6 Error-Correcting Codes

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COMP526 (Fall 2023)
University of Liverpool
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Learning Outcomes

1. Understand the context of *error-prone communication*.
2. Understand concepts of *error-detecting codes* and *error-correcting codes*.
3. Know and understand the *terminology of block codes*.
4. Know and understand *Hamming codes*, in particular (7,4) Hamming code.
5. Reason about the *suitability of a code* for an application.

Unit 6: ***Error-Correcting Codes***

Outline

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6.1 Introduction

Noisy Communication

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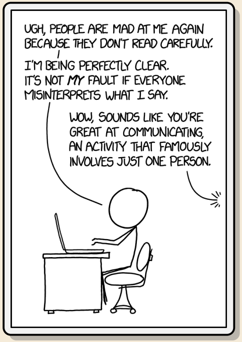
⇝ We can

1. **detect errors** "This sentence has aao pi dgsdho gioasghds."
2. **correct** (some) **errors** "Tiny errs ar corrrected automaticly."
   (sometimes too eagerly as in the Chinese Whispers / Telephone)



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- computers: copper cables &
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- We can aim at
  1. **error detection** ⇝ can request a re-transmit
  2. **error correction** ⇝ avoid re-transmit for common types of errors
- This will require *redundancy*: sending *more* bits than plain message
  ⇝ **goal:** robust code with lowest redundancy

that's the opposite of compression!

Clicker Question

What do you think, how many extra bits do we need to **detect** a **single bit error** in a message of 100 bits?

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What do you think, how many extra bits do we need to **correct** a **single bit error** in a message of 100 bits?

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6.2 Lower Bounds

Block codes

- **model:**
  - want to send message $S \in \{0,1\}^\star$ (bitstream) across a *(communication) channel*
  - any bit transmitted through the channel might ***flip*** ($0 \to 1$ resp. $1 \to 0$)
    no other errors occur (no bits lost, duplicated, inserted, etc.)
  - instead of S, we send *encoded bitstream* $C \in \{0,1\}^\star$
    sender *encodes* S to C, receiver *decodes* C to S (hopefully)
  - ⇝ what errors can be detected and/or corrected?

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- all codes discussed here are ***block codes***
  - divide S into *messages* $m \in \{0,1\}^k$ of k bits each (k = *message length*)
  - encode each message (separately) as $C(m) \in \{0,1\}^n$ (n = *block length*, $n \geq k$)
  - ⇝ can analyze everything block-wise

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  - ⇝ can analyze everything block-wise
- between 0 and n bits might be flipped
  - how many flipped bits can we definitely **detect**? (invalid code)
  - how many flipped bits can we **correct** without retransmit? (i. e. decoding m still possible)

Code distance

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- define $\mathcal{C}$ = set of all codewords = $C(\{0,1\}^k)$ $= \{ b \in \{0,1\}^n : \exists m \in \{0,1\}^k : b = C(m) \}$

$\rightsquigarrow \mathcal{C} \subseteq \{0,1\}^n$ $|\mathcal{C}| = 2^k$ out of 2^n n-bit strings are valid codewords

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d_H('aaba', 'abaa') = 2

- decoding = finding closest valid codeword

d_H = # positions where words differ

- ***distance of code:***

  d = minimal Hamming distance of any two codewords = $\min_{x,y \in \mathcal{C},\ x \neq y} d_H(x,y)$

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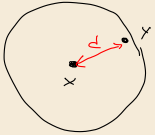
Implications for codes

1. Need distance d to **detect** all errors flipping up to $d-1$ bits.
2. Need distance d to **correct** all errors flipping up to $\left\lfloor \frac{d-1}{2} \right\rfloor$ bits.

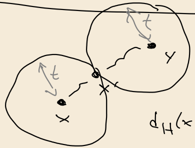


$x, y \in \mathcal{C}$

cannot distinguish receiving y because

(a) m was sent, $C(m) = y$

(b) m' was sent, $C(m') = x$ and d bits flipped



$d_H(x,y) \le 2t$

(a) $C(m) = x$ flip t bits $\rightsquigarrow x'$

(b) $C(m') = y$ flip t bits $\rightsquigarrow x'$

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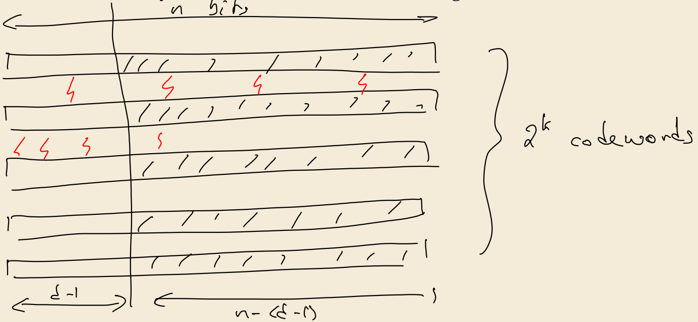
  Given block length n, message length k, code distance d, we must have:

