9 Range-Minimum Queries

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COMP526 (Fall 2023)
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Learning Outcomes

1. Know the *RMQ problem* and its *connection* to longest common extensions in strings.
2. Know and understand trivial RMQ solutions and *sparse tables*.
3. Know and understand the *Cartesian trees* data structure.
4. Know and understand the *exhaustive-tabulation technique* for RMQ with linear-time preprocessing.

Unit 9: *Range-Minimum Queries*

Outline

9 Range-Minimum Queries

9.1 Introduction

Range-minimum queries (RMQ)

array / numbers don't change

- **Given:** Static array $A[0..n)$ of numbers
- **Goal:** Find minimum in a range;
  A known in advance and can be preprocessed

$\mathrm{RMQ}(7, 15) = 10$

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	4	6	4	7	10	5	6	3	11	2	2	3	6	10	9	13	4	6	16	10

- **Nitpicks:**
  - Report *index* of minimum, not its value
  - Report *leftmost* position in case of ties

Clicker Question

Given the array from the slides, what is $\mathrm{RMQ}_A(1,6)$ = 1

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	4	6	4	7	10	5	6	3	11	2	2	3	6	10	9	13	4	6	16	10

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Rules of the Game

- comparison-based $\rightsquigarrow$ values don't matter, only relative order
- Two main quantities of interest:
  1. **Preprocessing time**: Running time $P(n)$ of the preprocessing step $\rightsquigarrow$ space usage $\le P(n)$
  2. **Query time**: Running time $Q(n)$ of one query (using precomputed data)
- Write $\langle P(n), Q(n) \rangle$ **time solution** for short

Clicker Question

What do you think, what running times can we achieve? For a $\langle P(n), Q(n) \rangle$ time solution, enter "<P(n),Q(n)>".

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9.2 RMQ, LCP, LCE, LCA — WTF?

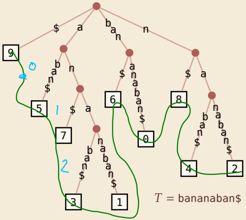
Recall Unit 8

Application 4: Longest Common Extensions

- We implicitly used a special case of a more general, versatile idea:

  The ***longest common extension (LCE)*** data structure:
  - **Given:** String $T[0..n-1]$
  - **Goal:** Answer LCE queries, i. e.,
    given positions i, j in T,
    how far can we read the same text from there?
    formally: $\mathrm{LCE}(i, j) = \max\{\ell : T[i..i+\ell) = T[j..j+\ell)\}$

$\rightsquigarrow$ use suffix tree of T!



T = bananaban$

longest common prefix of ith and jth suffix

- In $\mathcal{T}$: $\mathrm{LCE}(i, j) = \mathrm{LCP}(T_i, T_j)$ $\rightsquigarrow$ same thing, different name!
  $=$ string depth of *lowest common ancester (LCA)* of leaves $\boxed{i}$ and $\boxed{j}$

- in short: $\boxed{\mathrm{LCE}(i, j) = \mathrm{LCP}(T_i, T_j) = \mathrm{stringDepth}\big(\mathrm{LCA}(\boxed{i}, \boxed{j})\big)}$

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Recall Unit 8

Efficient LCA

How to find lowest common ancestors?

- Could walk up the tree to find LCA $\rightsquigarrow$ $\Theta(n)$ worst case
- Could store all LCAs in big table $\rightsquigarrow$ $\Theta(n^2)$ space and preprocessing

Amazing result: Can compute data structure in $\Theta(n)$ time and space that finds any LCA is **constant(!) time**.

- a bit tricky to understand
- but a theoretical breakthrough
- and useful in practice

$< \Theta(n), \Theta(1) >$

and suffix tree construction inside . . .

$\rightsquigarrow$ for now, use $O(1)$ LCA as black box.

$\rightsquigarrow$ After linear preprocessing (time & space), we can find LCEs in $O(1)$ time.

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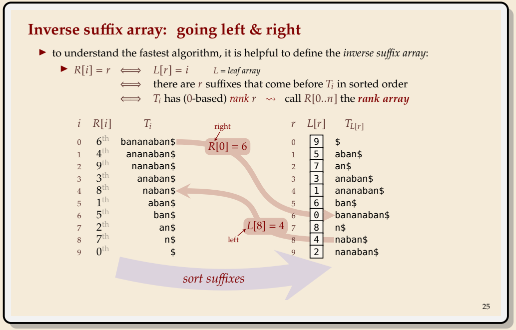
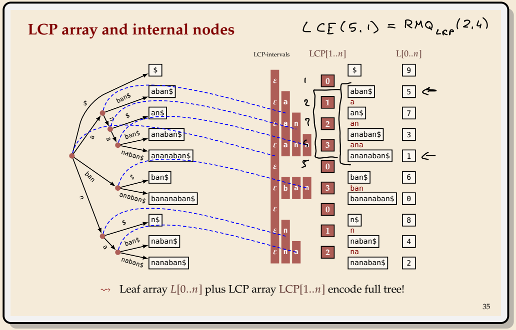
Finally: Longest common extensions

