3 Efficient Sorting

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Outline

Why study sorting?

- fundamental problem of computer science that is still not solved

Algorithm with optimal #comparisons in worst case?

- building brick of many more advanced algorithms
  - for preprocessing
  - as subroutine
- playground of manageable complexity to practice algorithmic techniques

Here:

- "classic" fast sorting method
- **parallel** sorting

Part I

The Basics

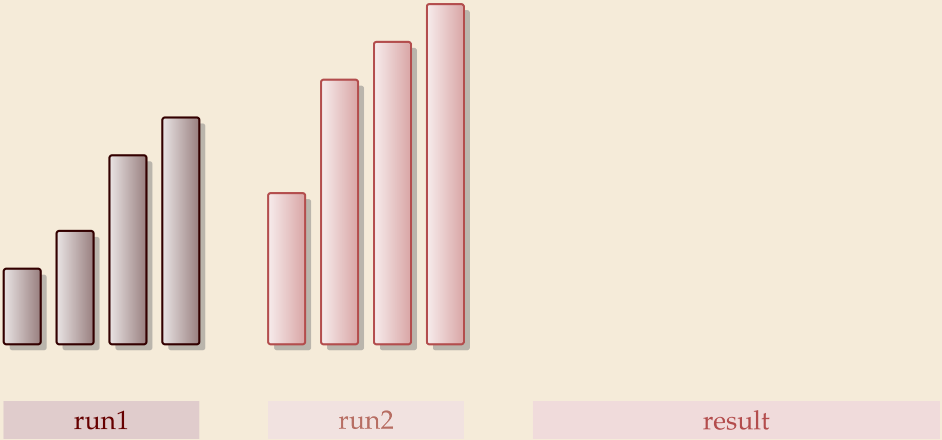
Rules of the game

- **Given:**
  - array $A[0..n-1]$ of n objects
  - a total order relation $\leq$ among $A[0], \ldots, A[n-1]$
    (a comparison function)
- **Goal:** rearrange (=permute) elements within A,
  so that A is *sorted*, i. e., $A[0] \leq A[1] \leq \cdots \leq A[n-1]$

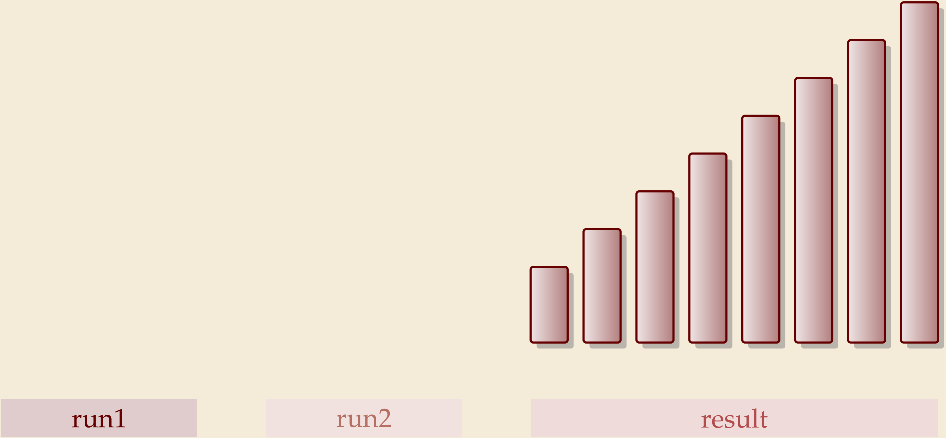
- for now: A stored in main memory (*internal sorting*)
  single processor (*sequential sorting*)

3.1 Mergesort

Merging sorted lists



Merging sorted lists



Mergesort

procedure mergesort($A[l..r]$)
  $n := r - l + 1$
  if $n \geq 1$ **return**
  $m := l + \left\lfloor \frac{n}{2} \right\rfloor$
  mergesort($A[l..m-1]$)
  mergesort($A[m..r]$)
  merge($A[l..m-1]$, $A[m..r]$, buf)
  copy buf to $A[l..r]$

- recursive procedure; *divide & conquer*
- merging needs
  - temporary storage for result of same size as merged runs
  - to read and write each element twice (once for merging, once for copying back)

Analysis: count "element visits" (read and/or write)

$$C(n) = \begin{cases} 0 & n \leq 1 \\ C(\lfloor n/2 \rfloor) + C(\lceil n/2 \rceil) + 2n & n \geq 2 \end{cases}$$

same for best and worst case!

