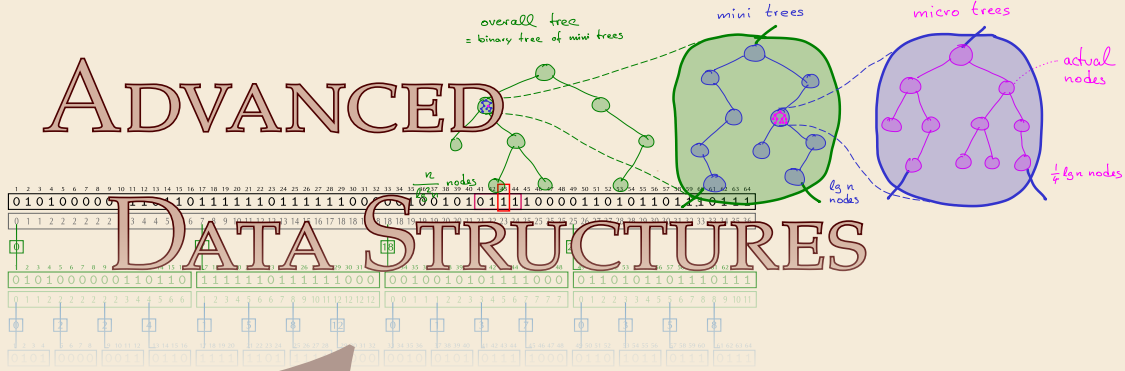
Advanced Data Structures



1 Randomized Trees

Advanced Data Structures · Summer 2026

Prof. Dr. Sebastian Wild

CS650 (Summer 2026)
Philipps-Universität Marburg
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Outline

1 Randomized Trees

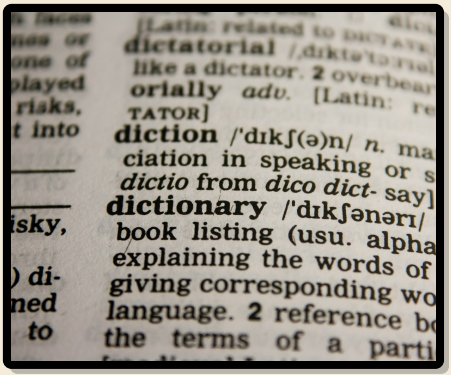
1.1 Recap: Sorted Dictionary and BSTs

Symbol table ADT

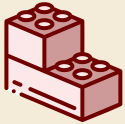
Java: `java.util.Map<K,V>`

Symbol table / Dictionary / Map / Associative array / key-value store:

Python dict `{k:v}`



- `put(`k`,`v`)` *Python* dict: `d[`k`] = `v
  Put key-value pair (k, v) into table
- `get(`k`)` *Python* dict: `d[`k`]`
  Return value associated with key k
- `delete(`k`)` *Python* dict: `del d[`k`]`
  Remove key k (any associated value) form table
- `contains(`k`)` *Python* dict: k `in d`
  Returns whether the table has a value for key k
- `isEmpty(),size()`
- `create()`

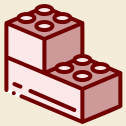


Most fundamental building block in computer science.

(Every programming library has a symbol table implementation.)

Ordered symbol tables

- `min()`, `max()`
  Return the smallest resp. largest key in the ST
- `floor(`x`)`, $\lfloor x \rfloor = \mathbb{Z}.$`floor(`$x$`)`
  Return largest key k in ST with $k \leq x$.
- `ceiling(`x`)`
  Return smallest key k in ST with $k \geq x$.
- `rank(`x`)`
  Return the number of keys k in ST $k < x$.
- `select(`i`)`
  Return the ith smallest key in ST (zero-based, i. e., $i \in [0..n)$)



With select, we can simulate access as in a truly dynamic array!

(Might not need any keys at all then!)

Binary search trees

Binary search trees (BSTs) $\approx$ dynamic sorted array

- binary tree
  - Each node has left and right child
  - Either can be empty (`null`)
- Keys satisfy ***search-tree property***

  all keys in left subtree $\leq$ root key $\leq$ all keys in right subtree

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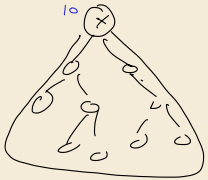
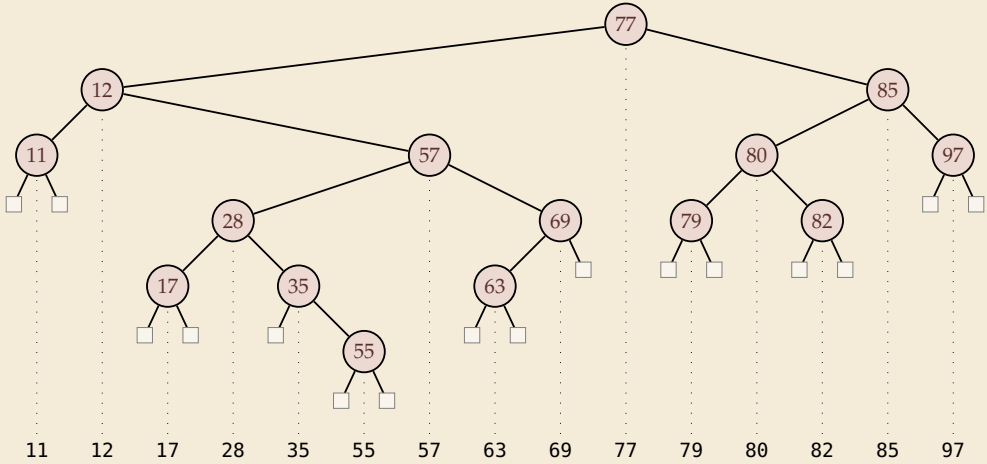
- Standard trick: ***Augmented BSTs***
  - Each node stores the **size** of its **subtree**
  - Easy to maintain upon updates
  - $\rightsquigarrow$ Allows to answer all ordered-symbol-table operations

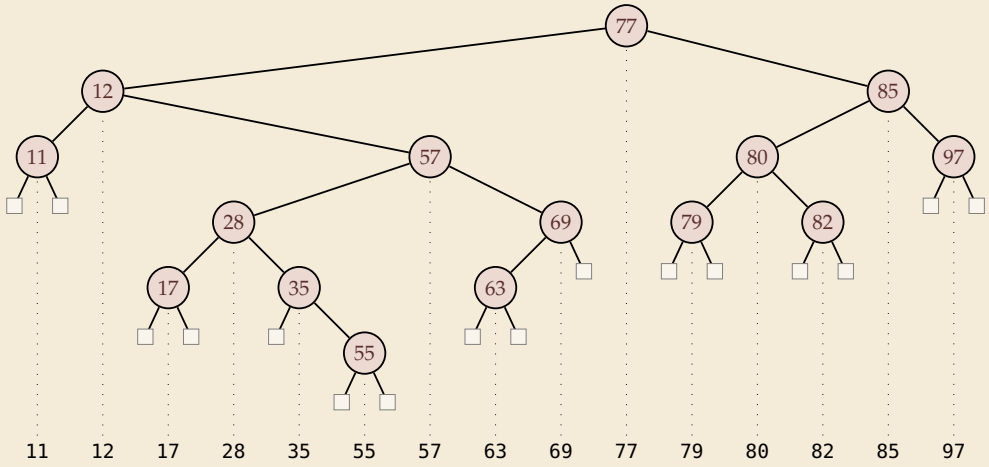


BST example & find



BST insert

Example: Insert 88



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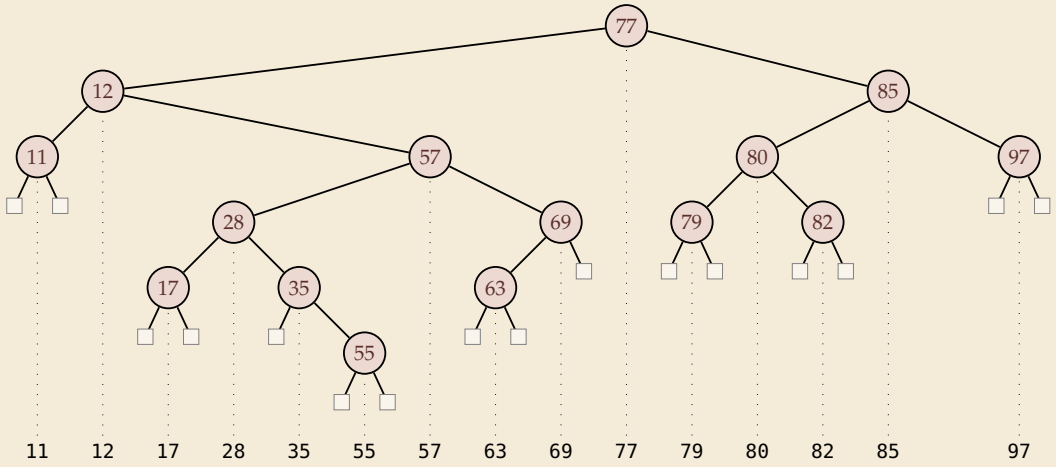