- In Unit 8: Left question open how to compute LCA in suffix trees
- But: Enhanced Suffix Array makes life easier!

$$\boxed{\mathrm{LCE}(i,j) \;=\; \mathrm{LCP}\Big[\mathrm{RMQ}_{\mathrm{LCP}}\big(\min\{R[i],R[j]\}+1,\ \max\{R[i],R[j]\}\big)\Big]}$$





RMQ Implications for LCE

- Recall: Can compute (inverse) suffix array and LCP array in $O(n)$ time

$\rightsquigarrow$ A $\langle P(n), Q(n)\rangle$ time RMQ data structure implies a $\langle P(n), Q(n)\rangle$ time solution for longest-common extensions

(given rank array & LCP array)

$\rightsquigarrow$ want $\langle \Theta(n), \Theta(1)\rangle$

9.3 Trivial Solutions & Sparse Tables

Trivial Solutions

RMQ$(7, 15) = 10$

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
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- Two easy solutions show extreme ends of scale:

Trivial Solutions

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- Two easy solutions show extreme ends of scale:

1. **Scan on demand**
   - no preprocessing at all
   - answer $\mathrm{RMQ}(i, j)$ by scanning through $A[i..j]$, keeping track of min

   $\rightsquigarrow \langle O(1), O(n) \rangle$

Trivial Solutions

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2. **Precompute all**
   - Precompute all answers in a big 2D array $M[0..n)[0..n)$
   - queries simple: $\mathrm{RMQ}(i, j) = M[i][j]$

   $\rightsquigarrow \langle O(n^3), O(1)\rangle$

```
for i = 0...n-1
    for j = i,...,n-1
        a = i
        for k := i...j
            if A[a] > A[k]
                a = k
        M[i][j] = a
```

Trivial Solutions

RMQ(7, 15) = 10

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$M[i][j] \rightsquigarrow M[i][j+1]$

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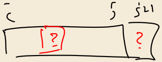
   $\rightsquigarrow \langle O(n^3), O(1) \rangle$
   - Preprocessing can reuse partial results $\rightsquigarrow \langle O(n^2), O(1) \rangle$



for $\ell := 1, \ldots, n$
for $i = 0, \ldots, n-\ell$
if $\ell == 1$: $M[i][i+\ell] = i$
else
$a = M[i][i+\ell-1]$
$b = j+1$
if $A[a] \le A[b]$
$M[i][i+\ell] = a$
else
$M[i][i+\ell] = b$

Sparse Table

- **Idea:** Like "precompute-all", but keep only some entries
- store $M[i][j]$ iff $\ell = j - i + 1$ is 2^k.
  - $\rightsquigarrow$ $\leq n \cdot \lg n$ entries
  - $\rightsquigarrow$ Can be stored as $M'[i][k]$

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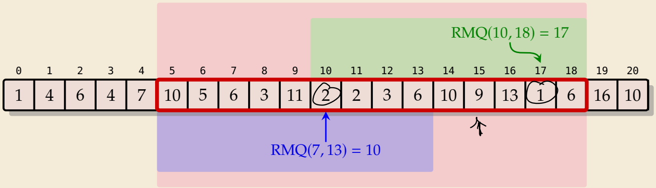

1. Find k with $\ell/2 \le 2^k \le \ell$
2. Cover range $[i..j]$ by
   2^k positions right from i and
   2^k positions left from j
3. $\mathrm{RMQ}(i,j) =$
   $\arg\min\{A[rmq_1], A[rmq_2]\}$

with $rmq_1 = \mathrm{RMQ}(i, i + 2^k - 1) = M'[i][k]$
$rmq_2 = \mathrm{RMQ}(j - 2^k + 1, j) = M'[j-2^k+1][k]$

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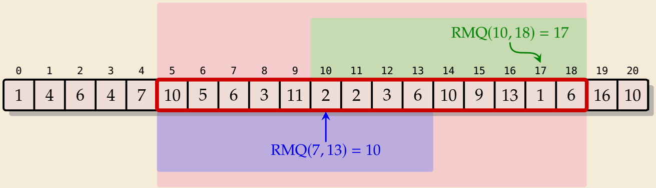

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   with $rmq_1 = \mathrm{RMQ}(i, i + 2^k - 1)$
   $rmq_2 = \mathrm{RMQ}(j - 2^k + 1, j)$

- Preprocessing can be done in $O(n \log n)$ times

$\rightsquigarrow$ $\langle O(n \log n), O(1) \rangle$ time solution!