Simplification $\boxed{n = 2^k}$

$$C(2^k) = \begin{cases} 0 & k \leq 0 \\ 2 \cdot C(2^{k-1}) + 2 \cdot 2^k & k \geq 1 \end{cases} = 2 \cdot 2^k + 2^2 \cdot 2^{k-1} + 2^3 \cdot 2^{k-2} + \cdots + 2^k \cdot 2^1 = 2k \cdot 2^k$$

$$C(n) = 2n \lg(n) = \Theta(n \log n)$$

Mergesort – Discussion

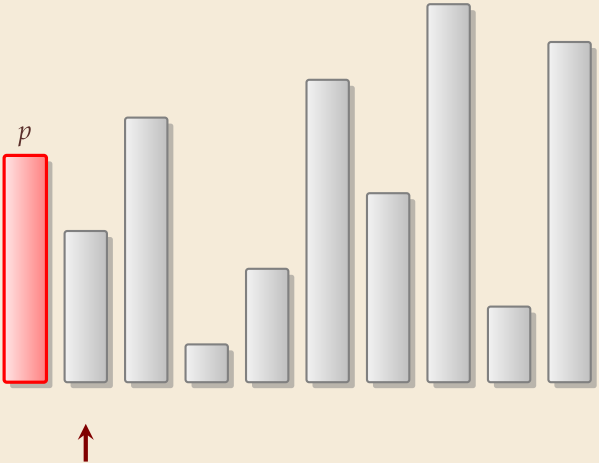
- optimal time complexity of $\Theta(n \log n)$ in the worst case
- *stable* sorting method   i. e., retains relative order of equal-key items
- memory access is sequential (scans over arrays)

- requires $\Theta(n)$ extra space

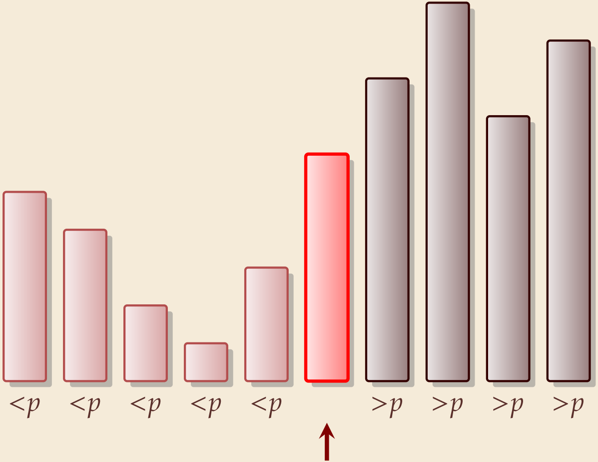
  there are in-place merging methods,
  but they are substantially more complicated
  and not (widely) used

3.2 Quicksort

Partitioning around a pivot



Partitioning around a pivot



- no extra space needed
- visits each element once
- returns rank/position of pivot

Partitioning – Detailed code

Beware: details easy to get wrong; use this code!

```
1  procedure partition(A, b)
2      // input: array A[0..n − 1], position of pivot b ∈ [0..n − 1]
3      swap(A[0], A[b])
4      i := 0,    j := n
5      while true do
6          do i := i + 1 while i < n and A[i] < A[0]
7          do j := j − 1 while j ≥ 1 and A[j] > A[0]
8          if i ≥ j then break     (goto 8)
9          else swap(A[i], A[j])
10     end while
11     swap(A[0], A[j])
12     return j
```

