,\ y_5 & \geq & 0
\end{array}
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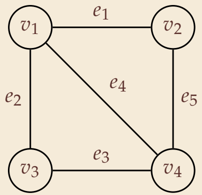
Graph G



Minimum Vertex Cover

$$
\begin{array}{llr}
\min & x_1 + x_2 + x_3 + x_4 & \\
\text{s.t.} & x_1 + x_2 & \geq 1 \\
 & x_1 + x_3 & \geq 1 \\
 & x_3 + x_4 & \geq 1 \\
 & x_1 + x_4 & \geq 1 \\
 & x_2 + x_4 & \geq 1 \\
 & x_1,\ x_2,\ x_3,\ x_4 & \geq 0
\end{array}
$$

$$
A^T \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}
$$

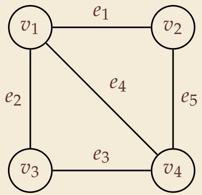
Maximum Matching

$$
\begin{array}{llr}
\max & y_1 + y_2 + y_3 + y_4 + y_5 & \\
\text{s.t.} & y_1 + y_2 + y_4 & \leq 1 \\
 & y_1 + y_5 & \leq 1 \\
 & y_2 + y_3 & \leq 1 \\
 & y_3 + y_4 + y_5 & \leq 1 \\
 & y_1,\ y_2,\ y_3,\ y_4,\ y_5 & \geq 0
\end{array}
$$

$$
A = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}
$$

Vertex Cover & Maximum Matching – Example

Graph G



Minimum Vertex Cover

$$
\begin{array}{lrcl}
\min & x_1 + x_2 + x_3 + x_4 & & \\
\text{s. t.} & x_1 + x_2 & \geq & 1 \\
& x_1 + x_3 & \geq & 1 \\
& x_3 + x_4 & \geq & 1 \\
& x_1 + x_4 & \geq & 1 \\
& x_2 + x_4 & \geq & 1 \\
& x_1 \,,\; x_2 \,,\; x_3 \,,\; x_4 & \geq & 0
\end{array}
$$

$$
\begin{pmatrix}
1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 \\
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\end{pmatrix}
$$

Maximum Matching

$$
\begin{array}{lrcl}
\max & y_1 + y_2 + y_3 + y_4 + y_5 & & \\
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& y_3 + y_4 + y_5 & \leq & 1 \\
& y_1 \,,\; y_2 \,,\; y_3 \,,\; y_4 \,,\; y_5 & \geq & 0
\end{array}
$$

$$
\begin{array}{c|ccccc}
 & e_1 & e_2 & e_3 & e_4 & e_5 \\
\hline
v_1 & 1 & 1 & 0 & 1 & 0 \\
v_2 & 1 & 0 & 0 & 0 & 1 \\
v_3 & 0 & 1 & 1 & 0 & 0 \\
v_4 & 0 & 0 & 1 & 1 & 1
\end{array}
$$

incidence matrix of G!

Vertex Cover & Maximum Matching – Dual Problems

Problems are *dual!*

- ⇝ Our earlier lemma "VC $\geq$ M" is just weak duality (on the ILPs)

Vertex Cover & Maximum Matching – Dual Problems

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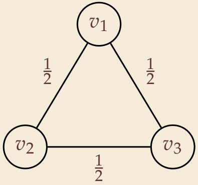
Vertex Cover & Maximum Matching – Dual Problems

Problems are *dual!*

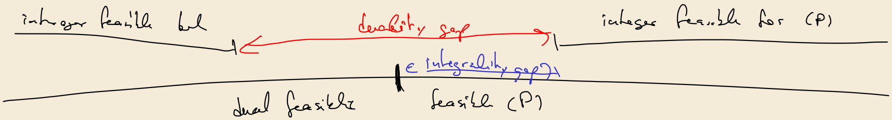
- ⇝ Our earlier lemma "VC ≥ M" is just weak duality (on the ILPs)

- ⇝ Can generally try to build approximation algorithm by constructing pair of primally/dually feasible solutions

Note: Dual **LPs** have **equal** optimal objective value;
For dual **ILPs**, can have a ***duality gap***



- ⇝ For VertexCover/MaximumMatching, duality gap is 2.