$[i, i+2^k-1] \rightsquigarrow [i, i+2^{k+1}-1]$

9.4 Cartesian Trees

RMQ & LCA

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
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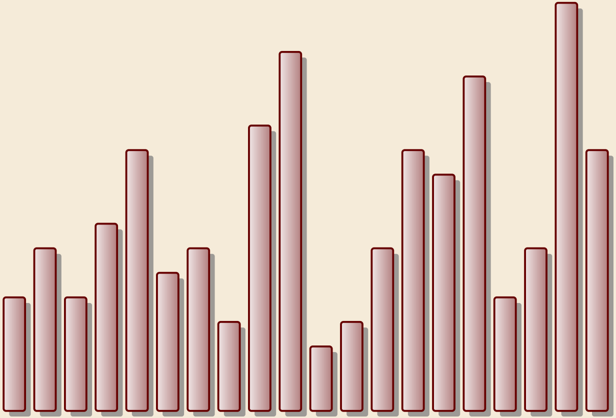
RMQ & LCA

range-maximum query



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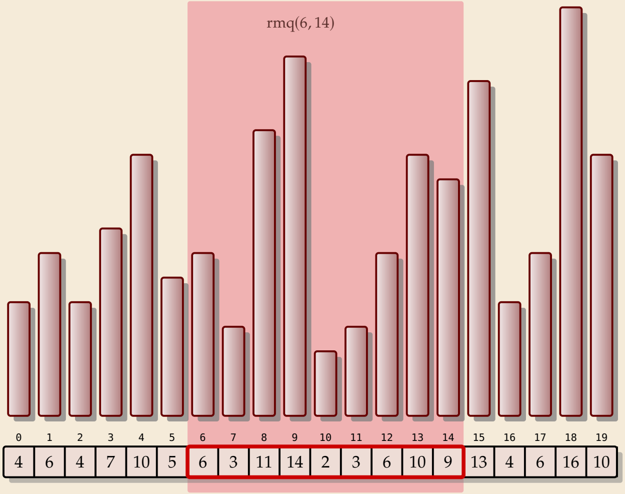
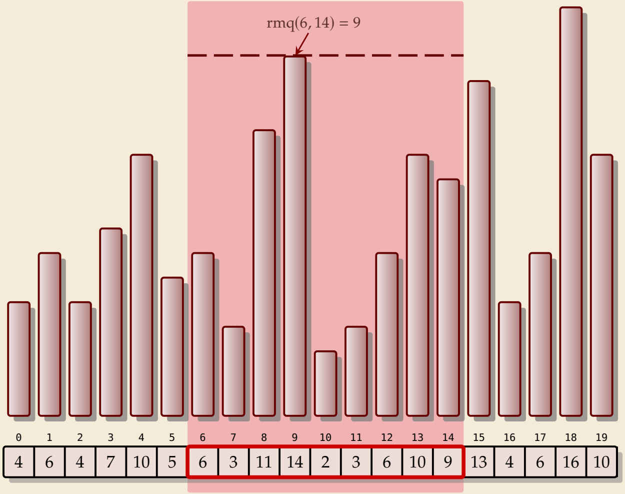
RMQ & LCA



- **Range-max queries** on array A:
  $$\mathrm{rmq}_A(i,j) = \arg\max_{i \le k \le j} A[k]$$
  $$= \textit{index} \text{ of max}$$

RMQ & LCA



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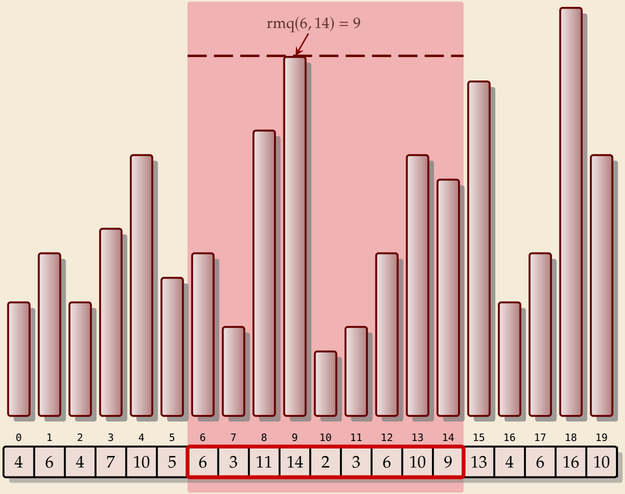

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- **Task:** Preprocess A,
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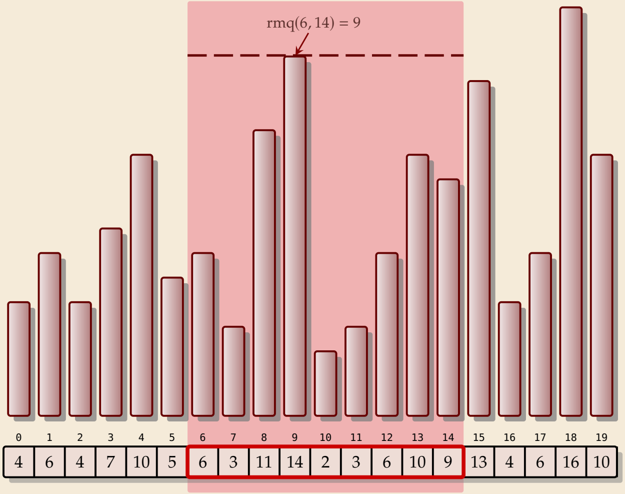
RMQ & LCA