Loop invariant (5–10): A

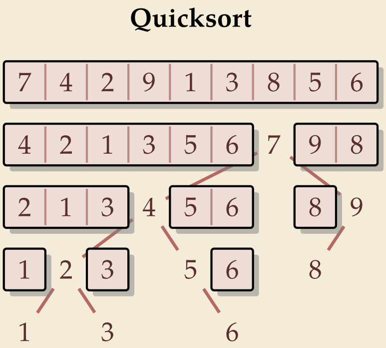
p	$\leq p$	?	$\geq p$
		i	j

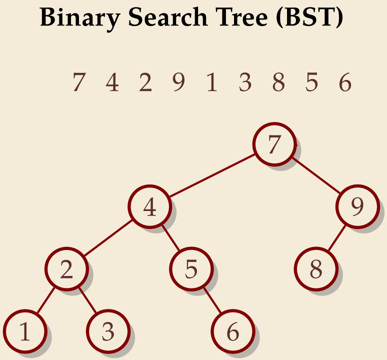
Quicksort

```
procedure quicksort(A[l..r])
    if l ≥ r then return
    b := choosePivot(A[l..r])
    j := partition(A[l..r], b)
    quicksort(A[l..j − 1])
    quicksort(A[j + 1..r])
```

- recursive procedure; *divide & conquer*
- choice of pivot can be
  - fixed position $\rightsquigarrow$ dangerous!
  - random
  - more sophisticated, e. g., median of 3

Quicksort & Binary Search Trees



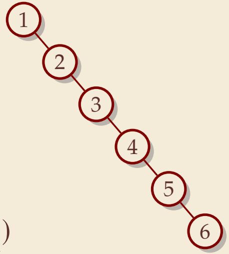


- recursion tree of quicksort = binary search tree from successive insertion
- comparisons in quicksort = comparisons to built BST
- comparisons in quicksort $\approx$ comparisons to search each element in BST

Quicksort – Worst Case

- Problem: BSTs can degenerate
- Cost to search for k is $k - 1$

⇝ Total cost $\sum_{k=1}^{n}(k-1) = \dfrac{n(n-1)}{2} \sim \frac{1}{2}n^2$



⇝ quicksort worst-case running time is in $\Theta(n^2)$

terribly slow!

But, we can fix this:

Randomized quicksort:

- choose a *random pivot* in each step

⇝ same as randomly *shuffling* input before sorting

Randomized Quicksort – Analysis

- $C(n)$ element visits (as for mergesort)

$\rightsquigarrow$ quicksort needs $\sim 2\ln(2) \cdot n \lg n \approx 1.39 n \lg n$ *in expectation*

- also: very unlikely to be much worse:
  e. g., one can prove: $\Pr[\text{cost} > 10 n \lg n] \;=\; O(n^{-2.5})$
  distribution of costs is "concentrated around mean"
- intuition: have to be constantly unlucky with pivot choice

Quicksort – Discussion

- 👍 fastest general-purpose method
- 👍 $\Theta(n \log n)$ average case
- 👍 works *in-place* (no extra space required)
- 👍 memory access is sequential (scans over arrays)
- 👎 $\Theta(n^2)$ worst case (although extremely unlikely)
- 👎 not a *stable* sorting method

Open problem: Simple algorithm that is stable in-place.

3.3 Comparison-Based Lower Bound

Lower Bounds

- **Lower bound:** mathematical proof that no algorithm can do better.
  - very powerful concept: bulletproof *impossibility* result
    $\approx$ *conservation of energy* in physics
  - **(unique?) feature of computer science:**
    for many problems, solutions are known that (asymptotically) *achieve the lower bound*

  $\rightsquigarrow$ can speak of "*optimal* algorithms"

- To prove a statement about *all algorithms*, we must precisely define what that is!
- already know one option: the word-RAM model
- Here: use a simpler, more restricted model.

The Comparison Model

- In the *comparison model* data can only be accessed in two ways:
  - comparing two elements
  - moving elements around (e. g. copying, swapping)
  - Cost: number of these operations.

That's good!
Keeps algorithms general!

