# 3 Efficient Sorting

*18 February 2020*

Sebastian Wild

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# Outline

## 3 Efficient Sorting

# Why study sorting?

- fundamental problem of computer science that is still not solved

Algorithm with optimal #comparisons in worst case?

- building brick of many more advanced algorithms
  - for preprocessing
  - as subroutine
- playground of manageable complexity
  to practice algorithmic techniques

Here:

- “classic” fast sorting method
- **parallel** sorting

# Part I

## *The Basics*

# Rules of the game

- **Given:**
  - array $A[0..n-1]$ of $n$ objects
  - a total order relation $\leq$ among $A[0], \ldots, A[n-1]$
    (a comparison function)
- **Goal:** rearrange (=permute) elements within $A$,
  so that $A$ is *sorted*, i. e., $A[0] \leq A[1] \leq \cdots \leq A[n-1]$

- for now: $A$ stored in main memory (*internal sorting*)
  single processor (*sequential sorting*)

## 3.1 Mergesort

# Clicker Question

How does mergesort work?

- **A** Split elements around median, then recurse on small / large elements.
- **B** Recurse on left / right half, then combine sorted halves.
- **C** Grow sorted part on left, repeatedly add next element to sorted range.
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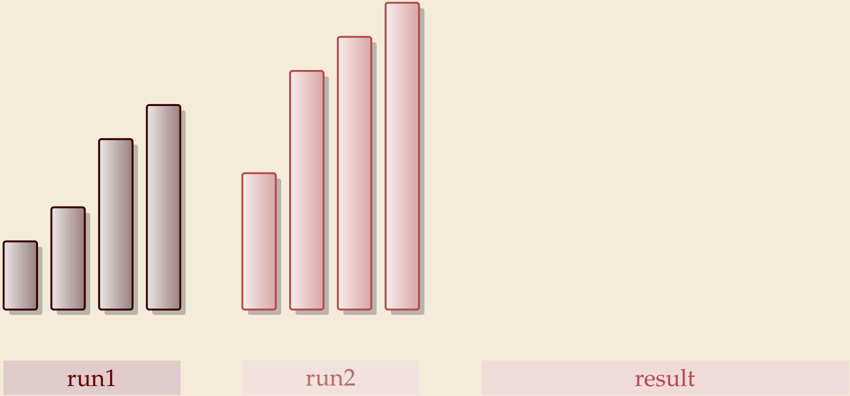

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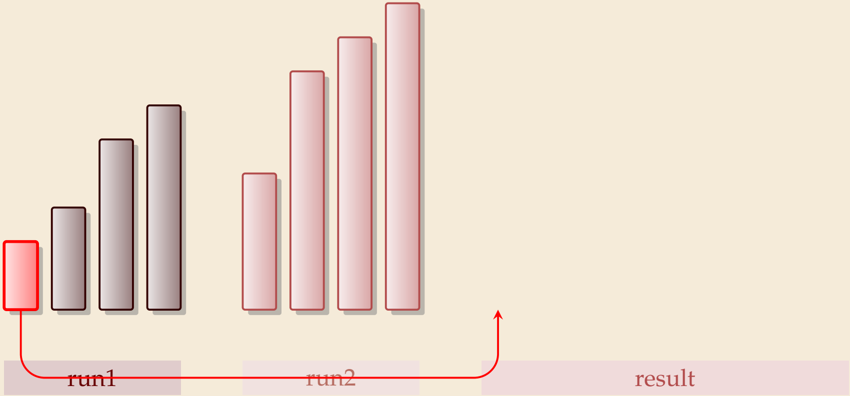

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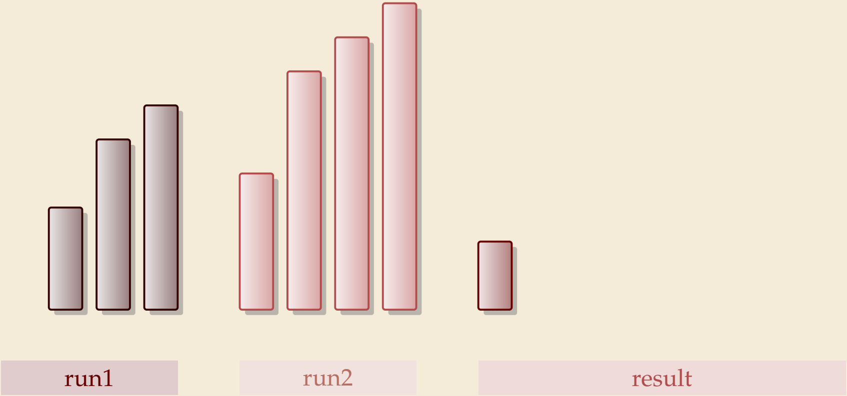