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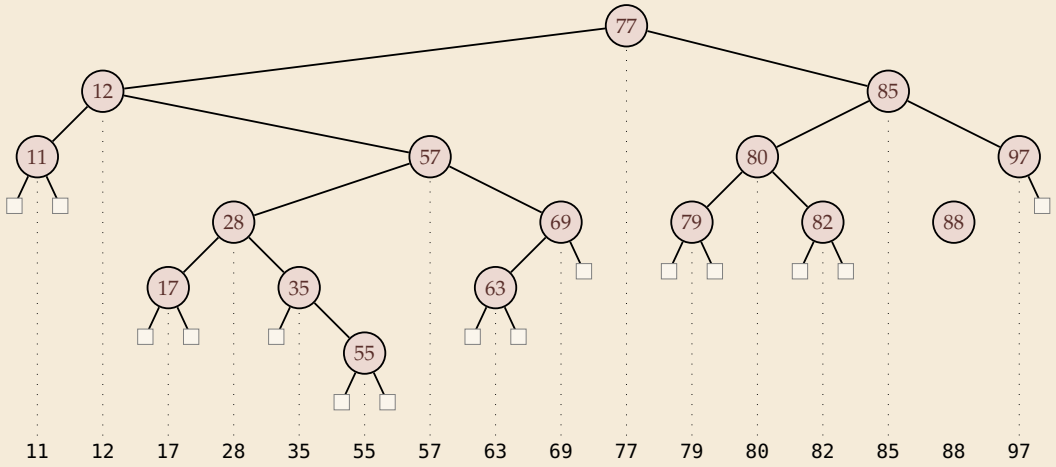

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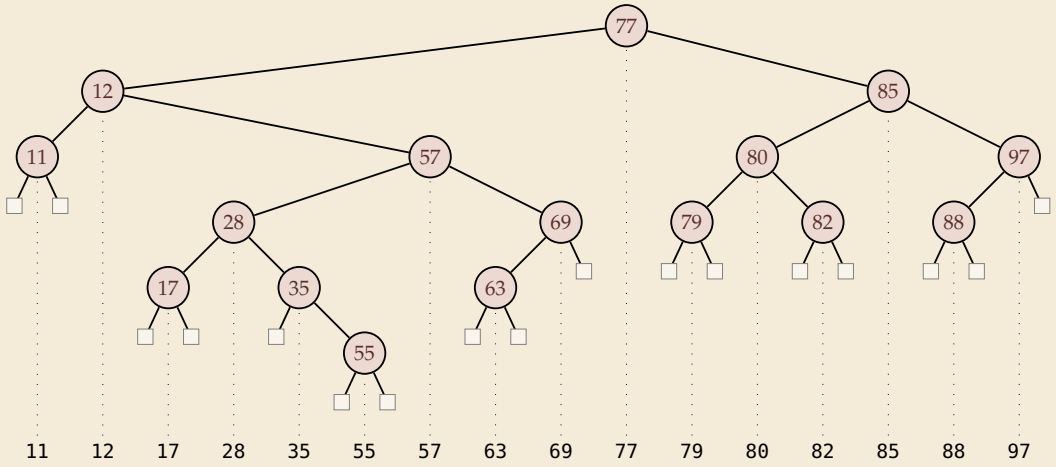
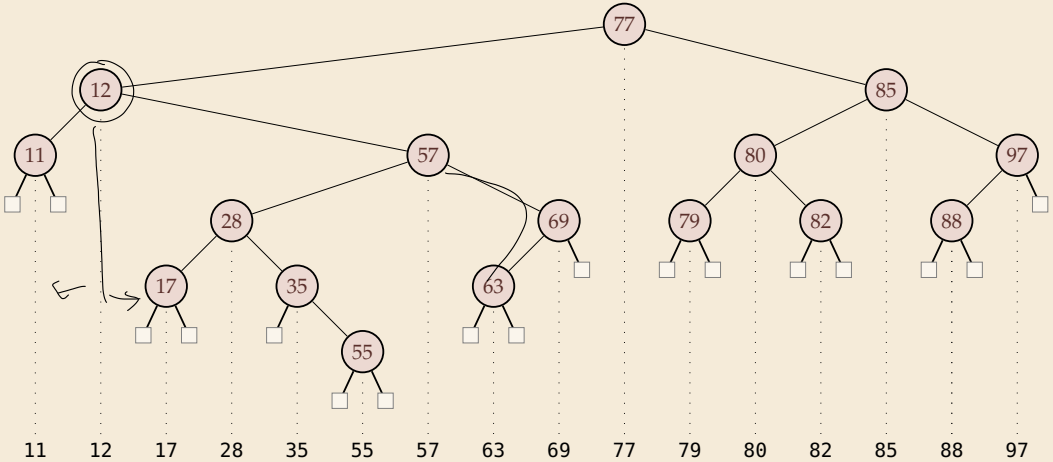


BST delete

Hibbard's deletion

- Easy case: remove leaf, e. g., 11 ⇝ replace by `null`
- Medium case: remove unary, e. g., 69 ⇝ replace by unique child
- Hard case: remove binary, e. g., 85 ⇝ swap with predecessor, recurse



Unbalanced Binary Search Trees

Operation	Running Time
`construct(`$A[1..n]$`)`	$O(n\boldsymbol{h})$
`put(`k`,`v`)`	$O(\boldsymbol{h})$
`get(`k`)`	$O(\boldsymbol{h})$
`delete(`k`)`	$O(\boldsymbol{h})$
`contains(`k`)`	$O(\boldsymbol{h})$
`isEmpty()`	$O(1)$
`size()`	$O(1)$
`min(), max()`	$O(1)$ (if stored)
`floor(`x`), ceiling(`x`)`	$O(\boldsymbol{h})$
`rank(`x`), select(`i`)`	$O(\boldsymbol{h})$

h = *height* of the BST

$1 - 2^{-n}$

- Height h of unbalanced BST depends on insertion order
- satisfies $\lg n \le h \le n$ $\quad \lg = \log_2$
- For **random** insertions, $h = O(\log n)$ in expectation and with high probability

Prob $= 1 - o(1)$

Balanced Binary Search Tree

Balanced binary search trees (BBSTs):

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  - *red-black trees*

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- **Properties**
  - All of them guarantee $h = O(\log n)$ at all times (constants differ!)
  - All of them work with *rotations* along the search path

  $\rightsquigarrow$ $O(\log n)$ extra cost for maintaining balance
  - Further guarantees specific to rule known

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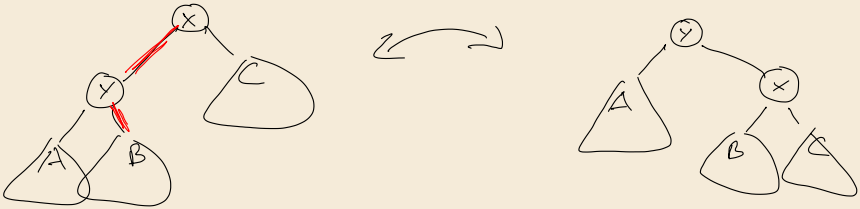
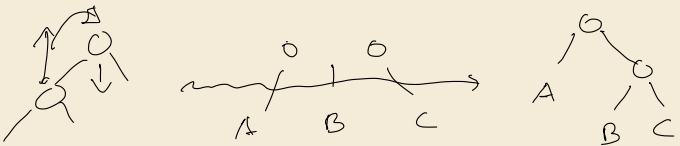
OPEN: *What is the exact average cost (constant in front of $\lg n$) of insertion in AVL trees / red-black trees?*