Bipartite Graphs

Except for **bipartite graphs!**

Bipartite graph: $V(G) = L \dot{\cup} R$, $E(G) \subset L \times R$

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Bipartite graph: $V(G) = L \dot{\cup} R$, $E(G) \subset L \times R$

Known:

every square submatrix has determinant 0, 1, or −1

- incidence matrix A of bipartite G is a *totally unimodular (TU)* matrix
- A TU $\rightsquigarrow$ LPs $\min\{c^T x : Ax \ge b, x \ge 0\}$ and $\max\{b^T y : A^T y \le c, y \ge 0\}$ with integral b and c have **integral** optimal solutions x^* and y^*

$\rightsquigarrow$ No integrality gap and no duality gap!

Bipartite Graphs

∉ exam

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$\rightsquigarrow$ No integrality gap and no duality gap!

Here, also easy to see directly:

- Maximum matching in bipartite graph must have one side (L or R) completely matched

$\rightsquigarrow$ Taking all of these vertices must be a VC

11.6 Set Cover Duality & Dual Fitting

Dual Fitting

Dual fitting uses (I)LPs for a minimization problem as follows:

- Simple algorithm maintains primally feasible and **integral** x.
- In the analysis, we show that $c^T x$,
  the cost of x, is at most the cost of an implicitly computed (nonintegral) dual y.
  However, y is not in general dually feasible.
- By *scaling* y down by a factor $\delta > 1$, we obtain a feasible dual solution: y/δ.

$\rightsquigarrow$ $OPT \geq OPT_{(P)} = c^T x^* = b^T y^* \geq b^T(y/\delta)$

$\rightsquigarrow$ $c^T x \leq b^T y \leq \delta \cdot OPT$

$b^T y^* = OPT_{(D)}$

($c^T x^* = b^T y^*$: weak duality)

$\min\ c^T x$ s.t. $Ax \geq b$, $x \in \mathbb{N}^n$ (I)

$\max\ b^T y$ s.t. $A^T y \leq c$, $y \geq 0$ (D)

$\min\ c^T x$ s.t. $Ax \geq b$, $x \geq 0$ (P)

Set Cover LP and its dual

Recall: Input: $S = (S_1, \ldots, S_k)$ over universe U; define $V(e) = \{j : e \in S_j\}$.

$$
\begin{aligned}
\min \quad & \sum_{j=1}^{k} c(S_j) \cdot x_j \\
\text{s.t.} \quad & \sum_{j \in V(e)} x_j \ \geq\ 1 \quad \forall e \in U \\
& \boldsymbol{x} \ \geq\ \boldsymbol{0}
\end{aligned}
$$

$\longrightarrow$ mechanical

$$
\begin{aligned}
\max \quad & \sum_{e \in U} y_e \\
\text{s.t.} \quad & \sum_{e \in S_j} y_e \ \leq\ c(S_j) \quad \forall j \in [k] \\
& \boldsymbol{y} \ \geq\ \boldsymbol{0}
\end{aligned}
$$

Intuition:

Pack as much (y_e) of good e as possible, so that for group S_j its capacity $c(S_j)$ is not exceeded.

Analysis of greedySetCover by dual fitting

Recall greedySetCover from Unit 10:

procedure greedySetCover(n, $\mathcal{S}$, c)
  $\mathcal{C} := \emptyset$; $C := \emptyset$
  // *For analysis:* $i := 1$
  while $C \neq [n]$
    $i^* := \arg\min_{i \in [n]} \dfrac{c(S_i)}{|S_i \setminus C|}$
    $\mathcal{C} := \mathcal{C} \cup \{i^*\}$
    $C := C \cup S_{i^*}$
    // *For analysis only:*
    // $\alpha_i := \dfrac{c(S_{i^*})}{|S_{i^*} \setminus C|}$
    // *for* $e \in S_{i^*} \setminus C$ *set* $price(e) := \alpha_i$
    // $i := i + 1$
  return $\mathcal{C}$

Lemma 11.15

$y_e = price(e)/H_n$ is dually feasible. ◀

Proof:

$price(e)$ essentially dual variable, but not directly feasible. (Recall $\sum_{e \in U} price(e) = c(\mathcal{C})$).

