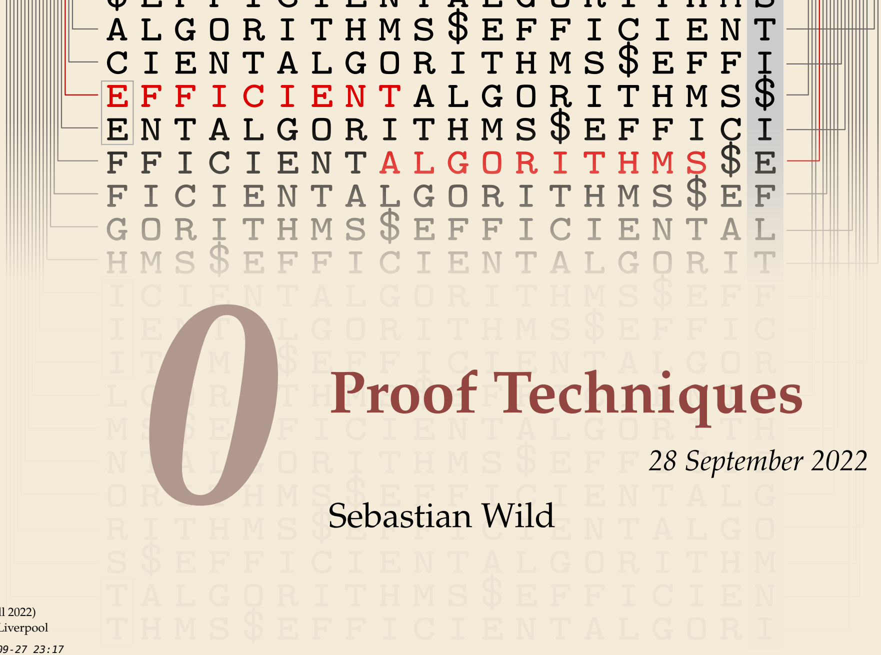

0

Proof Techniques

28 September 2022

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COMP526 (Fall 2022)
University of Liverpool

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Learning Outcomes

Unit 0: *Proof Techniques*

1. Know logical *proof strategies* for proving implications, set inclusions, set equalities, and quantified statements.
2. Be able to use *mathematical induction* in simple proofs.
3. Know techniques for *proving termination* and *correctness* of procedures.

Outline

0 Proof Techniques

0.1 Digression: Random Shuffle

Random shuffle

- Goal: Put an array $A[0..n)$ of n numbers into random order.
  More precisely: Any ordering of the elements $A[0], \ldots, A[n-1]$ should be equally likely.
- A natural approach yields the following code

```
procedure mySHuffle($A[0..n)$)
    for $i := 0, \ldots, n-1$
        $r$ := randomInt($[0..n)$) // A uniformly random number $r$ with $0 \leq r < n$.
        Swap $A[i]$ and $A[r]$ // Swap $A[i]$ to random position.
    end for
```

- Intuitively: *All elements are moved to a random index, so the order is random . . . right?*

Clicker Question

Select all statements that apply to myShuffle (for you).

- **A** I have seen this shuffling algorithm (or a very similar method) before.
- **B** I can understand the pseudocode for myShuffle (so I would be able to do an example by hand).
- **C** It can generate all possible orderings of A (depending on the random numbers).
- **D** myShuffle produces all possible orderings with the same probability.
- **E** Assuming randomInt gives (perfect) uniform random numbers in the given range, myShuffle generates any ordering with equal probability.

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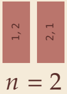

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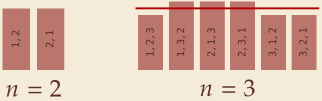

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procedure mySnuffle($A[0..n)$)
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    end for
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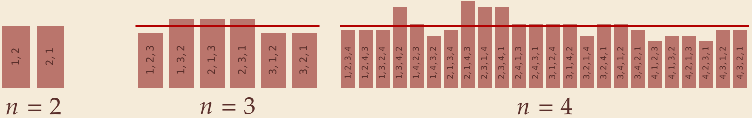