A Primitive for BSTs: Rotations



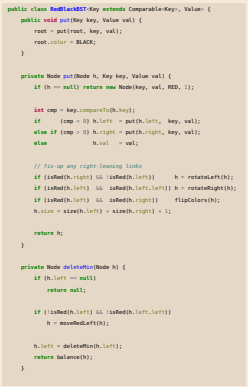


1.2 What's wrong with BBSTs?

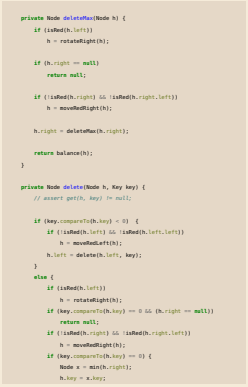
Red-Black Tree Implementation

`RedBlackTree` from Sedgewick & Wayne (excerpt) `algs4.cs.princeton.edu/code`

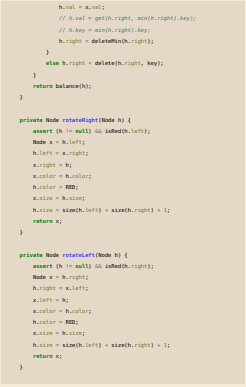
- showing only the bare-bones core: `put` (insert) and `delete`
- no rank-based operations
- full class has 750 lines (and this is a good and compact implementation!)



```
public class RedBlackBST<Key extends Comparable<Key>, Value> {
   public void put(Key key, Value val) {
      root = put(root, key, val);
      root.color = BLACK;
   }

   private Node put(Node h, Key key, Value val) {
      if (h == null) return new Node(key, val, RED, 1);

      int cmp = key.compareTo(h.key);
      if      (cmp < 0) h.left  = put(h.left,  key, val);
      else if (cmp > 0) h.right = put(h.right, key, val);
      else              h.val   = val;

      // fix-up any right-leaning links
      if (isRed(h.right) && !isRed(h.left))      h = rotateLeft(h);
      if (isRed(h.left)  &&  isRed(h.left.left)) h = rotateRight(h);
      if (isRed(h.left)  &&  isRed(h.right))     flipColors(h);
      h.size = size(h.left) + size(h.right) + 1;

      return h;
   }

   private Node deleteMin(Node h) {
      if (h.left == null)
         return null;

      if (!isRed(h.left) && !isRed(h.left.left))
         h = moveRedLeft(h);

      h.left = deleteMin(h.left);
      return balance(h);
   }
```



```
   private Node deleteMax(Node h) {
      if (isRed(h.left))
         h = rotateRight(h);

      if (h.right == null)
         return null;

      if (!isRed(h.right) && !isRed(h.right.left))
         h = moveRedRight(h);

      h.right = deleteMax(h.right);

      return balance(h);
   }

   private Node delete(Node h, Key key) {
      // assert get(h, key) != null;

      if (key.compareTo(h.key) < 0)  {
         if (!isRed(h.left) && !isRed(h.left.left))
            h = moveRedLeft(h);
         h.left = delete(h.left, key);
      }
      else {
         if (isRed(h.left))
            h = rotateRight(h);
         if (key.compareTo(h.key) == 0 && (h.right == null))
            return null;
         if (!isRed(h.right) && !isRed(h.right.left))
            h = moveRedRight(h);
         if (key.compareTo(h.key) == 0) {
            Node x = min(h.right);
            h.key = x.key;
```



```
            h.val = x.val;
            // h.val = get(h.right, min(h.right).key);
            // h.key = min(h.right).key;
            h.right = deleteMin(h.right);
         }
         else h.right = delete(h.right, key);
      }
      return balance(h);
   }

   private Node rotateRight(Node h) {
      assert (h != null) && isRed(h.left);
      Node x = h.left;
      h.left = x.right;
      x.right = h;
      x.color = h.color;
      h.color = RED;
      x.size = h.size;
      h.size = size(h.left) + size(h.right) + 1;
      return x;
   }

   private Node rotateLeft(Node h) {
      assert (h != null) && isRed(h.right);
      Node x = h.right;
      h.right = x.left;
      x.left = h;
      x.color = h.color;
      h.color = RED;
      x.size = h.size;
      h.size = size(h.left) + size(h.right) + 1;
      return x;
   }
```

```
   private void flipColors(Node h) {
      h.color = !h.color;
      h.left.color = !h.left.color;
      h.right.color = !h.right.color;
   }

   private Node moveRedLeft(Node h) {
      flipColors(h);
      if (isRed(h.right.left)) {
         h.right = rotateRight(h.right);
         h = rotateLeft(h);
         flipColors(h);
      }
      return h;
   }

   private Node moveRedRight(Node h) {
      flipColors(h);
      if (isRed(h.left.left)) {
         h = rotateRight(h);
         flipColors(h);
      }
      return h;
   }

   private Node balance(Node h) {
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Can we have something simpler?

In this and the next unit, we will look into alternatives to avoid lengthy case distinctions.

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Disclaimer: It seems that something has to give.

All known (much) more elegant BST alternatives are either *randomized* or *amortized*.

- However, this is often tolerable.
- Complexity of code can also come with running-time penalty;
  if we avoid that, might be an overall speedup.
  (Typically not the case for well-tested library implementations, though.)

Recall: Average-Case Analysis vs. Randomized Algorithms

This unit: *Randomized solutions*.

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- algorithm is **deterministic**
  same input, same computation

Randomized Algorithm (here)

- algorithm is **not** deterministic
  same input, potentially different comp.

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Confusingly enough, the analysis (technique) is often the same!

But: Implications are quite different; randomization is much more versatile and robust.

1.3 Random BSTs

Random BSTs are good (enough)

*Let's first do some wishful thinking: assume we're dealing with **random** inputs only.*

- ⇝ Let's first see if we like the performance in this case.
- ▶ If so, will try and see how to **enforce** this randomness later.

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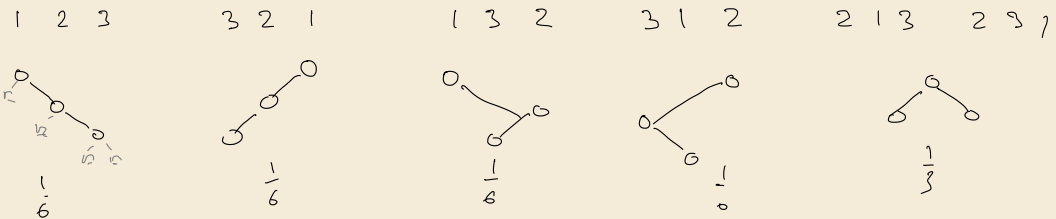
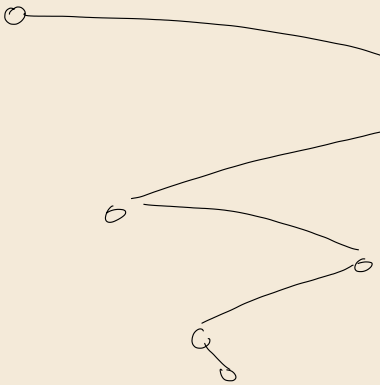
Random here means:

Definition 1.1 (Random BST)

A ***Random BST*** of size n is the (random) shape of an initially empty BST after successively inserting a random permutation of $[n]$. ◀

Example: $n = 3$





$\approx 2^n$ maximally unbalanced trees

#binary trees on n nodes

nth Catalan number

$$= \binom{2n}{n} \cdot \frac{1}{n+1}$$

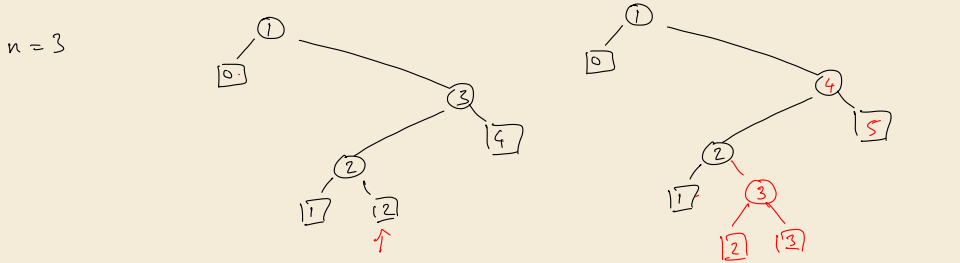
$$\approx 4^n / n^{3/2}$$

$$n! \approx \left(\frac{n}{e}\right)^n$$

Iterative randomness

Lemma 1.2 (Random insertion yields random BST)

Let $n \geq 0$ and T_n a random BST over n keys. Inserting an element equally likely in one of the $n+1$ *gaps* in T_n (external leaves) results in a new BST T_{n+1} that has the same shape as a random BST on $n+1$ keys.