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    while C ≠ [n]
        i* := arg min_{i∈[n]} c(S_i) / |S_i \ C|
        𝒞 := 𝒞 ∪ {i*}
        C := C ∪ S_{i*}
        // For analysis only:
        // α_i := c(S_{i*}) / |S_{i*} \ C|
        // for e ∈ S_{i*} \ C set price(e) := α_i
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    return 𝒞
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Consider the dual constraint for S_j:

$\sum_{e \in S_j} y_e \le c(S_j)$. Write $\ell = |S_j|$.

$$\begin{aligned} \max \quad & \sum_{e \in U} y_e \\ \text{s.t.} \quad & \sum_{e \in S_j} y_e \le c(S_j) \quad \forall j \in [k] \\ & \boldsymbol{y} \ge \boldsymbol{0} \end{aligned}$$

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$$\sum_{e \in S_j} y_e \le c(S_j). \qquad \text{Write } \ell = |S_j|.$$

Let $e_1, \ldots, e_n$ be elements in order as covered by algorithm.
When e_i covered, S_j contains $\ge \ell - (i-1)$ *uncovered* elements.

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        C := C ∪ S_{i*}
        // For analysis only:
        // α_i := c(S_{i*}) / |S_{i*} \ C|
        // for e ∈ S_{i*} \ C set price(e) := α_i
        // i := i + 1
    return 𝒞
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Let $e_1, \ldots, e_n$ be elements in order as covered by algorithm.

When e_i covered, S_j contains $\ge \ell - (i-1)$ *uncovered* elements.

$\rightsquigarrow$ S_j covers e_i at price $\le \dfrac{c(S_j)}{\ell - i + 1}$ per element.

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$price(e)$ essentially dual variable, but not directly feasible. (Recall $\sum_{e \in U} price(e) = c(\mathcal{C})$).

Consider the dual constraint for S_j:

$$\sum_{e \in S_j} y_e \le c(S_j). \qquad \text{Write } \ell = |S_j|.$$

Let $e_1, \ldots, e_n$ be elements in order as covered by algorithm.

When e_i covered, S_j contains $\ge \ell - (i-1)$ *uncovered* elements.

$\rightsquigarrow$ S_j covers e_i at price $\le \dfrac{c(S_j)}{\ell - i + 1}$ per element.

$\rightsquigarrow$ $price(e_i) \le \dfrac{c(S_j)}{\ell - i + 1}$

Analysis of greedySetCover by dual fitting

Recall greedySetCover from Unit 10:

```
procedure greedySetCover(n, 𝒮, c)
    𝒞 := ∅;  C := ∅
    // For analysis:  i := 1
    while C ≠ [n]
        i* := arg min_{i∈[n]} c(S_i) / |S_i \ C|
        𝒞 := 𝒞 ∪ {i*}
        C := C ∪ S_{i*}
        // For analysis only:
        // α_i := c(S_{i*}) / |S_{i*} \ C|
        // for e ∈ S_{i*} \ C set price(e) := α_i
        // i := i + 1
    return 𝒞
```

Lemma 11.15

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$\rightsquigarrow$ $price(e_i) \le \dfrac{c(S_j)}{\ell - i + 1}$ $\rightsquigarrow$ $y_{e_i} \le \dfrac{1}{H_n}\dfrac{c(S_j)}{\ell - i + 1}$

Analysis of greedySetCover by dual fitting [2]

Proof (cont.):

Consider dual constraint for S_j:

$$\sum_{e \in S_j} y_e = \sum_{m=1}^{\ell} y_{e_{i_m}}$$

$S_j = \{e_{i_1}, \ldots, e_{i_\ell}\}$

$$\rightsquigarrow \quad c(\mathcal{C}) \le H_n \cdot OPT_{(D)} = H_n \cdot OPT_{(P)}.$$

Also note: actually suffices to scale by H_ℓ for $\ell = \max |S_j|$.

Analysis of greedySetCover by dual fitting [2]

Proof (cont.):

Consider dual constraint for S_j:

$$\sum_{e \in S_j} y_e \;=\; \sum_{m=1}^{\ell} y_{e_{i_m}} \;\le\; \frac{c(S_j)}{H_n} \sum_{m=1}^{\ell} \frac{1}{m}$$

$$y_{e_i} \le \frac{1}{H_n} \frac{c(S_j)}{\ell - i + 1}$$

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Analysis of greedySetCover by dual fitting [2]

Proof (cont.):

Consider dual constraint for S_j:

$$\sum_{e \in S_j} y_e \;=\; \sum_{m=1}^{\ell} y_{e_{i_m}} \;\le\; \frac{c(S_j)}{H_n} \sum_{m=1}^{\ell} \frac{1}{m} \;=\; \underbrace{\frac{H_\ell}{H_n}}_{\le 1} c(S_j) \;\le\; c(S_j)$$