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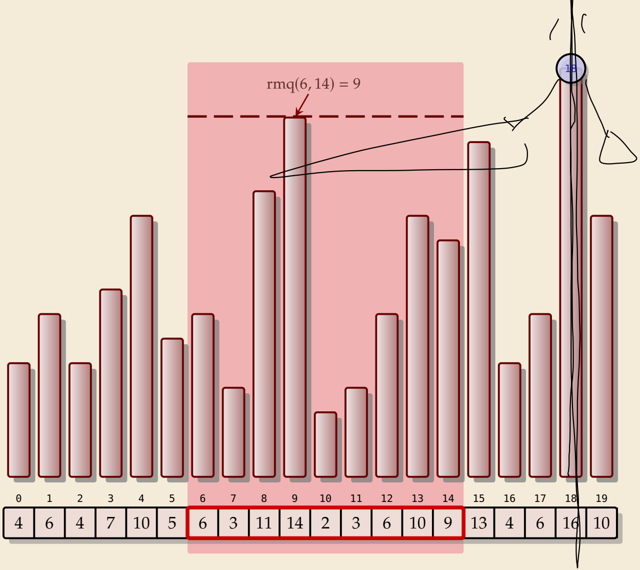
RMQ & LCA



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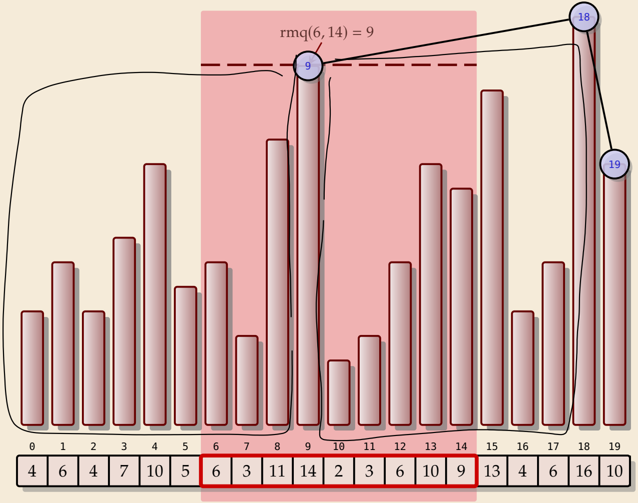
- **Cartesian tree:** (cf. *treap*) construct binary tree by sweeping line down

RMQ & LCA



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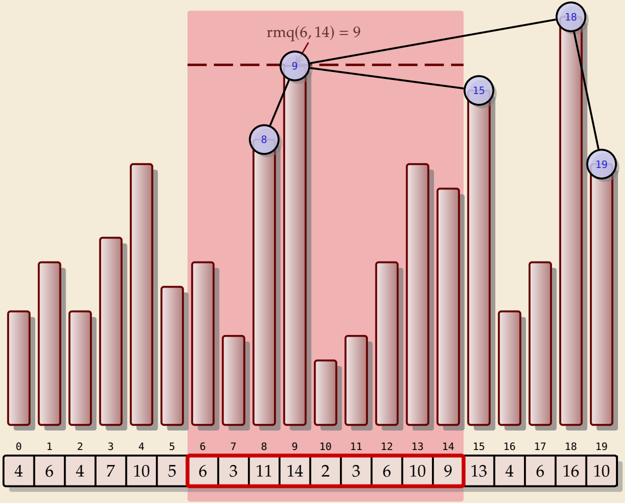

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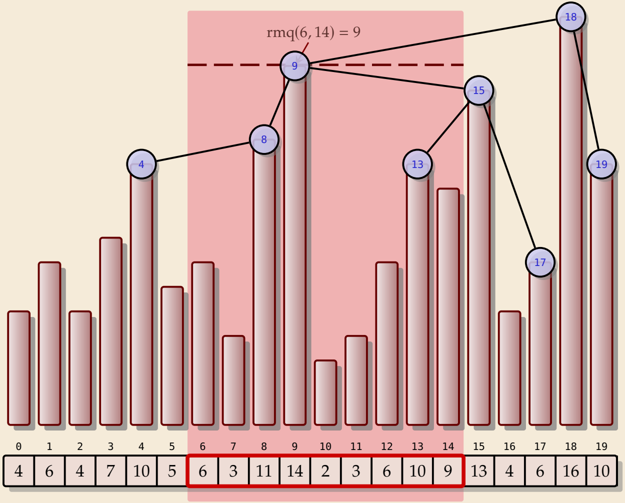

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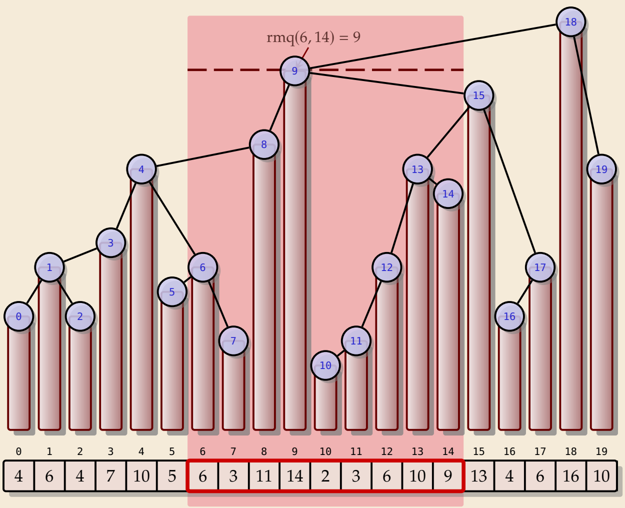