- ***Singleton bound:*** $2^k \leq 2^{n-(d-1)} \rightsquigarrow n \geq k + d - 1$
  - *proof sketch:* We have 2^k codeswords with distance d
    after deleting the first $d - 1$ bits, all are still distinct
    but there are only $2^{n-(d-1)}$ such shorter bitstrings.



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disjoint "Hamming-balls"

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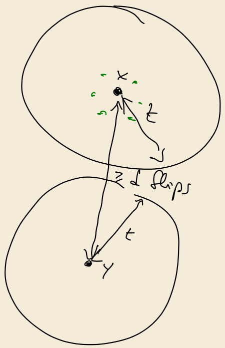
- ***Hamming bound:*** $2^k \le \dfrac{2^n}{\sum_{f=0}^{\lfloor (d-1)/2 \rfloor} \binom{n}{f}}$ $\qquad 2^n \ge 2^k \sum_{f=0}^{t} \binom{n}{f}$
  - *proof idea:* consider "balls" of bitstrings around codewords
    count bitstrings with Hamming-distance $\le t = \lfloor (d-1)/2 \rfloor$
    correcting t errors means all these balls are disjoint
    so $2^k \cdot \text{ball size} \le 2^n$



$\rightsquigarrow$ We will come back to these.

6.3 Hamming Codes

Parity Bit

- simplest possible error-detecting code: add a **parity bit**

`0 1 1 0 1 1 1 1 0 | 0`

$$= \begin{cases} 0 & \text{if number of ones is even} \\ 1 & \text{if number of ones is odd} \end{cases}$$

$$0 \oplus 1 \oplus 1 \oplus 0 \oplus 1 \oplus 1 \oplus 1 \oplus 1 \oplus 0 = 0$$

XOR

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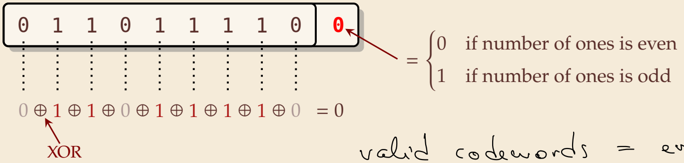

valid codewords = even # 1s

⇝ code distance 2

- can detect any single-bit error (actually, any odd number of flipped bits)
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  - early forms of main memory

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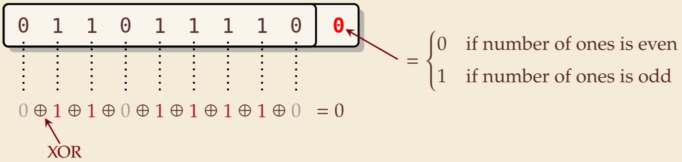

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👍 very simple and cheap

👎 cannot correct any errors

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Error-correcting codes

any downtime is expensive!

- typical application: heavy-duty server RAM
  - bits can randomly flip (e. g., by cosmic rays)
  - individually very unlikely,
    but in always-on server with lots of RAM, it happens!

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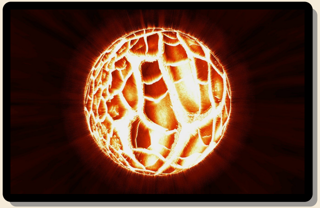

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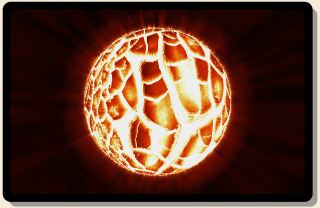

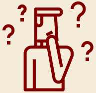

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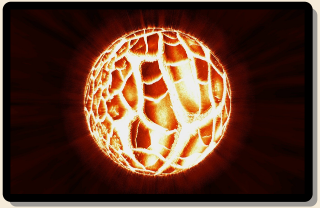

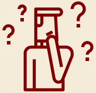

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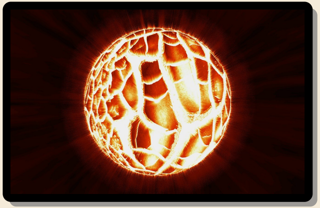

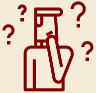

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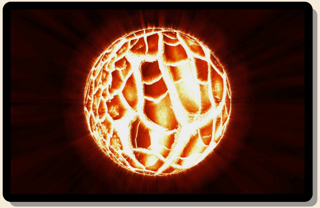

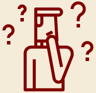

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Can do it with 11% extra memory! (instead of 200% (!))