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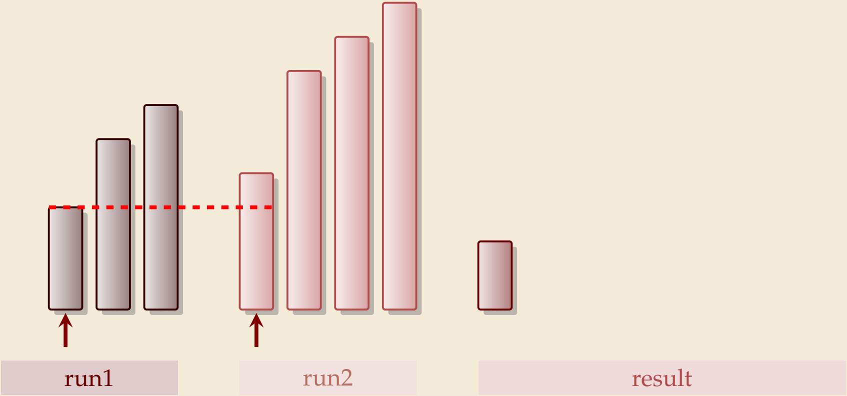

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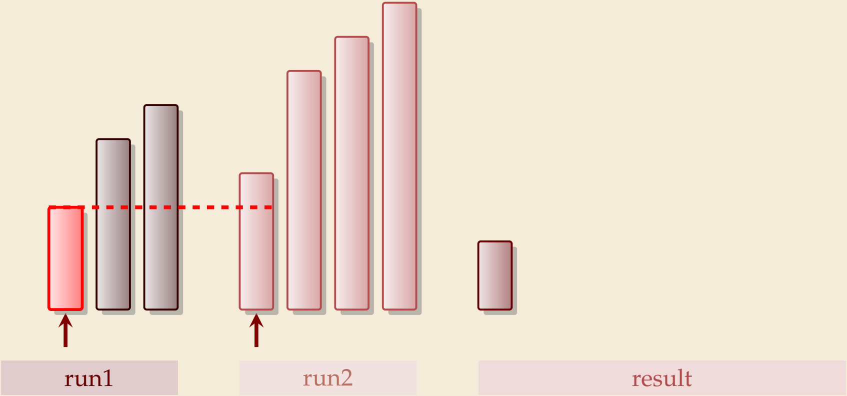

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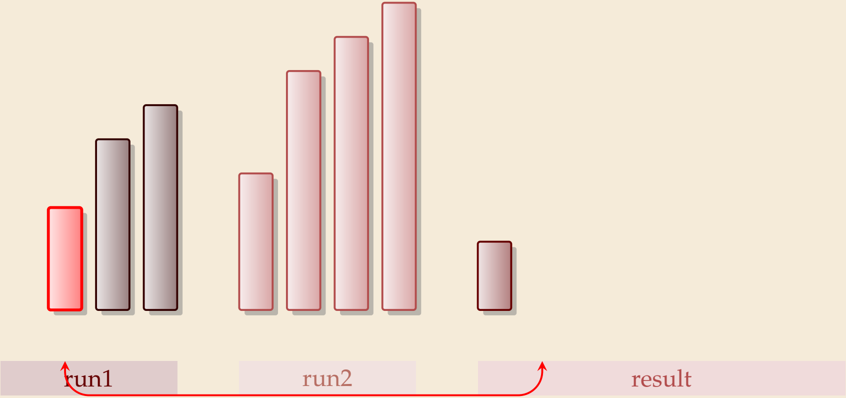

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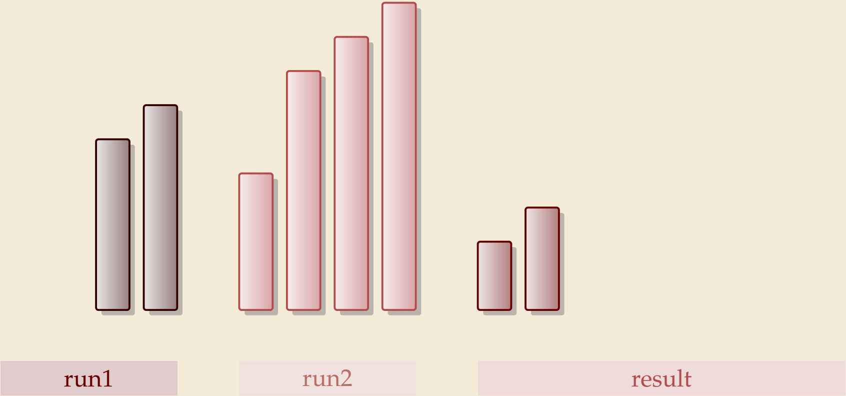