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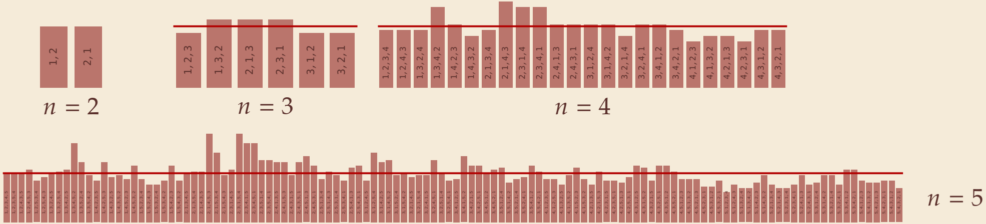

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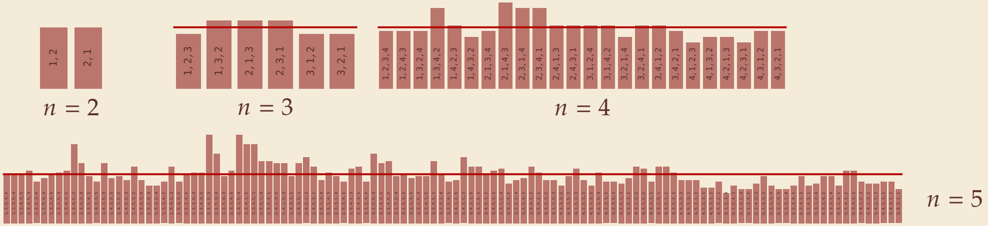
```

← **WRONG!**
DO NOT USE

- Intuitively: *All elements are moved to a random index, so the order is random . . . right????*



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- **C** It can generate all possible orderings of A (depending on the random numbers). ✓
- **D** ~~myShuffle produces all possible orderings with the same probability.~~
- **E** ~~Assuming randomInt gives (perfect) uniform random numbers in the given range, myShuffle generates any ordering with equal probability.~~

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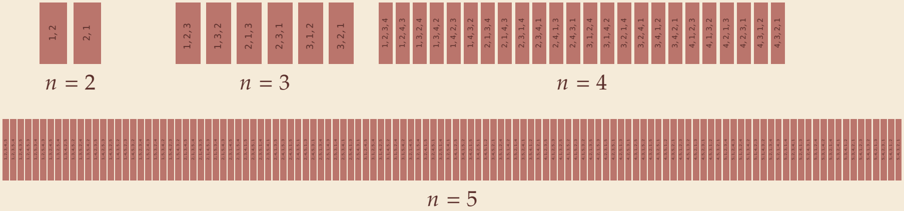
Correct shuffle

- interestingly, a very small change corrects the issue