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- **Idea:** Use several parity bits
  - each covers a **subset** of bits
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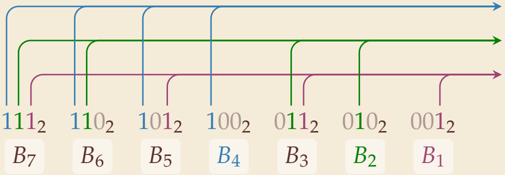

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codeword

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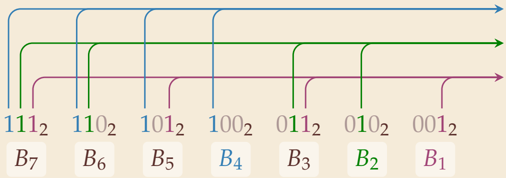

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↑ B_5 flipped $C_2 = C_0 = 1$ ⇝ $C = 101_2 = 5$

Observe:

- No error (all 7 bits correct) ⇝ $C = C_2C_1C_0 = 000_2 = 0$ ✓
- What happens if (exactly) 1 bit, say B_i flips?

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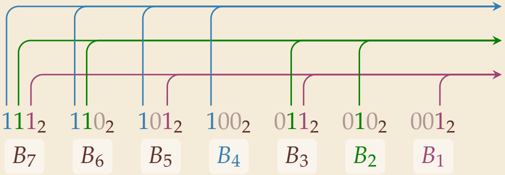

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$C_j = 1$ iff jth bit in binary representation of i is 1

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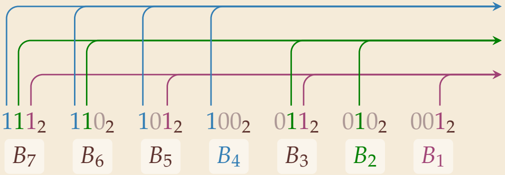

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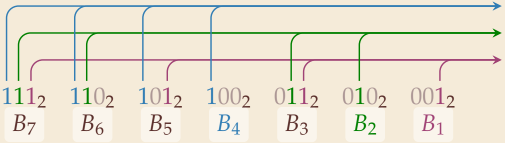
$\rightsquigarrow$ C *encodes* ***position of error!***

(7, 4) Hamming Code

- *How can we turn this into a code?*



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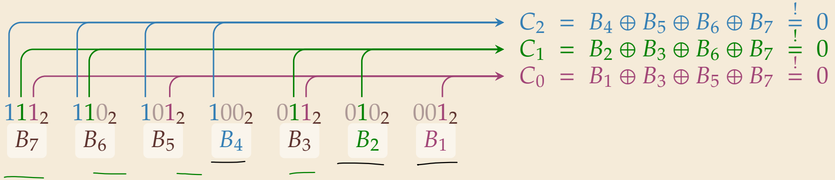

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- **(7,4) *Hamming Code* – Encoding**
  1. **Given:** message $D_3D_2D_1D_0$ of length $k = 4$

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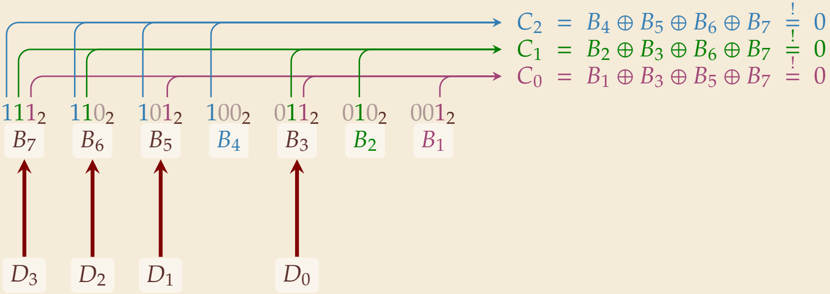