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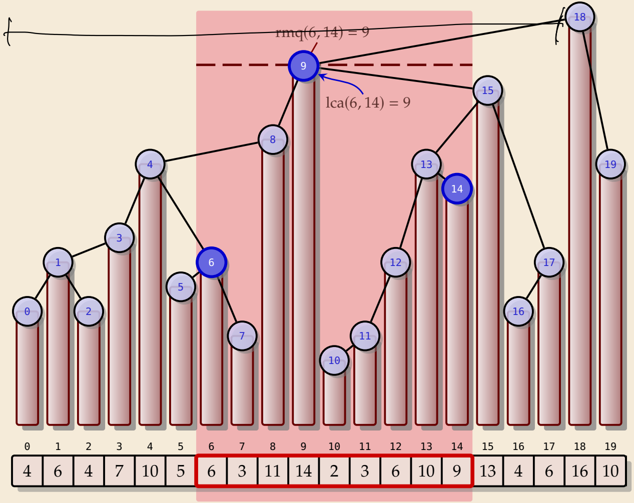

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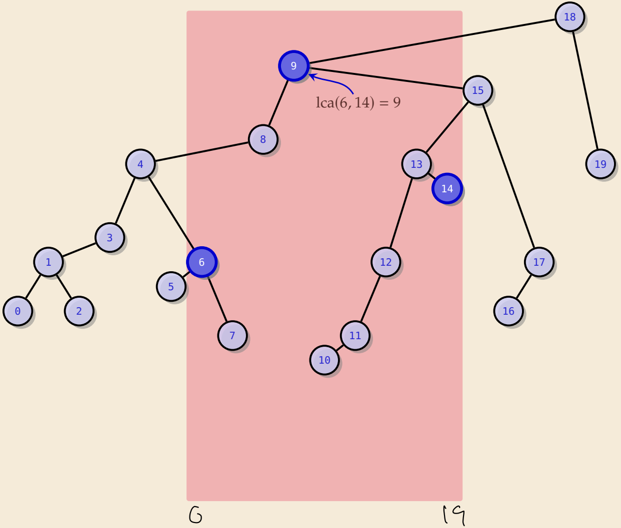
- $\mathrm{rmq}(i,j)$ = **l**owest **c**ommon **a**ncestor (LCA)

RMQ & LCA



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- $\text{rmq}(i, j)$ =
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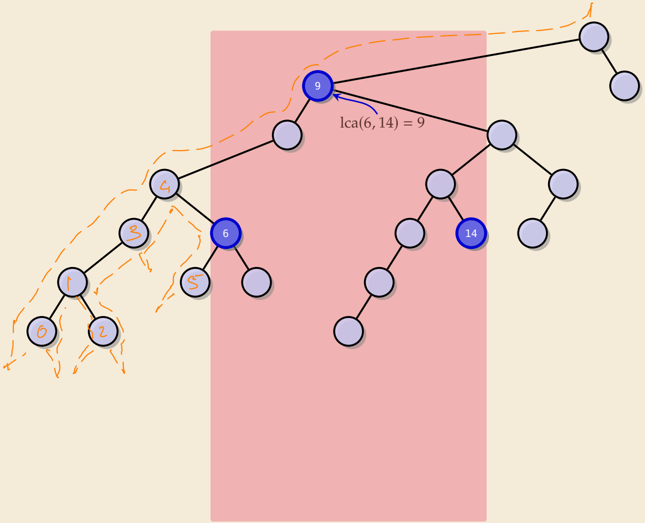
RMQ & LCA

inorder traversal: left - self - right



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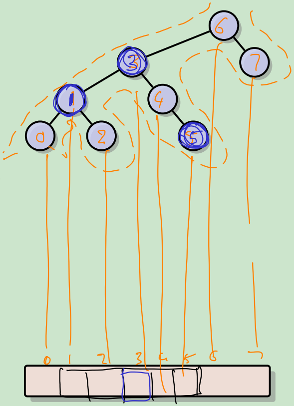
- $\mathrm{rmq}(i,j)$ = inorder of **lowest common ancestor** (LCA) of ith and jth node in inorder

Clicker Question



Given the (max-oriented) Cartesian tree for A on the left, what is $\mathrm{RMQ}_A(1,5)$?