```
procedure shuffleKnuthFisherYates($A[0..n)$)
    for $i := 0, \dots, n-1$
        $r := \mathrm{randomInt}([i..n))$
        Swap $A[i]$ and $A[r]$
    end for
```



- looks good . . .
- . . . but how can we convince ourselves that it is correct *beyond any doubt?*

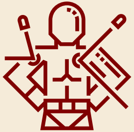
What is a *formal* proof?

A formal proof (in a logical system) is a **sequence of statements** such that each statement

1. is an *axiom* (of the logical system), or
2. follows from previous statements using the *inference rules* (of the logical system).

Among experts: Suffices to *convince a human* that a formal proof *exists*.

But: Use formal logic as guidance against faulty reasoning. ⇝ bulletproof



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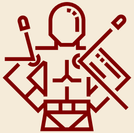

Notation:

- Statements: $A \equiv$ "it rains", $B \equiv$ "the street is wet".
- Negation: $\neg A$ "Not A"
- And/Or: $A \wedge B$ "A and B"; $A \vee B$ "A or B or both"
- Implication: $A \Rightarrow B$ "If A, then B"; $\neg A \vee B$
- Equivalence: $A \Leftrightarrow B$ "A holds true *if and only if ('iff')* B holds true."; $(A \Rightarrow B) \wedge (B \Rightarrow A)$

exclusive or XOR

$A \oplus B$

Clicker Question

Is the following statement true?
“If the Earth is flat, then ships can fall over its rim.”

A Yes **B** No **C** Neither

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Clicker Question

$A \Rightarrow B$ iff $\neg A \vee B$

Is the following statement true?

"If the $\underbrace{\textit{Earth is flat}}_{A}$*, then* $\underbrace{\textit{ships can fall over its rim}}_{B}$*."*

A Yes ✓

B ~~No~~

C ~~Neither~~

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0.2 Proof Templates

Implications

To prove $A \Rightarrow B$, we can

- directly derive B from A *direct proof*
- prove $(\neg B) \Rightarrow (\neg A)$ *indirect proof, proof by contraposition*
- assume $A \wedge \neg B$ and derive a contradiction *proof by contradiction, reductio ad absurdum*
- distinguish cases, i. e., separately prove
  $(A \wedge C) \Rightarrow B$ and $(A \wedge \neg C) \Rightarrow B$. *proof by exhaustive case distinction*

Clicker Question

Suppose we want to prove:
"If $n^2 \in \mathbb{N}_0$ is an even number, then n is also even."
For that we show that when n is odd, also n^2 is odd.
Which proof template do we follow?

- **A** direct proof: $A \Rightarrow B$
- **B** indirect proof: $(\neg B) \Rightarrow (\neg A)$
- **C** proof by contradiction: $A \wedge \neg B \Rightarrow \lightning$
- **D** proof by case distinction: $(A \wedge C) \Rightarrow B$ and $(A \wedge \neg C) \Rightarrow B$

n odd

$\Rightarrow n = 2k+1$ for some $k \in \mathbb{N}_0$

$\Rightarrow n^2 = (2k+1)^2$

$= 4k^2 + 4k + 1$

$= 2(\underbrace{2k^2+2k}_{k' \in \mathbb{N}_0}) + 1$

$\Rightarrow n^2$ odd

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Clicker Question

Suppose we want to prove: A B

"If $n^2 \in \mathbb{N}_0$ is an even number, then n is also even."

For that we show that when n is odd, also n^2 is odd. $\neg A$

Which proof template do we follow? $\neg B$

A ~~direct proof: $A \Rightarrow B$~~

B indirect proof: $(\neg B) \Rightarrow (\neg A)$ ✓

C ~~proof by contradiction: $A \wedge \neg B \Rightarrow \lightning$~~

D ~~proof by case distinction: $(A \wedge C) \Rightarrow B$ and $(A \wedge \neg C) \Rightarrow B$~~

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Equivalences

To prove $A \Leftrightarrow B$,
we prove both implications $A \Rightarrow B$ and $B \Rightarrow A$ separately.

(Often, one direction is much easier than the other.)

Set Inclusion and Equality

To prove that a set S contains a set R, i. e., $R \subseteq S$,
we prove the implication $x \in R \Rightarrow x \in S$.

$\forall x$

To prove that two sets S and R are equal, $S = R$,
we prove both inclusions, $S \subseteq R$ and $R \subseteq S$ separately.

continue

12:05

0.3 Mathematical Induction

Quantified Statements

Notation

- Statements with parameters: $A(x) \equiv$ "x is an even number."
- Existential quantifiers: $\exists x : A(x)$ "There exists some x, so that $A(x)$."
- Universal quantifiers: $\boxed{\forall x : A(x)}$ "For all x it holds that $A(x)$."

  Note: $\forall x : A(x)$ is equivalent to $\neg \exists x : \neg A(x)$

Quantifiers can be nested, e. g., *ε-δ-criterion for limits*:

$$\lim_{x \to \xi} f(x) = a \quad :\Leftrightarrow \quad \forall \varepsilon > 0 \ \exists \underset{\delta(\varepsilon)}{\underset{\parallel}{\delta}} > 0 \ : \ \left(|x - \xi| < \delta\right) \Rightarrow \left|f(x) - a\right| < \varepsilon.$$

To prove $\exists x : A(x)$, we simply list an example ξ such that $A(\xi)$ is true.

Clicker Question

Have you seen **proofs by *mathematical induction*** before?

A Yes, could do it

B Yes, but only vaguely remember

C I've heard this term before, but ...

D I have not heard "mathematical induction" before

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For-all statements

To prove $\forall x : A(x)$, we can

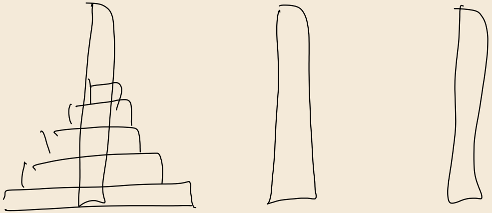
$\forall x \in \mathbb{N}_0 : A(x)$ $\forall x : x \in \mathbb{N}_0 \Rightarrow A(x)$

- derive $A(x)$ for an *"arbitrary but fixed value of x"*, or,
- for $x \in \mathbb{N}_0$, use ***induction***, i. e.,
  - prove $A(0)$, *induction basis*, and
  - prove $\forall n \in \mathbb{N}_0 : A(n) \Rightarrow A(n+1)$ *inductive step*

More general variants of induction:

- complete/strong induction
  inductive step shows $\underline{\big(A(0) \wedge \cdots \wedge A(n)\big)} \Rightarrow A(n+1)$
- structural/transfinite induction
  works on any *well-ordered* set, e. g., binary trees, graphs, Boolean formulas, strings, . . .

no infinite strictly decreasing chains



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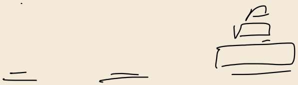
How many steps are needed?



$\forall n : \underbrace{\#\text{ steps needed is } 2^n - 1}_{A(n)}$

n disks

induction base: $n = 0$ 0 steps

$2^0 - 1 = 0$ ✓

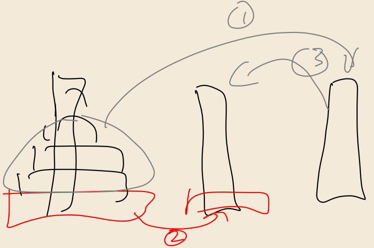
inductive step:

assuming inductive hypothesis $(= A(n))$

$\#$ steps for $n+1$ disks

$= 2 \cdot \#\text{ steps for } n + 1$

$\underset{IV}{=} 2(2^n - 1) + 1 = 2^{n+1} - 1$ $\square$



0.4 Correctness Proofs

Formal verification

- verification: prove that a program computes the correct result

⇝ **not** our focus in COMP 526
but some techniques are useful for *reasoning* about algorithms

Here:

1. Prove that loop or recursive call eventually *terminates*.
2. Prove that a *loop* computes the *correct* result.

precondition $\underset{\text{code}}{\Longrightarrow}$ post condition

Proving termination

To prove that a recursive procedure `proc`$(x_1, \ldots, x_m)$ eventually terminates, we

- define a *potential* $\Phi(x_1, \ldots x_m) \underline{\in \mathbb{N}_0}$ of the parameters
  (Note: $\Phi(x_1, \ldots x_m) \geq 0$ by definition!)

- prove that every recursive call decreases the potential, i. e.,
  any recursive call `proc`$(y_1, \ldots, y_m)$ inside `proc`$(x_1, \ldots, x_m)$ satisfies

$$\Phi(y_1, \ldots, y_m) < \Phi(x_1, \ldots, x_m) \qquad \text{which means}$$
$$\Phi(y_1, \ldots, y_m) \leq \Phi(x_1, \ldots, x_m) \mathbf{-1}$$

⇝ `proc`$(x_1, \ldots, x_m)$ terminates because
we can only strictly *decrease* the (integral) potential
a *finite* number of times from its initial value

- Can use same idea for a loop: show that potential decreases in each iteration.
  - ⇝ see tutorials for an example.

Loop invariants

Goal: Prove that a *post condition* holds after execution of a (terminating) loop.