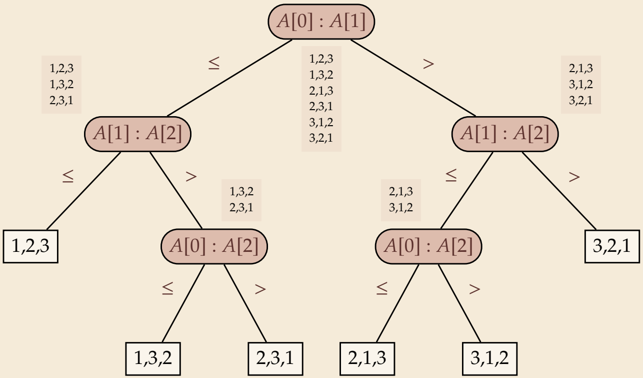
- This makes very few assumptions on the kind of objects we are sorting.

- Mergesort and Quicksort work in the comparison model.

$\rightsquigarrow$ Every comparison-based sorting algorithm corresponds to a ***decision tree***.
  - only model comparisons $\rightsquigarrow$ ignore data movement
  - nodes = comparisons the algorithm does
  - next comparisons can depend on outcomes $\rightsquigarrow$ different subtrees
  - child links = outcomes of comparison
  - leaf = unique initial input permutation compatible with comparison outcomes

Comparison Lower Bound

Example: Comparison tree for a sorting method for $A[0..2]$:



- Execution = follow a path in comparison tree.

$\rightsquigarrow$ height of comparison tree = worst-case # comparisons

- comparison trees are *binary* trees

$\rightsquigarrow$ ℓ leaves $\rightsquigarrow$ height $\geq \lceil \lg(\ell) \rceil$

- comparison trees for sorting method must have $\geq n!$ leaves

$\rightsquigarrow$ height $\geq \lg(n!) \sim n \lg n$

more precisely: $\lg(n!) = n \lg n - \lg(e)n + O(\log n)$

- Mergesort achieves $\sim n \lg n$ comparisons $\rightsquigarrow$ asymptotically comparison-optimal!
- Open (theory) problem: Can we sort with $n \lg n - \lg(e)n + o(n)$ comparisons?

≈ 1.4427

3.4 Integer Sorting

How to beat a lower bound

- Does the above lower bound mean, sorting always takes time $\Omega(n \log n)$?

- **Not necessarily;** only in the *comparison model!*
  $\rightsquigarrow$ Lower bounds show where to change the model!

- Here: sort n **integers**
  - can do *a lot* with integers: add them up, compute averages, . . . (full power of word-RAM)

  $\rightsquigarrow$ we are **not** working in the comparison model

  $\rightsquigarrow$ *above lower bound does not apply!*

  - but: a priori unclear how much arithmetic helps for sorting . . .

Counting sort

- Important parameter: size/range of numbers
  - numbers in range $[0..U) = \{0, \ldots, U-1\}$ typically $U = 2^b$ $\rightsquigarrow$ b-bit binary numbers
- We can sort n integers in $\Theta(n+U)$ time and $\Theta(U)$ space when $\boxed{b \le w}$

word size

Counting sort

```
procedure countingSort(A[0..n-1])
    // A contains integers in range [0..U).
    C[0..U-1] := new integer array, initialized to 0
    // Count occurrences
    for i := 0, ..., n-1
        C[A[i]] := C[A[i]] + 1
    i := 0 // Produce sorted list
    for k := 0, ... U-1
        for j := 1, ... C[k]
            A[i] := k; i := i + 1
```

- *count* how often each *possible* value occurs
- produce sorted result directly from counts
- circumvents lower bound by using integers as array index / pointer offset

$\rightsquigarrow$ Can sort n integers in range $[0..U)$ with $U = O(n)$ in time and space $\Theta(n)$.

Integer Sorting – State of the art

- $O(n)$ time sorting also possible for numbers in range $U = O(n^c)$ for constant c.
  - *radix sort* with radix 2^w

- **algorithm theory**
  - suppose $U = 2^w$, but w can be arbitrary function of n
  - how fast can we sort n such w-bit integers on a w-bit word-RAM?
    - for $w = O(\log n)$: linear time (*radix/counting sort*)
    - for $w = \Omega(\log^{2+\varepsilon} n)$: linear time (*signature sort*)
    - for w in between: can do $O(n\sqrt{\lg \lg n})$ (very complicated algorithm)
      don't know if that is best possible!