$\ell \le n$

■

Analysis of greedySetCover by dual fitting [2]

Proof (cont.):

Consider dual constraint for S_j:

$$\sum_{e \in S_j} y_e \;=\; \sum_{m=1}^{\ell} y_{e_{i_m}} \;\le\; \frac{c(S_j)}{H_n} \sum_{m=1}^{\ell} \frac{1}{m} \;=\; \frac{H_\ell}{H_n} c(S_j) \;\le\; c(S_j)$$

$\Rightarrow \frac{y}{H_n}$ dual feasible

$$\rightsquigarrow \quad c(\mathcal{C}) \;\le\; H_n \cdot OPT_{(D)} \;=\; H_n \cdot OPT_{(P)}.$$

$$\rightsquigarrow \; OPT \;\ge\; OPT_{(P)} \;=\; c^T x^* \;=\; b^T y^* \;\ge\; b^T (y/\delta)$$

$$\rightsquigarrow \; c^T x \;\le\; b^T y \;\le\; \delta \cdot OPT$$

$OPT_{(D)}$

Analysis of greedySetCover by dual fitting [2]

Proof (cont.):

Consider dual constraint for S_j:

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Also note: actually suffices to scale by H_ℓ for $\ell = \max |S_j|$.

Integrality Gap of Set Cover

Previous result shows that integrality gap $\frac{OPT}{OPT_{(P)}} \leq H_n$.

Can we give a lower bound?

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Theorem 11.16 (Integrality Gap of Set Cover)

For the set cover ILP and its relaxation holds

$$\frac{OPT}{OPT_{(P)}} \;\ge\; \frac{\log_2(n+1)}{2\frac{n}{n+1}} \;\sim\; \frac{1}{2\ln 2}H_n \;\approx\; 0.721H_n$$

$\rightsquigarrow$ not possible to improve worst case using LP tricks alone

Proof:

We construct a concrete example family.

Given $n = 2^\ell - 1$ for $\ell \in \mathbb{N}_{\ge 1}$ $\rightsquigarrow$ $U = [1..2^\ell)$ all ℓ-bit binary numbers (except 0)

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$11 = 1011_2$

$\leadsto$ not possible to improve worst case using LP tricks alone

$11 = (1,0,1,1)$

Proof:

We construct a concrete example family.

Given $n = 2^\ell - 1$ for $\ell \in \mathbb{N}_{\ge 1}$ $\leadsto$ $U = [1..2^\ell)$ all ℓ-bit binary numbers (except 0)

View $i \in U$ as binary vector $\boldsymbol{i} \in \{0,1\}^\ell$ using binary digits of number i.

Set $S_j = \{i \in U : \boldsymbol{i}^T \boldsymbol{j} \equiv 1 \pmod 2\}$ for $j = 0, \ldots, n-1$; $c(S_j) = 1$

$1, \ldots, n$

Integrality Gap of Set Cover [2]

Proof (cont.): $= 2^{\ell-1}$

Can show: $|S_j| = \frac{n+1}{2}$ and $|V(i)| = \frac{n+1}{2}$

Given $\boldsymbol{j}$, can arbitrarily fill $\ell - 1$ digits of $\boldsymbol{i}$; for last p where $\boldsymbol{j}_p = 1$, exactly one choice for $\boldsymbol{i}_p$ makes $\boldsymbol{i}^T\boldsymbol{j} \equiv 1$.

$j = (0, 1, 0, 0, 0, 1)$ $\quad$ $i = (0, 1, 0, 1, 0, 0)$

$i^T j = 1$

Integrality Gap of Set Cover [2]

Proof (cont.):

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Given $\boldsymbol{j}$, can arbitrarily fill $\ell - 1$ digits of $\boldsymbol{i}$; for last p where $\boldsymbol{j}_p = 1$, exactly one choice for $\boldsymbol{i}_p$ makes $\boldsymbol{i}^T \boldsymbol{j} \equiv 1$.

Setting all $x_j = \frac{2}{n+1}$ is primally feasible for set cover LP (fractional set cover)

$\rightsquigarrow \quad OPT_{(P)} \leq n \cdot \frac{2}{n+1} \sim 2.$

#sets

primal constraint

$$\sum_{j \in V(e)} x_j \geq 1 \qquad e \in U$$

Integrality Gap of Set Cover [2]

Proof (cont.):

Can show: $|S_j| = \frac{n+1}{2}$ and $|V(i)| = \frac{n+1}{2}$

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However, integral set cover needs ℓ sets.