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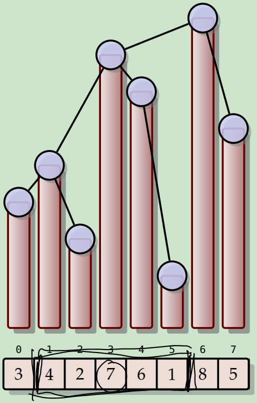
Clicker Question



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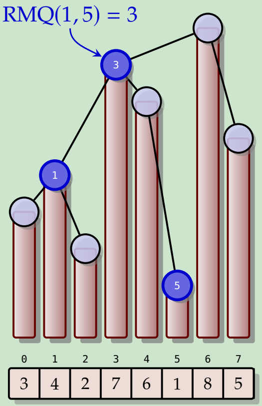

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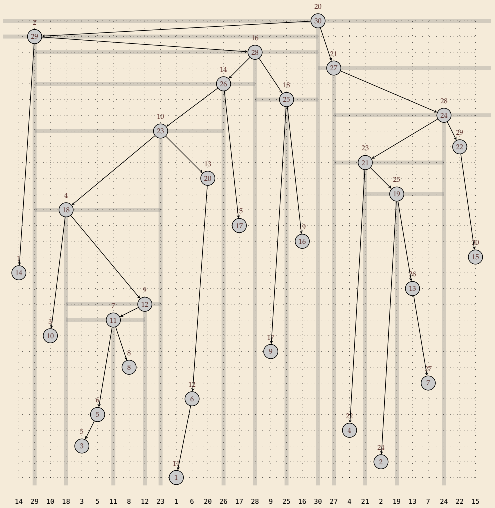
Cartesian Tree – Larger Example



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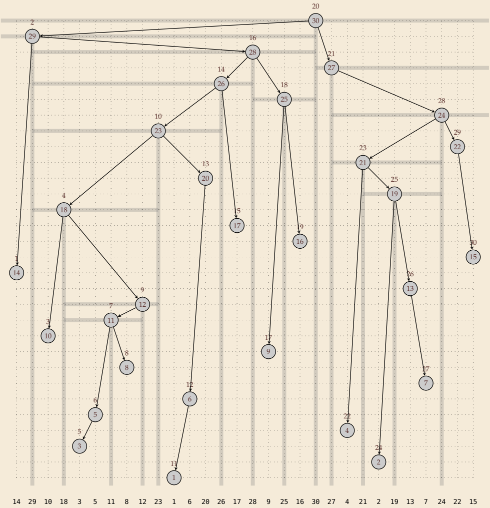

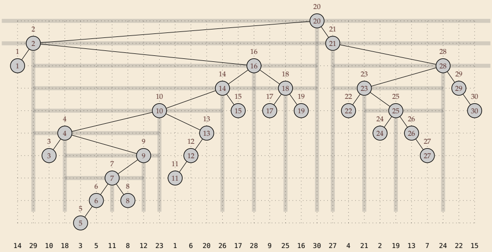

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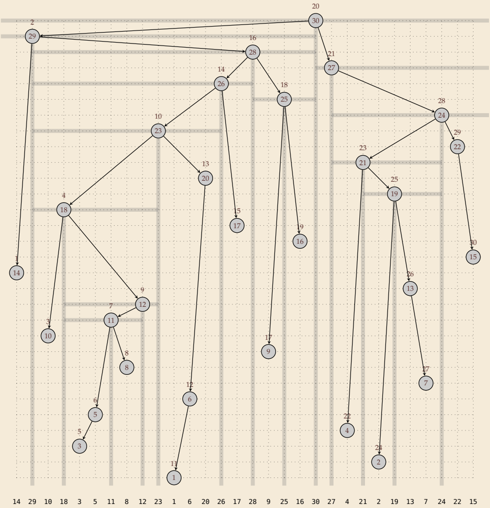
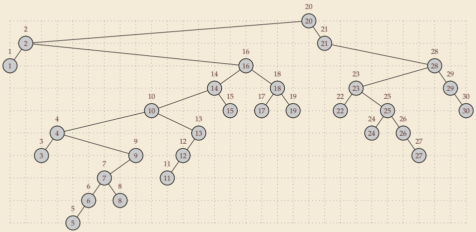
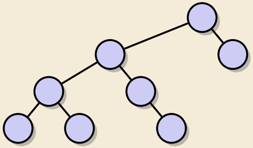




Counting binary trees



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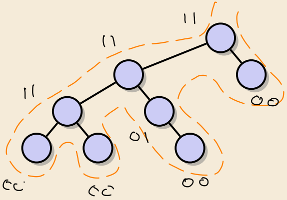

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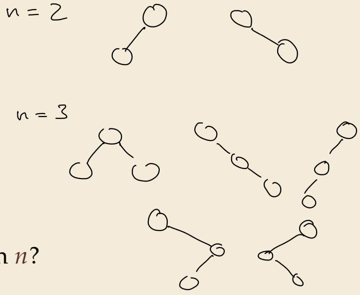
  and vice versa!



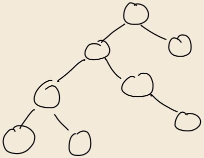
11 11 11 00 00 01 00 00

- How many different Cartesian trees are there for arrays of length n?
  - known result: *Catalan numbers* $\frac{1}{n+1}\binom{2n}{n}$
  - easy to see: $\leq 2^{2n}$ can encode binary tree with $2n$

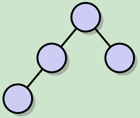
$\rightsquigarrow$ many arrays will give rise to the same Cartesian tree

Can we exploit that?