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Proof:

Insertion order for T_{n+1} is a permutation of $[n+1] = \{1, \ldots, n+1\}$, which is in bijection with a permutation of $[n]$ and a *insertion point (leaf)* in $[0..n]$, where we increment all values right of the insertion leaf by 1. ■

Example: $1,2,4,7,6,5 \mid 3 \;\widehat{=}\; ((1,2,3,6,5,4),2)$

Recall: Events

something we can assign a probability to

$A \in \mathcal{F}$ is called an *event* of probability space $(\Omega, \mathcal{F}, \mathbb{P})$; also a *measurable set*.

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Basic properties

- $\mathbb{P}\left[\overline{A}\right] = 1 - \mathbb{P}[A]$ counter-probability $\quad (\overline{A} = \Omega \setminus A)$
- $\mathbb{P}\left[\bigcup A_i\right] \leq \sum_i \mathbb{P}[A]$ the *union bound* (a.k.a. Boole's inequality a.k.a. σ-subadditivity)

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- $\{A_1, \ldots, A_k\}$ *(mutually) independent* $\iff \mathbb{P}\left[\bigcap_i A_i\right] = \prod_i \mathbb{P}[A_i]$

  An infinite set of events is mutually independent if every finite subset is so.

  k-wise independence means that only all size-k subsets are independent.

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discrete, $\mathcal{F} = 2^{\Omega}$ (powerset of Ω)

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  An infinite set of events is mutually independent if every finite subset is so.

  k-wise independence means that only all size-k subsets are independent.
- *conditional probability* for A given B: $\mathbb{P}[A \mid B] = \mathbb{P}[A \cap B] / \mathbb{P}[B]$
  generally undefined if $\mathbb{P}[B] = 0$
- *law of total probability*: If $\Omega = B_1 \dot\cup B_2 \dot\cup \cdots$ is a partition of Ω, we have

$$\mathbb{P}[A] = \sum_{\substack{i \\ \mathbb{P}[B_i] \neq 0}} \mathbb{P}[A \mid B_i] \cdot \mathbb{P}[B_i].$$

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Random variables (r.v.) $X : \Omega \to \mathcal{X}$; often $\mathcal{X} = \mathbb{R}$ (in general spaces: only *measurable* functions)

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Basic properties and conventions:

- event $\{X = x\}$ is defined as $\{\omega \in \Omega : X(\omega) = x\}$.
- For event A define the indicator r.v. $\mathbb{1}_A$ via $\mathbb{1}_A(\omega) = [\omega \in A] = \begin{cases} 1 & \omega \in A \\ 0 & \text{else} \end{cases}$

Recall: Random Variables

Random variables (r.v.) $X : \Omega \to \mathcal{X}$; often $\mathcal{X} = \mathbb{R}$ (in general spaces: only *measurable* functions)

Basic properties and conventions:

- event $\{X = x\}$ is defined as $\{\omega \in \Omega : X(\omega) = x\}$.
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Independence:

- X and Y independent $\iff \mathbb{P}[X = x \wedge Y = y] = \mathbb{P}[X = x] \cdot \mathbb{P}[Y = y]$ for all x, y.
  (Naturally follows from independent events)
- a sequence of *i.i.d.* r.v. $X_1, X_2, \ldots$ (*independent and identically distributed*)
  has $X_i \stackrel{\mathcal{D}}{=} X_1$ and $\{X_i\}_{i \geq 1}$ are mutually independent
  - typical example: sequence of coin tosses (with same coin)

Alternative Random Model

Corollary 1.3

A BST built by inserting n i.i.d. Uniform$(0, 1)$ r.v. has the shape of a random BST. ◀

1.4 Properties of Random BSTs

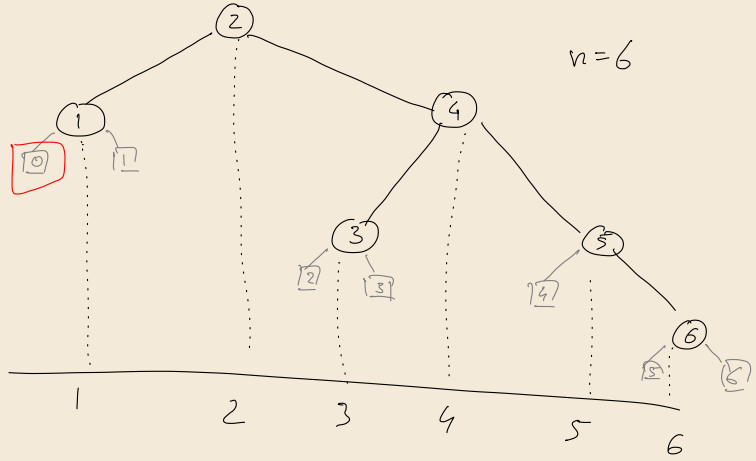


Figure 6.16 A binary search tree built from 237 randomly ordered keys

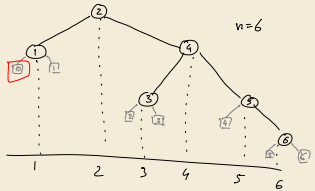
Analysis of Random BSTs I

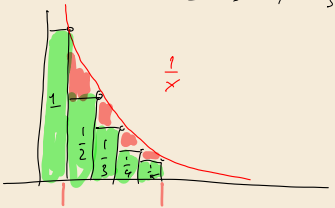
$\frac{H_n}{\ln n} \to 1 \quad (n \to \infty)$

Theorem 1.4 (Expected depth of leftmost leaf)

The *expected depth* (number of edges from root) of the leftmost external leaf (leaf for $-\infty$) in a random BST on $n \geq 1$ nodes is $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \sim \ln n$.

$= \sum_{i=1}^{n} \frac{1}{i}$





$\int_1^n \frac{1}{x}\,dx \approx \ln(n) - \ln(1)$

Analysis of Random BSTs I

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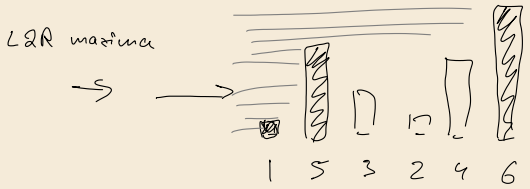
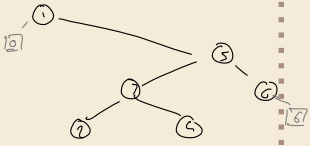
Proof:

$\text{depth}(\boxed{0}_n) =$ #left-to-right minima L2R_n in insertion sequence

$\text{L2R}_n = \sum_{i=1}^{n} X_i$ for X_i = [position i is a l2r min]





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First i inserted elements are in bijection with random permutation π of $[i]$

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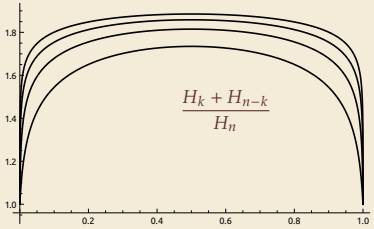
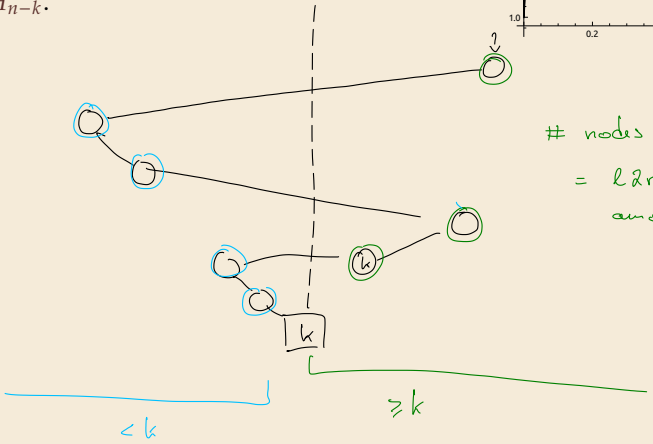
$\mathbb{E}[\text{L2R}_n] = \sum_{i=1}^{n} \mathbb{E}[X_i] = \sum_{i=1}^{n} \frac{1}{i} = H_n.$ ■

Analysis of Random BSTs II

Theorem 1.5 (Expected depth of kth leaf)

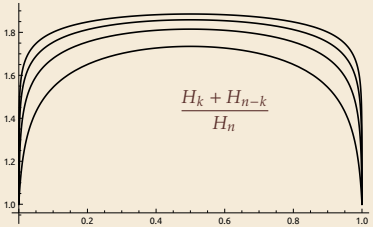
The *expected depth* of the *kth external leaf* (for $k = 0, \ldots, n$) in a random BST on $n \geq 1$ keys is $H_k + H_{n-k}$.





nodes right of k
= k2r minima among keys ≥ k

Analysis of Random BSTs II



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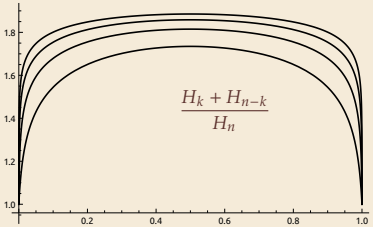
depth($\boxed{k}$) = #comparisons for unsuccessful search for a key x that terminates in $\boxed{k}$.

Analysis of Random BSTs II



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Proof:

$\text{depth}(\boxed{k})$ = #comparisons for unsuccessful search for a key x that terminates in $\boxed{k}$.

$k - \frac{1}{2}$

= #comparisons with keys $< x$ + #comparisons with keys $> x$

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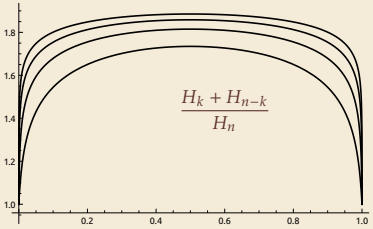
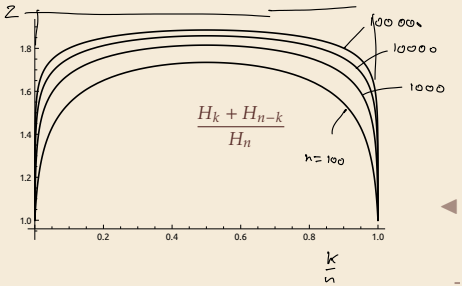


Proof:

$$\begin{aligned}
\text{depth}(\boxed{k}) &= \#\text{comparisons for unsuccessful search for a key } x \text{ that terminates in } \boxed{k}. \\
&= \#\text{comparisons with keys} < x \;+\; \#\text{comparisons with keys} > x \\
&= \text{l2r-}\overset{\text{max}}{\cancel{\text{min}}}\text{ among keys} < x \;+\; \text{l2r-}\overset{\text{min}}{\cancel{\text{max}}}\text{s with keys} > x
\end{aligned}$$

Analysis of Random BSTs II



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Proof:

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= #comparisons with keys $< x$ + #comparisons with keys $> x$

= #l2r-mins among keys $< x$ + #l2r-maxs with keys $> x$

Since there are k keys $< x$ and $n - k$ keys $> x$, the same derivation as above applies. ■

Analysis of Random BSTs III

Corollary 1.6 (Depth of typical leaf)

Consider a random BST T_n of n keys.

(a) The *expected external path length* of T_n is
$2(n+1)(H_{n+1}-1) \;=\; 2n\ln n - 2(1-\gamma)n \;\pm\; O(\log n)$. ($\gamma \approx 0.5772$ the *Euler-Mascheroni constant*)

(b) The depth of the αnth leaf in a random BST of n keys $\sim 2\ln n$ as $n \to \infty$ for any fixed $\alpha \in (0,1)$.

$$\mathbb{E}\left[\sum_{k=0}^{n} \text{depth}(\boxed{k})\right] = \sum_{k=0}^{n}(H_k + H_{n-k})$$

$$\sum_{i=1}^{n} H_i = (n+1)H_n - n,$$

$$= 2\sum_{k=0}^{n} H_k$$

$$H_{n+1} = H_n + \frac{1}{n+1}$$

recursion tree of Quicksort

$1.39 \log_2 n$

$$H_k + H_{n-k} = H_{\alpha n} + H_{(1-\alpha)n}$$

$$\sim \ln(\alpha n) + \ln((1-\alpha)n)$$

$$= 2\ln(n) + \ln(\alpha) + \ln(1-\alpha)$$

$$\sim 2\ln(n)$$

Detour: When Expectation Isn't Enough

- Two hypothetical algorithms:
  - A takes 1 step in half the cases and 3 steps otherwise
  - B takes 1 step in 99% of cases and **101** steps otherwise
  - $\rightsquigarrow$ both have expected costs of 2 steps.
  - probably want A ... certainly would want to be able to distinguish them!

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  - $\rightsquigarrow$ both have expected costs of 2 steps.
  - probably want A ... certainly would want to be able to distinguish them!

- **Goal:** Strengthen algorithms so $time(x)$ rarely far from $\mathbb{E}[time(x)]$
  - formally: bound probability that X (far) exceeds $\mathbb{E}[X]$
    $\rightsquigarrow$ *concentration bounds* a.k.a. *tail inequalities*
  - can then compare these typical times again
  - also obtain more reliable algorithms

  $\rightsquigarrow$ Let's establish some tools for that!

Detour: With High Probability

Definition 1.7 (With high probability)

We say

- an event $X = X(n)$ happens *with high probability (w.h.p.)* when $\forall c : \mathbb{P}[X(n)] \doteq 1 \pm O(n^{-c})$ as $n \to \infty$. $\quad 1 - d(c) \cdot n^{-c}$
- a random variable $X = X(n)$ is *in $O(f(n))$ with high probability (w.h.p.)* when $\forall c\ \exists d : \mathbb{P}[X \le d f(n)] = 1 \pm O(n^{-c})$ as $n \to \infty$.
  (This means, the constant in $O(f(n))$ may depend on c.)

Height / Depth of a leaf in Random BST $H = O(\log n)$ w.h.p.

$c = 10 \quad \rightsquigarrow \quad d_1, d_2 : \quad \mathbb{P}[H \ge d_1 \ln n] \le d_2 n^{-10}$

(for example, $\mathbb{P}[\text{height} \ge 42 \ln n] \le 2 n^{-7.4}$)

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  (This means, the constant in $O(f(n))$ may depend on c.) ◀

- Very strong notion: failure probability smaller than any polynomial

$\rightsquigarrow$ If A succeeds w.h.p. then also polynomially many repetitions of A succeed w.h.p.

Lemma 1.8 (Repetitions w.h.p.)

Suppose A is an algorithm that w.h.p. does not fail.
In n^d *independent* repetitions of A on inputs of size n, w.h.p. *no* repetition fails. ◀

Detour: With High Probability [2]

Proof (Lemma 1.8):
Let c from the definition of w.h.p. be given.

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Detour: Chernoff Bounds

Strong concentration inequalities require assumptions on distribution of X.
A classical one is that X consists of many **small and independent** parts.

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Theorem 1.9 (Chernoff Bound for Bernoulli trials)

Let $X_1, \ldots, X_n \in \{0,1\}$ be *(mutually) independent* with $X_i \overset{\mathcal{D}}{=} \mathrm{B}(p_i)$.
Define $X = X_1 + \cdots + X_n$ and $\mu = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n] = p_1 + \cdots + p_n$. Then holds

$$\forall \delta > 0 \;:\; \mathbb{P}[X \geq (1+\delta)\mu] < \left(\frac{e^\delta}{(1+\delta)^{1+\delta}}\right)^\mu$$

$$\forall \delta \in (0,1] \;:\; \mathbb{P}[X \geq (1+\delta)\mu] \leq \exp(-\mu\delta^2/3)$$

◀

Proof (Sketch):

Apply Markov's Inequality to $Y = e^{tX}$, then choose convenient t. ■

(more details in *Advanced Algorithms*)

Detour: Chernoff Bound for Binomial Distribution

The algorithmically most widely used special case has identical coin flips.

Corollary 1.10 (Chernoff Bound for Binomial Distribution)

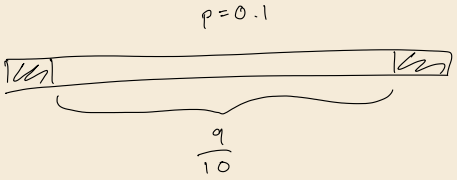
Let $X \stackrel{\mathcal{D}}{=} \mathrm{Bin}(n,p)$. Then

$$\forall \delta \geq 0 \;:\; \mathbb{P}\left[\left|\frac{X}{n} - p\right| \geq \delta\right] \leq 2\exp(-2\delta^2 n)$$

$\to \infty$

$\delta > \frac{1}{\sqrt{n}} = n^{-0.5}$ $\quad n^{-0.499}$

$\rightsquigarrow$ $\mathrm{Bin}(n,p) \in np \pm n^{0.501}$ w.h.p.



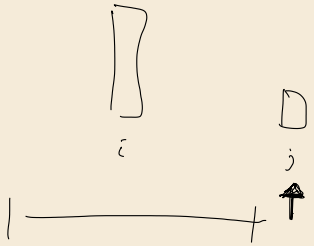
Analysis of Random BSTs – Concentration

Theorem 1.11 (Concentration of left-to-right minima)

The number of left-to-right minima in a permutation of length n is in $O(\log n)$ w.h.p.
Hence, the above expected results hold with high probability (up to constant factors). ◀

Proof (Idea):
Via *inversion table* of permutation:
$a_1, \ldots, a_n$ permutation of $[n]$ in bijection with $b_1, \ldots, b_n$ where

$$b_j = \#\text{inversions of form } (\bullet, j)$$
$$= \#\text{values left of } j \text{ that are} > j$$



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$$= \#\text{values left of } \underset{j}{\square} \text{ that are } > \overset{i}{\square}$$

random permutation $\rightsquigarrow$ b_j **independent** and $b_j \overset{\mathcal{D}}{=} \text{Uniform}[0..n-j]$

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Proof (Idea): *greater and left of "number" j*

Via *inversion table* of permutation:

$a_1, \ldots, a_n$ permutation of $[n]$ in bijection with $b_1, \ldots, b_n$ where

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random permutation $\rightsquigarrow$ b_j **independent** and $b_j \stackrel{\mathcal{D}}{=} \text{Uniform}[0..n-j]$

$$\text{L2RMax}(a) = \sum_{j=1}^{n} [b_j = 0] \rightsquigarrow \text{a sum of independent Bernoulli trials.}$$ + symmetry ■

Hook-Length Formula for Random BSTs