* * *

- for the rest of this unit: back to the comparisons model!

Part II

Sorting with of many processors

3.5 Parallel computation

Types of parallel computation

£££ can't buy you more time, but more computers!

$\rightsquigarrow$ Challenge: Algorithms for parallel computation.

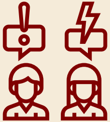
There are two main forms of parallelism

1. **shared-memory parallel computer** $\leftarrow$ *focus of today*
   - p processing elements (PEs, processors) working in parallel
   - single big memory, accessible from every PE
   - communication via shared memory
   - think: a big server, 128 CPU cores, terabyte of main memory
2. **distributed computing**
   - p PEs working in parallel
   - each PE has private memory
   - communication by sending messages via a network
   - think: a cluster of individual machines, supercomputers

PRAM – Parallel RAM

- extension of the RAM model (see unit 1)
- the p PEs are identified by ids $1, \ldots, p$
  - like w (the word size), p is a parameter of the model that can grow with n
  - $p = \Theta(n)$ is not unusual   maaany processors!
- the PEs all **independently** run a RAM-style program
  (they can use their id there)
- each PE has its own registers, but `MEM` is shared among all PEs
- computation runs in **synchronous** steps:
  in each time step, every PE executes one instruction

PRAM – Conflict management



Problem: What if several PEs simultaneously overwrite a memory cell?

- **EREW**-PRAM (exclusive read, exclusive write)
  any **parallel access** to same memory cell is **forbidden** (crash if happens)
- **CREW**-PRAM (concurrent read, exclusive write)
  parallel **write** access to same memory cell is *forbidden*, but reading is fine
- **CRCW**-PRAM (concurrent read, concurrent write)
  concurrent access is allowed,
  need a rule for write conflicts:
  - common CRCW-PRAM:
    all concurrent writes to same cell must write *same* value
  - arbitrary CRCW-PRAM:
    some unspecified concurrent write wins
  - (more exist . . .)

- no single model is always adequate, but our default is CREW

PRAM – Execution costs

Cost metrics in PRAMs

- **space:** total amount of accessed memory
- **time:** number of steps till all PEs finish   assuming sufficiently many PEs!
  sometimes called *depth* or *span*
- **work:** total #instructions executed on all PEs

Holy grail of PRAM algorithms:

- minimal time
- work (asymptotically) no worse than running time of best sequential algorithm
  - *work-efficient* algorithm: work in same Θ-class as best sequential

The number of processors

Hold on, my computer does not have $\Theta(n)$ processors! Why should I care for span and work!?

Theorem 3.1 (Brent's Theorem:)
If an algorithm has span T and work W (for an arbitrarily large number of processors), it can be run on a PRAM with p PEs in time $O(T + \frac{W}{p})$ (and using $O(W)$ work). ◀

Proof: schedule parallel steps in round-robin fashion on the p PEs.

$\rightsquigarrow$ span and work give guideline for *any* number of processors

3.6 Parallel primitives

Prefix sums

Before we come to parallel sorting, we study some useful building blocks.

Prefix-sum problem (also: cumulative sums, running totals)

- Given: array $A[0..n-1]$ of numbers
- Goal: compute all prefix sums $A[0] + \cdots + A[i]$ for $i = 0, \ldots, n-1$
  may be done "in-place", i. e., by overwriting A

Example:

input:	3	0	0	5	7	0	0	2	0	0	0	4	0	8	0	1
output:	3	3	3	8	15	15	15	17	17	17	17	21	21	29	29	30

Σ

Prefix sums – Sequential

- sequential solution does $n - 1$ additions
- but: cannot parallelize them
  data dependencies!