```
// (A) before loop
while cond do
    // (B) before body
    body
    // (C) after body
end while
// (D) after loop
```

For that, we

- find a *loop invariant I*   (that's the tough part!)
- prove that I holds at (A)
- prove that $I \wedge cond$ at (B) imply I at (C)
- prove that $I \wedge \neg cond$ imply the desired post condition at (D)

Note: I holds before, during, and after the loop execution, hence the name.

Loop invariant – Example

- loop condition: $cond \equiv i < n$
- post condition (in line 13):
  $curMax = \max_{k \in [0..n-1]} A[k]$
- loop invariant:
  $I \equiv curMax = \max_{k \in [0..i-1]} A[k] \wedge i \leq n$

```
1  procedure arrayMax($A$,$n$)
2      // input: array of $n$ elements, $n \geq 1$
3      // output: the maximum element in $A[0..n-1]$
4      $curMax := A[0]$; $i = 1$
5      // (A)
6      while $i < n$ do
7          // (B)
8          if $A[i] > curMax$
9              $curMax := A[i]$
10         $i := i + 1$
11         // (C)
12     end while
13     // (D)
14     return $curMax$
```

We have to proof:

(i) I holds at (A)

(ii) $I \wedge cond$ at (B) $\Rightarrow$ I at (C)

(iii) $I \wedge \neg cond$ $\Rightarrow$ post condition