Suppose not, let $i_1, \ldots, i_k$ yield cover with $k < \ell$.

$\rightsquigarrow \quad A = \begin{pmatrix} — \boldsymbol{i}_1 — \\ \vdots \\ — \boldsymbol{i}_k — \end{pmatrix}$ is $k \times \ell$ matrix

$\rightsquigarrow$ rank of A is $\le k$

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$i^T j \equiv 1$

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Can show: $|S_j| = \frac{n+1}{2}$ and $|V(i)| = \frac{n+1}{2}$

Given $\boldsymbol{j}$, can arbitrarily fill $\ell - 1$ digits of $\boldsymbol{i}$; for last p where $\boldsymbol{j}_p = 1$, exactly one choice for $\boldsymbol{i}_p$ makes $\boldsymbol{i}^T\boldsymbol{j} \equiv 1$.

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$\rightsquigarrow$ $j \notin S_{i_1}, \ldots, S_{i_k}$ ↯ SC.

$OPT \ge \ell = \lg(n+1).$

■

11.7 The Primal-Dual Schema

The Primal-Dual Schema

So far:

- ad hoc methods, a posteriori justified by LP arguments
- rounding algorithms, must solve primal LP to optimality (polytime, but expensive!)

Can we use duality more directly?

The Primal-Dual Schema

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Can we use duality more directly?

Idea: Use complementary slackness conditions to guide us
On ILPs, need suitably *relaxed CSC*

- maintain (x, y) throughout that satisfy relaxed CSC
- x is always integral, but initially **not** primal feasible
- y is dual feasible, but not integral
- To make x "more feasible" modify it

$\rightsquigarrow$ let CSCs guides adjustment to y

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$\rightsquigarrow$ let CSCs guides adjustment to y

$\rightsquigarrow$ self-certifying algorithm: y gives bound on OPT, so proofs approx. ratio for x

Relaxed CSCs

Recall: **LP Complementary Slackness Conditions:**

1. $\forall j \in [n] : x_j = 0 \vee \sum_{1 \le i \le m} a_{i,j} y_i = c_j$ and
2. $\forall i \in [m] : y_i = 0 \vee \sum_{1 \le j \le n} a_{i,j} x_j = b_i$.

(

multiplier for ith constraint in (P)

min $c^T x$

$Ax \geq b$ (P)

$x \geq 0$

max $b^T y$

$A^T y \leq c$ (D)

$y \geq 0$

Relaxed CSCs

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min $c^T x$ $Ax \ge b$ (P) $x \ge 0$

max $b^T y$ $A^T y \le c$ (D) $y \ge 0$

(α, β)-Relaxed CSCs: With $\boxed{\alpha \ge 1 \text{ and } \beta \ge 1}$

1. $\forall j \in [n] : x_j = 0 \vee \frac{c_j}{\alpha} \le \sum_{1 \le i \le m} a_{i,j} y_i \le c_j$ and
2. $\forall i \in [m] : y_i = 0 \vee b_i \le \sum_{1 \le j \le n} a_{i,j} x_j \le \beta \cdot b_i$.

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2. $\forall i \in [m] : y_i = 0 \lor b_i \le \sum_{1\le j\le n} a_{i,j} x_j \le \beta \cdot b_i$.

strong duality $c^T x = b^T y$

weak duality $c^T x \ge b^T y$

Lemma 11.17 (Relaxed CSC duality)

If x and y and primal resp. dual feasible and satisfy the (α,β)-relaxed CSCs then $c^T x \le \alpha\beta \cdot b^T y$

Proof:

Compute $c^T x \le \alpha(A^T y)^T x = \alpha y^T (Ax) \le \alpha y^T \beta b = \alpha\beta \cdot b^T y$. ■

CSC for Set Cover

Complementary Slackness Conditions for Set Cover

$$x_j = 0 \quad \vee \quad \sum_{u \in S_j} y_u = c(S_j) \qquad \forall j \in [k]$$

$$y_e = 0 \quad \vee \quad \sum_{j \in V(e)} x_j = 1 \qquad \forall e \in U$$

Problem: In general only simultaneously fulfilled by *fractional solutions*

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Problem: In general only simultaneously fulfilled by *fractional solutions*

⇝ Initially relax dual constraints via $\beta = f$ to $\alpha = 1$

$$y_e = 0 \ \vee \ \sum_{j \in V(e)} x_j \le f \qquad \forall e \in U$$

i. e., every element at most f times ⇝ trivially fulfilled . . .