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- **(7, 4)** ***Hamming Code*** **– Encoding**
  1. **Given:** message $D_3D_2D_1D_0$ of length $k = 4$
  2. copy $D_3D_2D_1D_0$ to $B_7B_6B_5B_3$

$(7,4)$ Hamming Code

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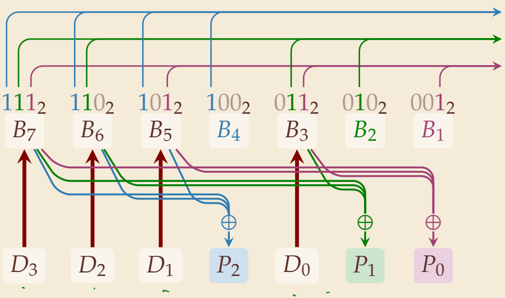

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- B_4, B_2 and B_1 occur only in one constraint each $\rightsquigarrow$ **define** them based on rest!
- **$(7,4)$ *Hamming Code* – Encoding**
  1. **Given:** message $D_3D_2D_1D_0$ of length $k = 4$
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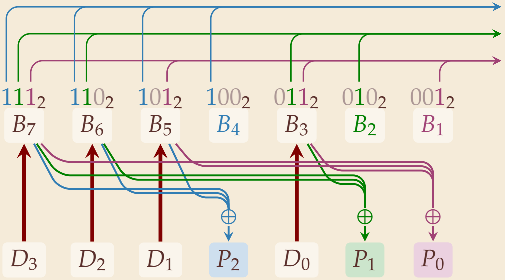
(7, 4) Hamming Code

- *How can we turn this into a code?*



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  4. send $D_3D_2D_1P_2D_0P_1P_0$

(7, 4) Hamming Code – Decoding

- **(7, 4) *Hamming Code* – Decoding**
  1. **Given:** block $B_7B_6B_5B_4B_3B_2B_1$ of length $n = 7$
  2. compute C (as above)
  3. if $C = 0$ no (detectable) error occurred
     otherwise, flip B_C (the Cth bit was twisted)
  4. return 4-bit message $B_7B_6B_5B_3$

Clicker Question

What is the code distance of $(7, 4)$ Hamming code?

- **A** 0
- **B** 1
- **C** 2
- **D** 3
- **E** 4
- **F** 5
- **G** 6
- **H** ≥ 7

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(7, 4) Hamming Code – Properties

$$2^n \geq 2^k \sum_{f=0}^{\lfloor \frac{d-1}{2} \rfloor} \binom{n}{f}$$

$n = 7$

$k = 4$

$d = 3$

- **Hamming bound:**
  - 2^4 valid 7-bit codewords (on per message)
  - any of the 7 single-bit errors corrected towards valid codeword
  - ⇝ each codeword covers 8 of all possible 7-bit strings
  - $2^4 \cdot 2^3 = 2^7$ ⇝ exactly cover space of 7-bit strings

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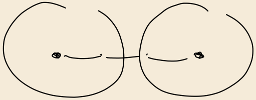

- distance $d = 3$
- can *correct* any 1-bit error
- How about 2-bit errors?
  - We can *detect* that *something* went wrong.
  - **But:** above decoder mistakes it for a (different!) 1-bit error and “corrects” that
  - Variant: store one additional parity bit for entire block
  - ⇝ Can *detect* any 2-bit error, but *not correct* it.

Hamming Codes – General recipe

- construction can be generalized:
  - Start with $n = 2^\ell - 1$ bits for $\ell \in \mathbb{N}$ (we had $\ell = 3$)
  - use the ℓ bits whose index is a power of 2 as parity bits
  - the other $n - \ell$ are data bits

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👍 simple and efficient coding / decoding

👍 fairly space-efficient

Outlook

- Indeed: $(2^\ell - 1,\ 2^\ell - \ell - 1)$ Hamming Code is *"perfect"* code

= matches Hamming lower bound

$\rightsquigarrow$ cannot use fewer bits . . .

  - **if** message length is $2^\ell - \ell - 1$ for $\ell \in \mathbb{N}_{\geq 2}$
    i. e., one of $1, 4, 11, 26, 57, 120, 247, 502, 1013, \ldots$
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- For other scenarios, finding good codes is an active research area
  - information theory predicts that *almost all* randomly chosen codes are good(!)
  - but these are inefficient to decode

  $\rightsquigarrow$ clever tricks and constructions needed