For a given tree, the probability to see this shape can be computed recursively over the tree structure:

Theorem 1.12 (Random BST Distribution)

Let T_n be a binary tree. The probability that Random BSTs attains the shape T_n after n insertions is

$$\Pr[T_n] = \begin{cases} 1 & n = 0, \\ \frac{1}{n} \cdot \Pr[T_L] \cdot \Pr[T_R] & n \geq 1, \end{cases} \quad \text{(for } T_L \text{ and } T_R \text{ the left resp. right subtrees of } T\text{).}$$

randomness preservation! filtering out values $< x$

$\frac{1}{3}$ again a random permutation

$$\Pr\left[\begin{array}{c} \circ \\ \circ\ \ \circ \end{array}\right] = \frac{1}{3}$$

$$\Pr\left[\begin{array}{c} \circ \\ \circ \\ \circ \end{array}\right] = \frac{1}{3} \cdot \Pr\left[\begin{array}{c} \circ \\ \circ \end{array}\right] = \frac{1}{3} \cdot \frac{1}{2} \cdot 1 = \frac{1}{6}$$

More AofA (Analysis of Algorithms)

Remark 1.13 (Knowledge on Random BSTs)

Random BSTs are extremely well-studied in Analysis of Algorithms.
A few more results (all proven in the literature):

(a) The **expected height** is $\alpha \ln n - \beta \ln \ln n \pm O(1)$ with $\alpha \approx 4.311$ and $\beta \approx 1.953$.

(b) The **height** divided by $\ln n$ **converges in probability** to the constant α.

(c) The number X_{nk} of **external leaves at depth** k satisfies $\mathbb{E}[X_{nk}] = \frac{2^k}{n!}\left[{n \atop k}\right]$.

(d) The **depth** of a typical **leaf** divided by $\ln n$ **converges in probability** to 2.

(e) The standardized **depth** of a random leaf **converges** in distribution to a standard **normal distribution**.

(f) The same is true for the standardized depth of a random internal node.

(g) Let D_n be the **depth of the nth inserted node**. Then $(D_n - \ln n)/\sqrt{\ln n}$ converges in distribution to a standard **normal distribution**. ◀

⇝ In many ways, random BSTs are close to perfectly balanced.

⇝ If only we can get the performance of random BSTs, we're happy.

Caveat: Random Deletions ≠ Random BSTs!

$\rightsquigarrow$ *Unbalanced BSTs have **great** performance **if** insertions come in random order.*

Interesting fact: *no longer true* if there are also *deletions*!

After long sequence of random inserts and deletes: expected height $\Theta(\sqrt{n})$, not $\Theta(\log n)$ (!)

Reason: Hibbard's deletion algorithm destroys randomness!

Animations: `http://algs4.cs.princeton.edu/32bst`

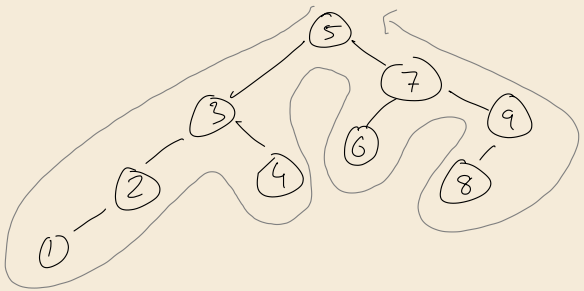
1.5 Treaps

Treaps

Observation: The *preorder* sequence of the keys fully determines a BST since

- each BST has unique preorder, and
- each preorder generates a unique BST by inserting keys in preorder into an initially empty tree.

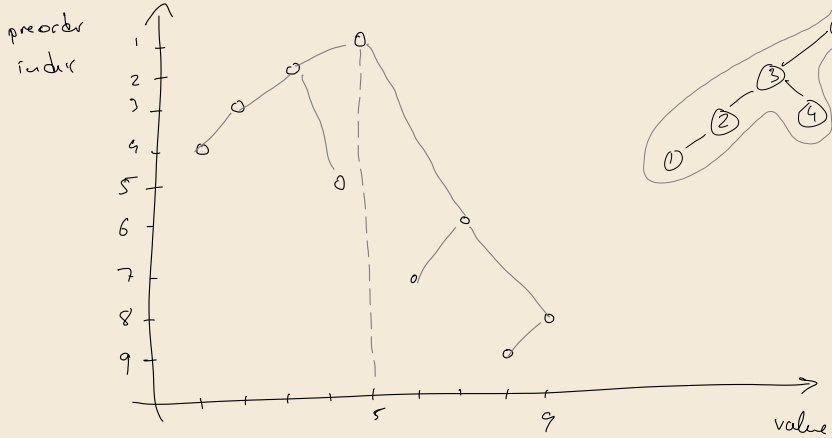
⇝ Enforcing the preorder corresponding to a **random** BST suffices!
. . . but how? We have no control over the insertion of keys!



5 3 2 1 4 7 6 9 8
1 2 3 4 5 6 7 8 9

5 3 2 1 4 7 6 9 8

1 2 3 4 5 6 7 8 9



Treaps

Observation: The *preorder* sequence of the keys fully determines a BST since

- each BST has unique preorder, and
- each preorder generates a unique BST by inserting keys in preorder into an initially empty tree.

$\rightsquigarrow$ Enforcing the preorder corresponding to a **random** BST suffices!
. . . but how? We have no control over the insertion of keys!

Idea: Separate *key values* from *rank in insertion order* using random **"priorities"**.

Definition 1.14 (Treap)

Let $S = \{(k_1, p_1), \ldots, (k_n, p_n)\}$ be a set of *key-priority pairs* where $k_i \in K$ and $p_i \in [0, 1]$ for K some totally ordered universe.
A *treap* for S is a binary tree with n internal nodes labeled by the key-priority pairs so that

(a) the *search tree property* holds w.r.t. the keys, and

(b) the *heap property* holds w.r.t. the priorities.

There can be only one

Theorem 1.15 (Treaps are unique)

Let S be a set of n key-priority pairs where all keys and all priorities are distinct. Then there is *exactly one treap* for S.

Proof:

existence: use algorithm w/ sliding down horizontal line

uniqueness: by induction

IB $n = 1$ ✓

IS: find $i = \arg\min p_j$ k_i = root

$\Rightarrow$ treaps for all (k_j, p_j) with $k_j < k_i$

resp. (k_j, p_j) " $>$

are unique

Randomized Treaps

Definition 1.16 (Randomized Treaps)

A *randomized treap* is the unique treap that results from given keys $k_1, k_2, \ldots$ where (upon insertion) we assign k_i a priority $p_i \overset{\mathcal{D}}{=} \text{Uniform}(0, 1)$ independent of all previous priorities. ◀

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Theorem 1.17 (Shape of randomized treaps)

The (random) shape of a randomized treap for n keys has the *same distribution* as random BST with n keys. ◀

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Corollary 1.18 (Search Costs)

All results for random BSTs apply, in particular:

(a) Expected search costs (#comparisons) $< 2 \ln n + 1$.

(b) Search costs in $O(\log n)$ w.h.p. ◀

1.6 Updates in Treaps

Insertions and Deletions in Randomized Treaps

Up to now: *static* view on treaps.

But can we efficiently turn a randomized treap for $k_1, \ldots, k_n$ into one for $k_1, \ldots, k_{n+1}$?
And vice versa?

Insertions and Deletions in Randomized Treaps

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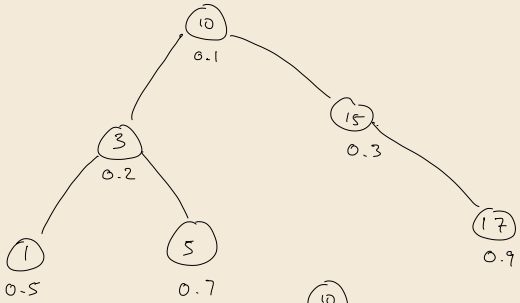
But can we efficiently turn a randomized treap for $k_1, \ldots, k_n$ into one for $k_1, \ldots, k_{n+1}$?
And vice versa?

Yes!

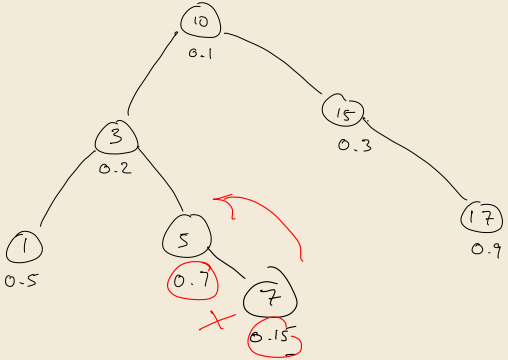
- **Insert:** Start as in plain BST, then *rotate up* until heap property holds.
- **Delete:** Rotate node down (as if priority was $+\infty$) until it is a leaf, then remove it.

Conceptually very simple!

$\rightsquigarrow$ *all operations in $O(\log n)$ time w.h.p.!* in particular # rotations = $O(\log n)$ whp


10
0.1
3
0.2
15
0.3
1
0.5
5
0.7
17
0.9

insert 7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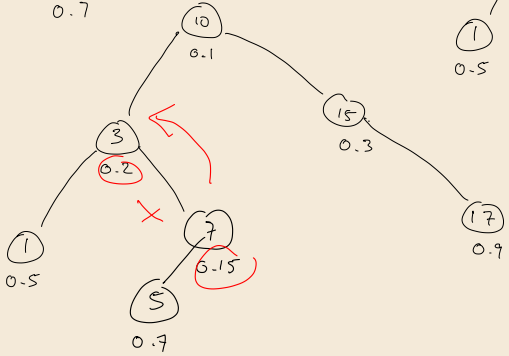
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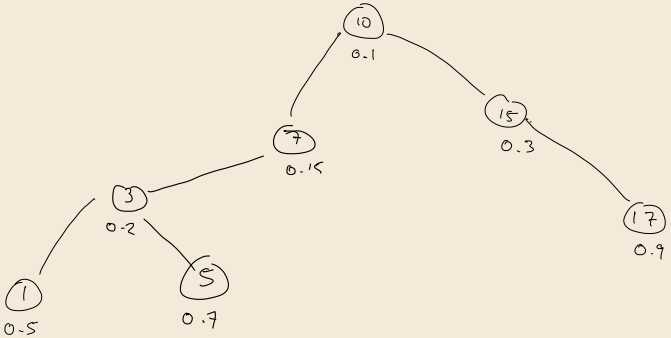


$ exam : Zip trees ≈ treap w/ lies for priorities are allowed / exploited