11 11 1100 00 01 00 00



Clicker Question



What binary string corresponds to the tree shown on the left?

(using the encoding just discussed)

→ sli.do/comp526

9.5 Exhaustive Tabulation

Four Russians?

The exhaustive-tabulation technique to follow is often called "Four Russians trick" . . .

- The algorithmic technique was published 1970 by
  V. L. Arlazarov, E. A. Dinitz, M. A. Kronrod, and I. A. Faradžev
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- American authors coined the slightly derogatory “Method of Four Russians” . . . name in widespread use

Bootstrapping

- We know a $\langle O(n \log n), O(1) \rangle$ time solution
- If we use that for $m = \Theta(n/\log n)$ elements, $O(m \log m) = O(n)$!

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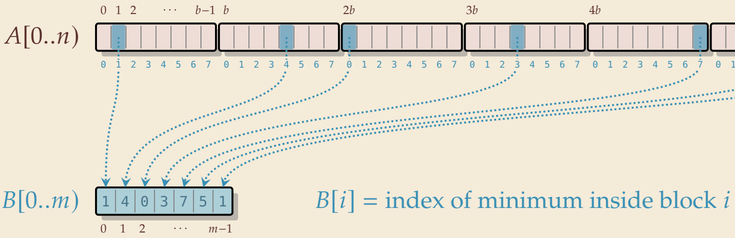

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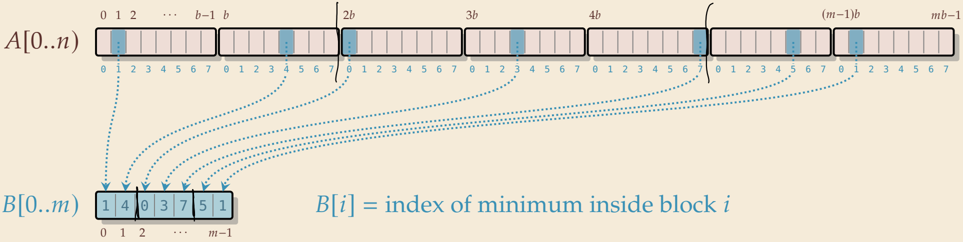


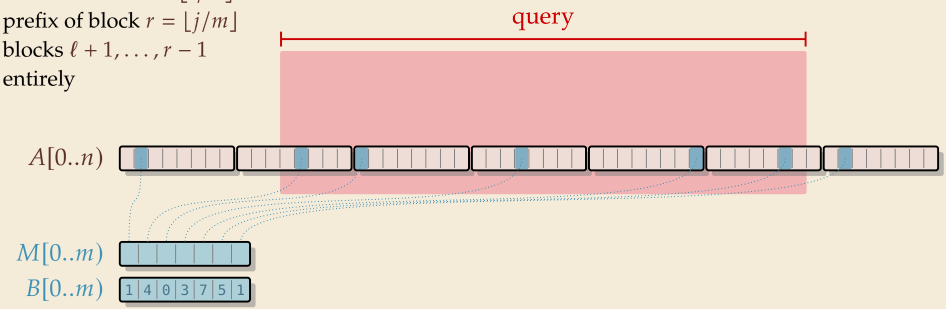
$B[i]$ = index of minimum inside block i

$\rightsquigarrow$ Use sparse tables for B.

$\rightsquigarrow$ Can solve RMQs in $B[0..m)$ in $\langle O(n), O(1) \rangle$ time

Query decomposition

- Query $\mathrm{RMQ}_A(i, j)$ covers
  - suffix of block $\ell = \lfloor i/m \rfloor$
  - prefix of block $r = \lfloor j/m \rfloor$
  - blocks $\ell + 1, \ldots, r - 1$ entirely



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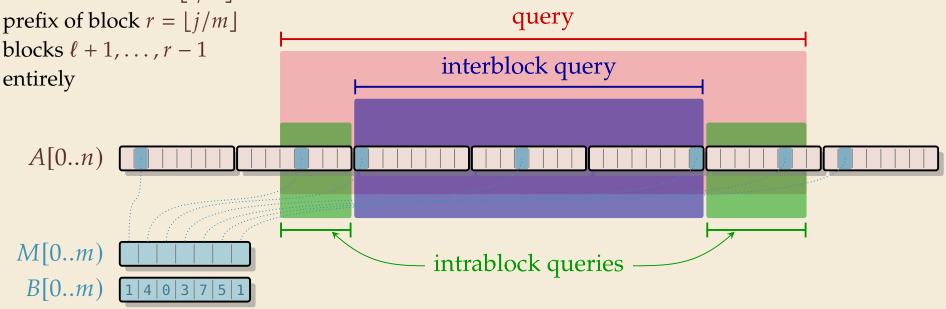