$\rightsquigarrow$ need a different approach

```
procedure prefixSum(A[0..n-1])
    for i := 1, ..., n-1 do
        A[i] := A[i-1] + A[i]
```

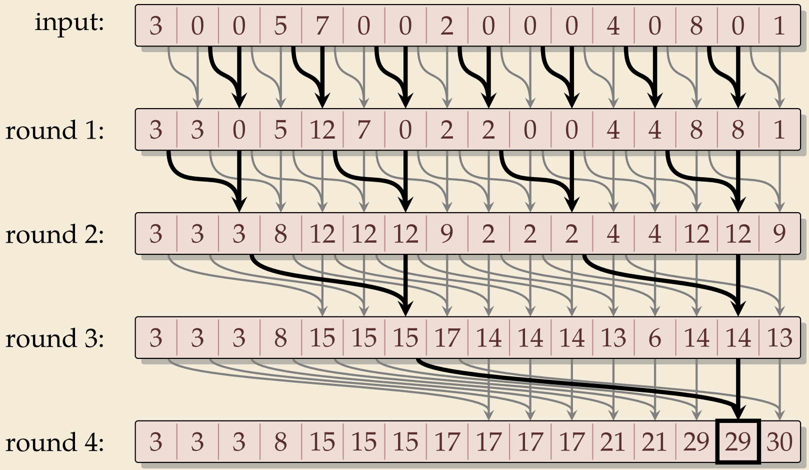
Let's try a simpler problem first.

Excursion: Sum

- Given: array $A[0..n-1]$ of numbers
- Goal: compute $A[0] + A[1] + \cdots + A[n-1]$
  (solved by prefix sums)

Parallel prefix sums

- Idea: Compute all prefix sums with balanced trees in parallel
  Remember partial results for reuse



Parallel prefix sums – Code

- can be realized in-place (overwriting A)
- assumption: in each parallel step, all reads precede all writes

```
procedure parallelPrefixSums(A[0..n − 1])
    for r := 1, . . . ⌈lg n⌉ do
        step := 2^(r−1)
        for i := step, . . . n − 1 do in parallel
            A[i] := A[i] + A[i − step]
        end parallel for
    end for
```

Parallel prefix sums – Analysis

- **Time:**
  - all additions of one round run in parallel
  - $\lceil \lg n \rceil$ rounds
  - ⇝ $\Theta(\log n)$ time best possible!
- **Work:**
  - $\geq \frac{n}{2}$ additions in all rounds (except maybe last round)
  - ⇝ $\Theta(n \log n)$ work
  - more than the $\Theta(n)$ sequential algorithm!
- Typical trade-off: greater parallelism at the expense of more overall work
- For prefix sums:
  - can actually get $\Theta(n)$ work in *twice* that time!
  - ⇝ algorithm is slightly more complicated
  - instead here: linear work in *thrice* the time using "blocking trick"

Work-efficient parallel prefix sums

standard trick to improve work: compute small blocks sequentially

1. Set $b := \lceil \lg n \rceil$
2. For blocks of b consecutive indices, i. e., $A[0..b)$, $A[b..2b)$, … do in parallel: compute local prefix sums sequentially
3. Use previous work-inefficient algorithm only on leftmost elements of block, i. e., to compute prefix sums of $A[0]$, $A[b]$, $A[2b]$, …
4. For blocks $A[0..b)$, $A[b..2b)$, … do in parallel: Add block-prefix sums to local prefix sums

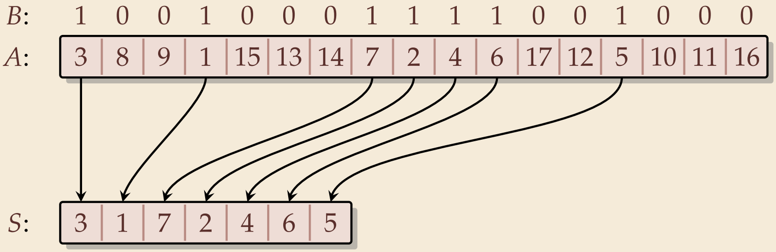
Analysis:

- **Time:**
  - 2. & 4.: $\Theta(b) = \Theta(\log n)$ time
  - 3. $\Theta(\log(n/b)) = \Theta(\log n)$ times
- **Work:**
  - 2. & 4.: $\Theta(b)$ per block $\times$ $\lceil \frac{n}{b} \rceil$ blocks $\rightsquigarrow$ $\Theta(n)$
  - 3. $\Theta\big(\frac{n}{b} \log(\frac{n}{b})\big) = \Theta(n)$

Compacting subsequences

How do prefix sums help with sorting?