Primal Dual Set Cover

```
procedure primalDualSetCover(n,S,c)
    f := global frequency
    x := 0;  y := 0;  T := [n]
    while T ≠ ∅
        Choose e ∈ T arbitrarily
        Increase y_e until CSC holds for one more set S_j
        for all S_j with ∑_{e∈S_j} y_e = c(S_j)
            T = T \ S_j
            x_j = 1 // fix S_j for solution
    return 𝒞 := {j ∈ [k] : x_j = 1}
```

Primal Set Cover LP

$$\min \sum_{j=1}^{k} c(S_j) \cdot x_j$$
$$\text{s.t.} \sum_{j \in V(e)} x_j \geq 1 \quad \forall e \in U$$
$$\boldsymbol{x} \geq \boldsymbol{0}$$

Dual Set Cover LP

$$\max \sum_{e \in U} y_e$$
$$\text{s.t.} \sum_{e \in S_j} y_e \leq c(S_j) \quad \forall j \in [k]$$
$$\boldsymbol{y} \geq \boldsymbol{0}$$

(α, β)-Relaxed CSCs: With $\alpha \geq 1$ and $\beta \geq 1$

1. $\forall j \in [n] : x_j = 0 \vee \frac{c_j}{\alpha} \leq \sum_{1 \leq i \leq m} a_{i,j} y_i \leq c_j$ and (CSC 1)
2. $\forall i \in [m] : y_i = 0 \vee b_i \leq \sum_{1 \leq j \leq n} a_{i,j} x_j \leq \beta \cdot b_i$.

Theorem 11.18

primalDualSetCover is an f-approximation for SetCover.

Proof:

The algorithm only terminates once $\mathcal{C}$ is a set cover $\rightsquigarrow$ x primal feasible.

Primal Dual Set Cover

procedure primalDualSetCover(n,S,c)
  f := global frequency
  $\boldsymbol{x} := \mathbf{0}$; $\boldsymbol{y} := \mathbf{0}$; $T := [n]$
  while $T \neq \emptyset$
    Choose $e \in T$ arbitrarily
    Increase y_e until CSC holds **for** one more set S_j
    for all S_j with $\sum_{e \in S_j} y_e = c(S_j)$
      $T = T \setminus S_j$
      $x_j = 1$ // *fix S_j for solution*
  return $\mathcal{C} := \{j \in [k] : x_j = 1\}$

Primal Set Cover LP

$$\begin{aligned} \min \quad & \sum_{j=1}^{k} c(S_j) \cdot x_j \\ \text{s.t.} \quad & \sum_{j \in V(e)} x_j \geq 1 \quad \forall e \in U \\ & \boldsymbol{x} \geq \mathbf{0} \end{aligned}$$

Dual Set Cover LP

$$\begin{aligned} \max \quad & \sum_{e \in U} y_e \\ \text{s.t.} \quad & \sum_{e \in S_j} y_e \leq c(S_j) \quad \forall j \in [k] \\ & \boldsymbol{y} \geq \mathbf{0} \end{aligned}$$

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For j with $x_j = 1$, must have $\sum_{e \in S_j} y_e = c(S_j)$ (CSC1)
(was true when x_j set to 1, not modified later)

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    return 𝒞 := {j ∈ [k] : x_j = 1}
```

Primal Set Cover LP

$$\begin{aligned} \min \quad & \sum_{j=1}^{k} c(S_j) \cdot x_j \\ \text{s.t.} \quad & \sum_{j \in V(e)} x_j \ \ge\ 1 \quad \forall e \in U \\ & \boldsymbol{x} \ \ge\ \boldsymbol{0} \end{aligned}$$

Dual Set Cover LP

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Theorem 11.18

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y feasible invariantly

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The algorithm only terminates once $\mathcal{C}$ is a set cover $\rightsquigarrow$ x primal feasible.

For j with $x_j = 1$, must have $\sum_{e \in S_j} y_e = c(S_j)$ (CSC1)
(was true when x_j set to 1, not modified later)

(x, y) satisfies $(1, f)$-relaxed CSCs $\rightsquigarrow$ $c(\mathcal{C}) = c^T x \le 1 \cdot f \cdot b^T y \le f \cdot OPT$ ■

Summary

LP-based Approximation design patterns

- deterministic rounding
- randomized rounding
- dual fitting
- primal-dual schema