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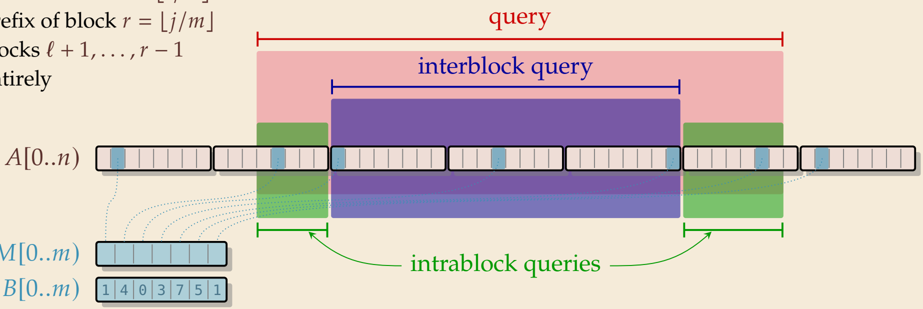

- $\mathrm{RMQ}_A(i,j) = \underset{k\in K}{\arg\min} A[k]$ with $K = \left\{ \begin{array}{l} \mathrm{RMQ}_{\text{block } \ell}\big(i-\ell b,\ (\ell+1)b-1\big), \\ b \cdot \mathrm{RMQ}_M\big(\ell+1, r-1\big) + \\ \qquad B\big[\mathrm{RMQ}_M\big(\ell+1, r-1\big)\big], \\ \mathrm{RMQ}_{\text{block } r}\big(rb,\ j-rb\big) \end{array} \right\}$

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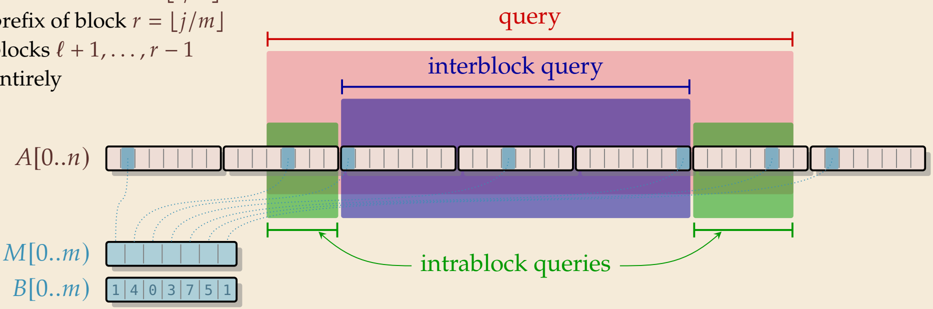

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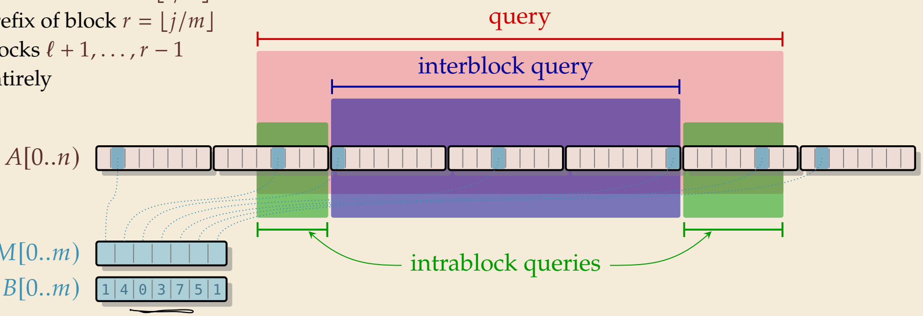

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Intrablock queries [1]

$\rightsquigarrow$ It remains to solve the intrablock queries!

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  must include preprocessing for all $m = \left\lceil \frac{n}{b} \right\rceil = \Theta\left(\frac{n}{\log n}\right)$ blocks!

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- $\rightsquigarrow$ ***Exhaustive Tabulation Technique:***
  1. represent each subproblem by storing its *type* (here: encoding of Cartesian tree)
  2. *enumerate* all possible subproblem types and their solutions
  3. use type as index in a large *lookup table*

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   - can be done in linear time

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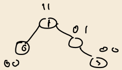
$b = 4$



Block type	i	j	$\mathrm{RMQ}(i, j)$
⋮			
11000100	0	3	1
"	1	3	1
"	2	3	2
⋮			

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   - can be done in linear time
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Block type	i	j	$\mathrm{RMQ}(i,j)$
$\vdots$			
$\vdots$			

- $\le \sqrt{n}$ block types
- $\le b^2$ combinations for i and j

$\rightsquigarrow$ $\Theta\left(\sqrt{n} \cdot \log^2 n\right)$ rows

- each row can be computed in $O(\log n)$ time

$\rightsquigarrow$ overall preprocessing: $O(n)$ time!

Discussion

- $\langle O(n), O(1) \rangle$ time solution for RMQ

$\leadsto$ $\langle O(n), O(1) \rangle$ time solution for LCE in strings!

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Research questions:

- Reduce the space usage
- Avoid access to A at query time