Goal: *Compact* a subsequence of an array



Use prefix sums on bitvector B

⇝ offset of selected cells in S

parallelPrefixSums(B)
for $j := 0, \ldots, n-1$ **do in parallel**
    if $B[j] == 1$ **then** $S[B[j]-1] := A[j]$
end parallel for

3.7 Parallel sorting

Parallel quicksort

Let's try to parallelize quicksort

- recursive calls can run in parallel (data independent)
- our sequential partitioning algorithm seems hard to parallelize
- but can split partitioning into rounds:
  1. comparisons: compare all elements pivot (in parallel), store bitvector
  2. compute prefix sums of bit vectors (in parallel as above)
  3. compact subsequences of small and large elements (in parallel as above)

Parallel quicksort – Code

```
procedure parQuicksort(A[l..r])
    b := choosePivot(A[l..r])
    j := parallelPartition(A[l..r], b)
    in parallel { parQuicksort(A[l..j − 1]), parQuicksort(A[j + 1..r]) }

procedure parallelPartition(A[l..r], b)
    swap(A[n − 1], A[b]);  p := A[n − 1]
    for i = 0, . . . , n − 2 do in parallel
        S[i] := [A[i] ≤ p]     // A[i] is 1 or 0
        L[i] := 1 − S[i]
    end parallel for
    in parallel { parallelPrefixSum(S[0..n − 2]);  parallelPrefixSum(L[0..n − 2]) }
    j := S[n − 2] + 1
    for i = 0, . . . , n − 2 do in parallel
        x := A[i]
        if x ≤ p then A[S[i] − 1] := x
        else A[j + L[i]] := x
    end parallel for
    A[j] := p
    return j
```

Parallel quicksort – Analysis

- **Time:**
  - partition: all $O(1)$ time except prefix sums $\rightsquigarrow$ $\Theta(\log n)$ time
  - quicksort: expected depth of recursion tree is $\Theta(\log n)$

  $\rightsquigarrow$ total time $O(\log^2(n))$ in expectation
- **Work:**
  - partition: $O(n)$ time except prefix sums $\rightsquigarrow$ $\Theta(n \log n)$ work

  $\rightsquigarrow$ quicksort $O(n \log^2(n))$ work in expectation
  - using a work-efficient prefix-sums algorithm yields (expected) work-efficient sorting!

Parallel mergesort

- As for quicksort, recursive calls can run in parallel ✓
- how about merging sorted halves $A[l..m-1]$ and $A[m..r]$?
- Must treat elements independently.
  - correct position of x in sorted output = *rank* of x #elements $\le x$
  - # elements $\le x$ = # elements from $A[l..m-1]$ that are $\le x$
    + # elements from $A[m..r]$ that are $\le x$
  - Note: rank in own run is simply the index of x in that run
  - find rank in *other* run by binary search

  $\rightsquigarrow$ can move it to correct position

Parallel mergesort – Analysis

- **Time:**
  - merge: $\Theta(n)$ from binary search, rest $O(1)$
  - mergesort: depth of recursion tree is $\Theta(\log n)$

  $\rightsquigarrow$ total time $O(\log^2(n))$
- **Work:**
  - merge: n binary searches $\rightsquigarrow$ $\Theta(n \log n)$

  $\rightsquigarrow$ mergesort: $O(n \log^2(n))$ work

  - work can be reduced to $\Theta(n)$ for merge
    - do full binary searches only for regularly sampled elements
    - ranks of remaining elements are sandwiched between sampled ranks
    - use a sequential method for small blocks, treat blocks in parallel
    - (detailed omitted)

Parallel sorting – State of the art

- more sophisticated methods can sort in $O(\log n)$ parallel time on CREW-RAM
- practical challenge: small units of work add overhead
- need a lot of PEs to see improvement from $O(\log n)$ parallel time

$\rightsquigarrow$ implementations tend to use simpler methods above

- check the Java library sources for interesting examples!
  `java.util.Arrays.parallelSort(int[])`