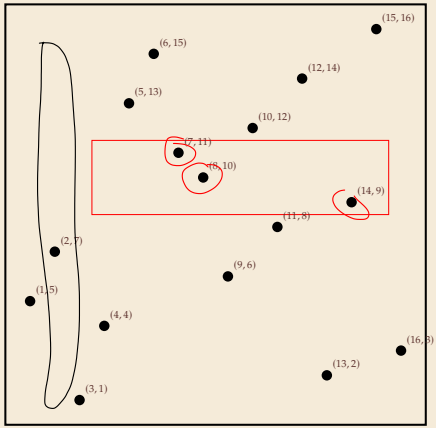
Excursion: Why care about counting rotations?

For secondary data structures, cost of a rotation can be *linear in subtree size*.

Example: *Range trees*

kd-trees, quad trees

2d-orthogonal range search



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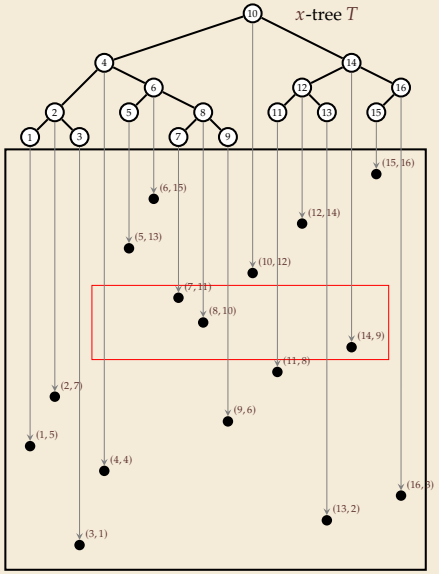

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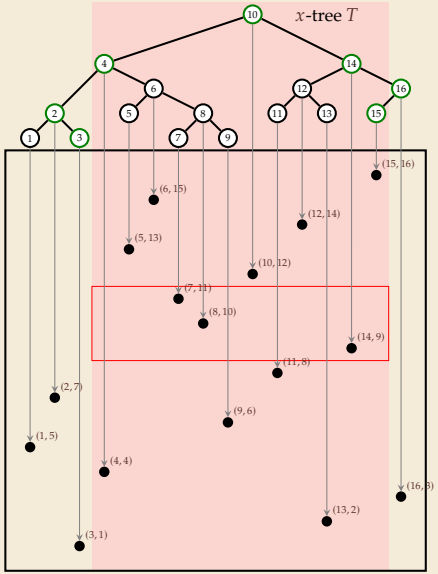

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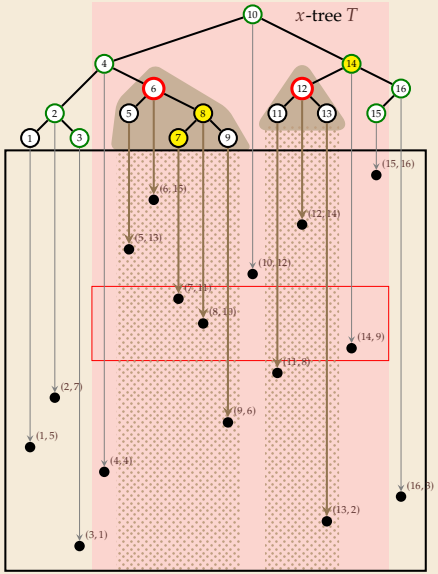

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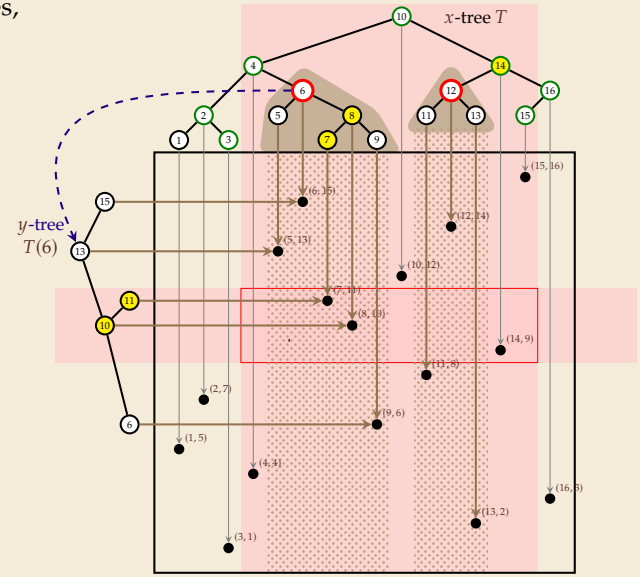
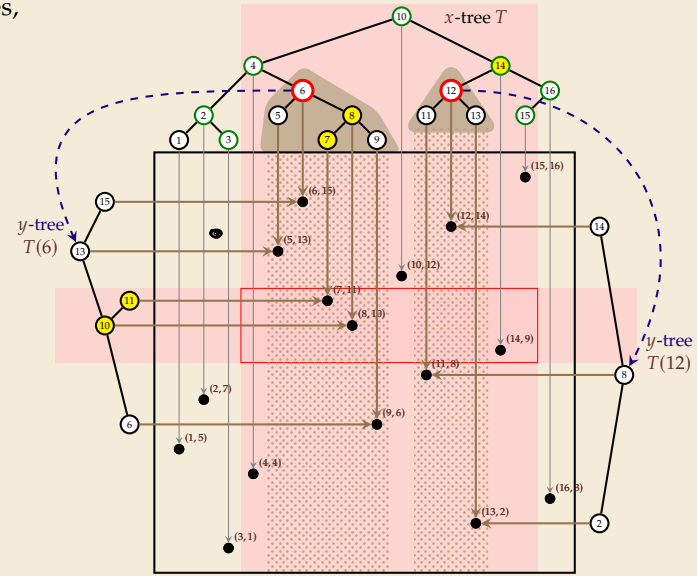


Excursion: Why care about counting rotations?

For secondary data structures, cost of a rotation can be *linear in subtree size*.

Example: *Range trees*

insert (x,y)
needs (potentially)
to rotate in x-treap



Ancestor Indicators

Can actually bound number of rotations much more tightly!

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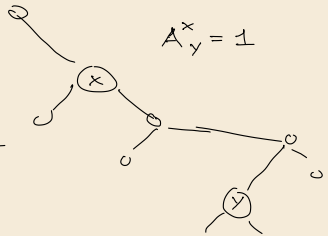
For that, we need closer look at depths of *internal nodes* in random BSTs.
Analysis possible based on handy notion:

Ancestor Indicators



Can actually bound number of rotations much more tightly!

For that, we need closer look at depths of *internal nodes* in random BSTs.
Analysis possible based on handy notion:

Lemma 1.19 (Ancestor indicators)

Let T_n be a random BST with keys $[n]$ and denote by $A^x_y = [x \text{ is a } \textit{proper} \text{ ancestor of } y]$ for $x, y \in [n]$.
(This means $A^x_x = 0$ and for $x \neq y$, $A^x_y = 1$ iff x lies on the path from the root to y.)
Then holds:

(a) $A^x_y = 1$ iff x was the *first* among the keys $[x..y] \cup [y..x]$ that was inserted into T_n.

(b) $A^x_y = 1$ iff x and y are *directly compared* by randomized Quicksort during a partitioning step using pivot x.

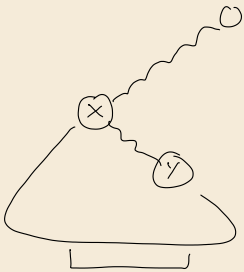
(c) $\Pr[A^x_y = 1] = \Pr[A^y_x = 1] = \dfrac{1}{|y - x| + 1}$ for $x \neq y$. ◀

Ancestor Indicators – Proof

(a) $A_y^x = 1$ iff x was the *first* among the keys $[x..y] \cup [y..x]$ that was inserted into T_n.

Proof: $x < y$ wlog

(a) "⇒" $A_y^x = 1$



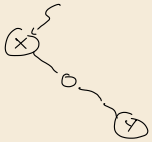
consider keys $[x..y]$

any other key $z \in (x..y)$

⇒ z separates x from y

⇒ x not an ancestr. of y ↯

"⇐" x inserted first among $[x..y]$



(c)

(c) $\Pr[A_y^x = 1] = \Pr[A_x^y = 1] = \dfrac{1}{|y-x|+1}$ for $x \neq y$.

from (a) $\mathbb{P}[A^*_{x,y} = 1) = \mathbb{P}[x \text{ inserted first among } [x..y]]$

$$= \frac{1}{|y-x|+1}$$

$\Rightarrow$ analysis of Quicksort ?

never compare same pair twice

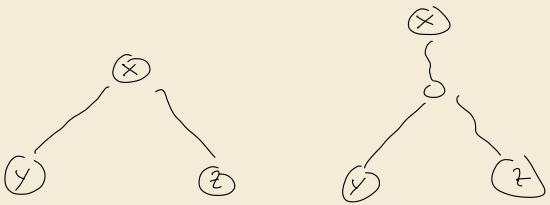
$$C = \sum_{1 \le i < j \le n} [i \text{ compared to } j) \underset{\text{(b)}}{=} A^i_j + A^j_i \underset{\text{(c)}}{=} \sum_{1 \le i < j \le n} \frac{2}{j-i+1}$$

Common Ancestor Indicators

Remark 1.20 (Common ancestor indicators)

Idea generalizes to $C^x_{y,z} = [x \text{ is common ancestor of } y \text{ and } z]$:

$$\Pr[C^x_{y,z} = 1] \;=\; \frac{1}{\max\{x,y,z\} - \min\{x,y,z\} + 1}.$$



Depth of Internal Nodes

Theorem 1.21 (Expected depth of kth node)

The *expected depth* of the *kth internal node* (for $k = 1, \ldots, n$) in a random BST on $n \geq 1$ nodes is $H_k + H_{n-k+1} - 2$. ◀

Recall: $\mathbb{E}[\text{depth}(\boxed{k})] = H_k + H_{n-k}$.

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Recall: $\mathbb{E}[\text{depth}(\boxed{k})] = H_k + H_{n-k}$.

Proof:

$$\text{depth}(\textcircled{k}) = \sum_{x=1}^{n} A_k^x$$

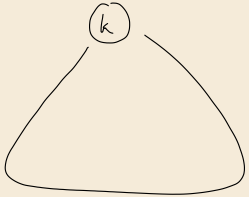
$$\mathbb{E}[\text{depth}(\textcircled{k})] = \sum_{x=1}^{n} \mathbb{P}[A_k^x = 1] = \sum_{x=1}^{k-1} \frac{1}{k-x+1} + \sum_{x=k+1}^{n} \frac{1}{x-k+1}$$

$$= \sum_{i=2}^{k} \frac{1}{i} + \sum_{i=2}^{n-k+1} \frac{1}{i} = H_k + H_{n-k+1} - 2$$

Subtree sizes

Remark 1.22 (Expected subtree size)

The *expected size* of the *subtree* rooted at the kth internal node is also $H_k + H_{n-k+1} - 1$. ◀



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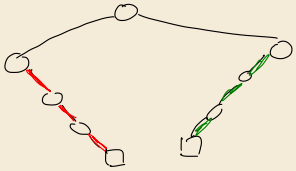
The *expected size* of the *subtree* rooted at the kth internal node is also $H_k + H_{n-k+1} - 1$. ◀

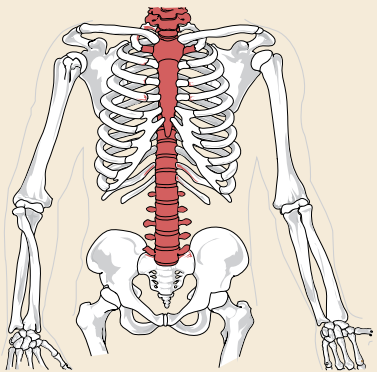
Proof:

Subtree size of $\textcircled{k} = 1 + \sum_{x=1}^{n} A_x^k$

Similar calculation ■

Spines of Trees





Lemma 1.23 (Bound on Rotations)
The number of *rotations* to insert or delete a node x in a randomized treap is at most $LS(x) + RS(x)$, where $LS(x)$ and $RS(x)$ are the *lengths of the left resp. right spine* of (the subtree of) x in the treap (after insertion resp. before deletion). ◀

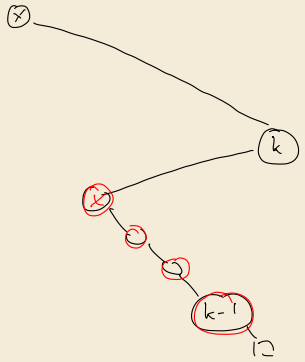
Lemma 1.24 (Expected Spine Lengths)
The expected length of the left and right spine of (the subtree of) the kth internal node (for $k = 1, \ldots, n$) in random BST of n keys are given by

$$\mathbb{E}[LS(k)] = 1 - \frac{1}{k} \qquad \mathbb{E}[RS(k)] = 1 - \frac{1}{n-k+1}$$ ◀

Spines of Trees – Proof

Proof:

$$LS(k) = \sum_{x=1}^{k-1} \left(A^x_{k-1} - C^x_{k-1,k} \right)$$



1.7 Insertion at Root

Simplerer!

- The details of the implementation of Treaps still need
  - cases distinctions for which rotation to use
  - both code to bubble up nodes (insert) and trickle down (delete)
- We can indeed simplify this further
  (at the expense for more pointer writes)

Simplerer!

- The details of the implementation of Treaps still need
  - cases distinctions for which rotation to use
  - both code to bubble up nodes (insert) and trickle down (delete)
- We can indeed simplify this further
  (at the expense for more pointer writes)
- Idea actually an alternative to standard BST insert / delete
  - Instead of adding a new leaf upon insert,
    we force the new key to **become the root** upon insert

  $\rightsquigarrow$ need a method to ***split*** a BST into $< x$ and $> x$ for a key x.
  (assume that x not in the tree)
  - if we also have a complementary ***join***,
    deleting an arbitrary node works by joining its children.

Branchless Children

A neat trick makes the following code more compact: store children in 2-element array

```
class Node {
    int key, size = 1;
    Node[] child = new Node[2]; // child[0] is Left, child[1] is Right
}
```

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}
```

For rank-based operations, we store the subtree size of a node

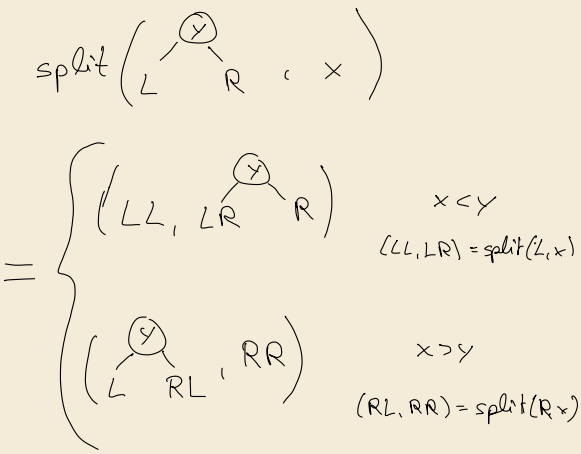
The following helper method is used to keep subtree size up to date

```
void updateSize(Node t) {
    if (t != null) t.size = 1 + size(t.child) + size(t.child);
}
```

Split

Split BST into two BSTs containing keys $< x$ and $\geq x$ resp.

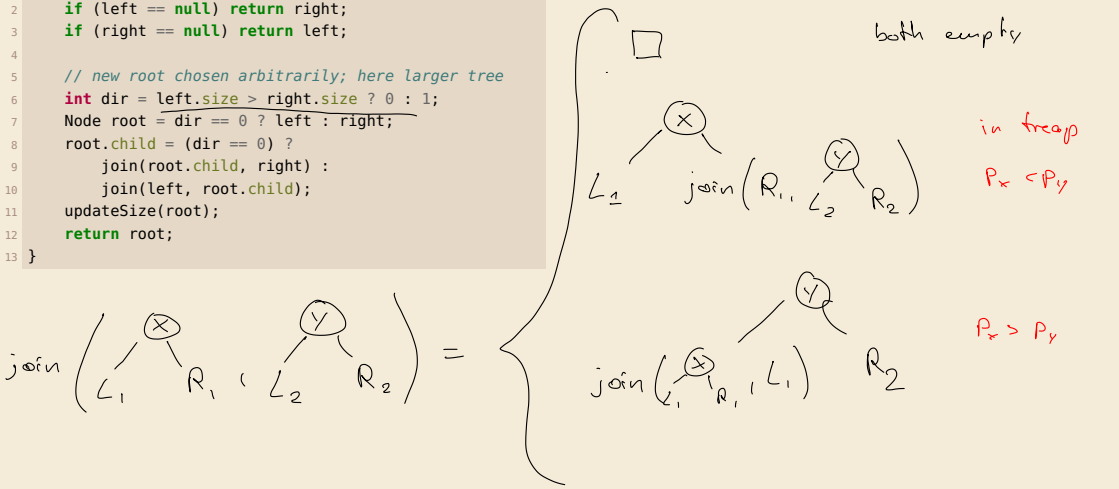
```
Node[] split(Node t, int x) {
    if (t == null) return new Node[]{null, null};

    int dir = x < t.key ? 0 : 1;
    Node[] res = split(t.child[dir], x);

    // Opposite child receives the remainder of split
    t.child[dir] = res[1 - dir];
    updateSize(t);

    // Construct result dynamically based on dir
    Node[] ans = new Node[2];
    ans[dir] = res[dir];
    ans[1 - dir] = t;
    return ans;
}
```



Join

Given two BSTs where all keys in the left are smaller than all keys in the right, merge them into a single BST.

```
1  private Node join(Node left, Node right) {
2      if (left == null) return right;
3      if (right == null) return left;
4
5      // new root chosen arbitrarily; here larger tree
6      int dir = left.size > right.size ? 0 : 1;
7      Node root = dir == 0 ? left : right;
8      root.child = (dir == 0) ?
9          join(root.child, right) :
10         join(left, root.child);
11     updateSize(root);
12     return root;
13 }
```



Insert at Root, Delete

Based on split and merge, insert and delete easy to implement.

```
Node insertAtRoot(Node t, int x) {
    Node[] splits = split(t, x);
    Node root = new Node(x);
    root.child = splits;
    updateSize(root);
    return root;
}

Node delete(Node t, int x) {
    if (t == null) return null;
    if (x == t.key) return join(t.child, t.child);
    int dir = x < t.key ? 0 : 1;
    t.child[dir] = delete(t.child[dir], x);
    updateSize(t);
    return t;
}
```

Tamio Nakajima's Treap Implementation

```cpp
#include <random>
using namespace std;

struct node {
    node *l, *r;
    int prio, key, sum;
};
node nil_node = {&nil_node, &nil_node, 0, 0, 0};
node *nil = &nil_node; // Sentinel node

node *mod_child(node *n0, int id, node *child) {
    node *n = new node;
    *n = *n0;
    (id == 1 ? n->r : n->l) = child;
    n->sum = n->l->sum + n->key + n->r->sum;
    return n;
}

node *join(node *l, node *r) {
    return l == nil ? r : r == nil ? l :
        l->prio > r->prio ?
            mod_child(l, 1, join(l->r, r)) :
            mod_child(r, 0, join(l, r->l));
}

pair<node*,node*> split(node* n, int x) {
    pair<node*, node*> ret = {nil, nil};
    return n == nil ? ret : n->key < x ?
        (ret = split(n->r, x), ret.first =
            mod_child(n, 1, ret.first), ret) :
        (ret = split(n->l, x), ret.second =
            mod_child(n, 0, ret.second), ret);
}

node *mk_node(int x){
    static mt19937 mt(random_device{}());
    int prio = uniform_int_distribution<int>(
        1, 1000000000)(mt);
    return new node {nil, nil, prio, x };
}


node *insert(node *root, int x){
    auto p = split(root, x);
    return join(join(p.first, mk_node(x)), p.second);
}

node *empty_treap(){ return nil; }
```