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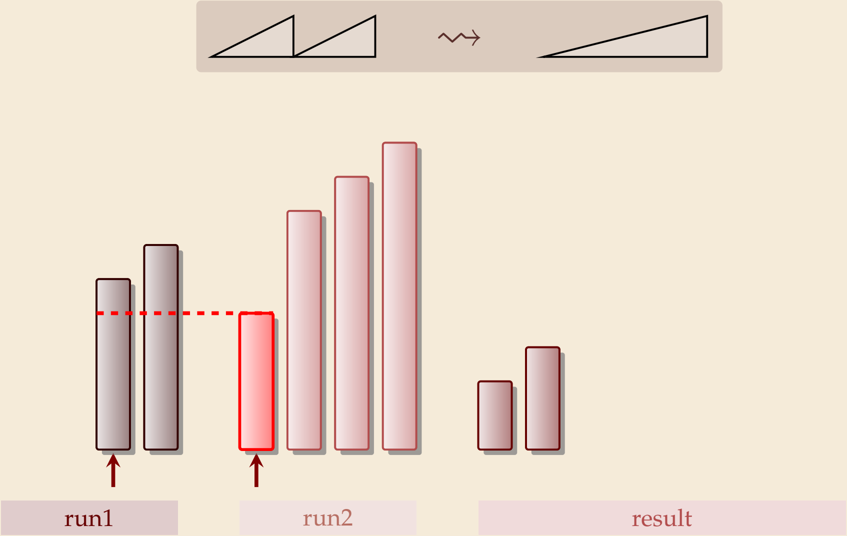

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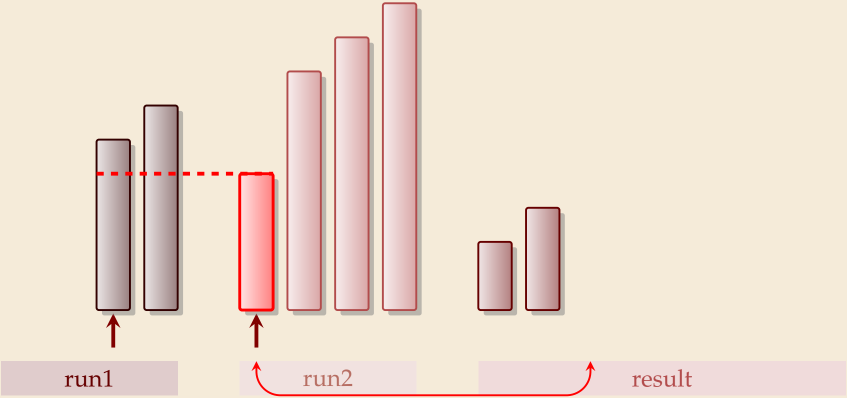

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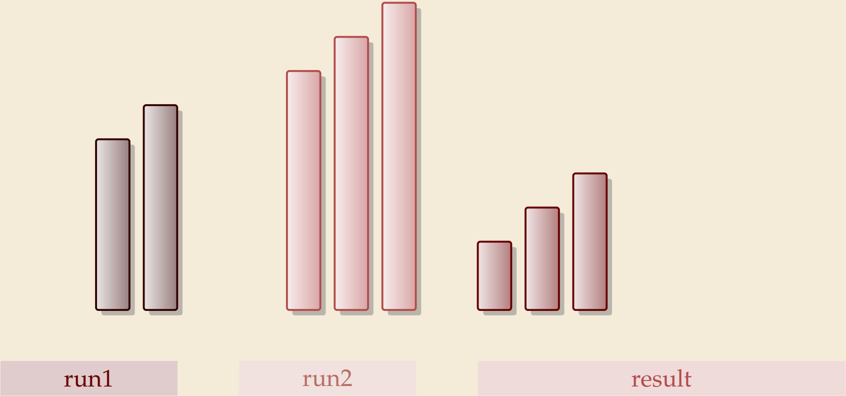

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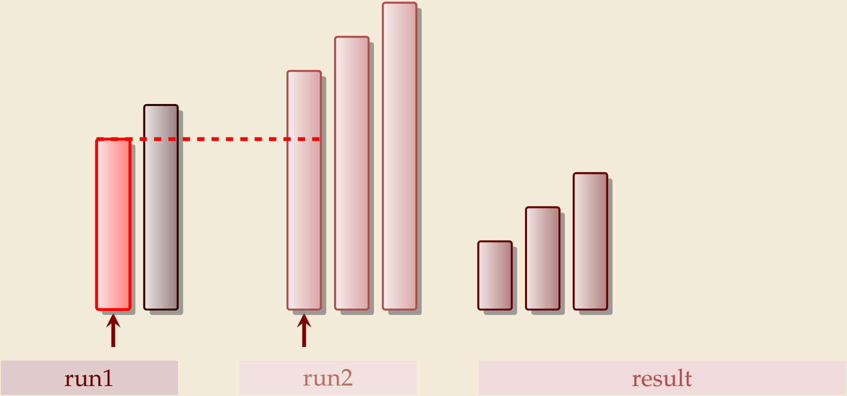

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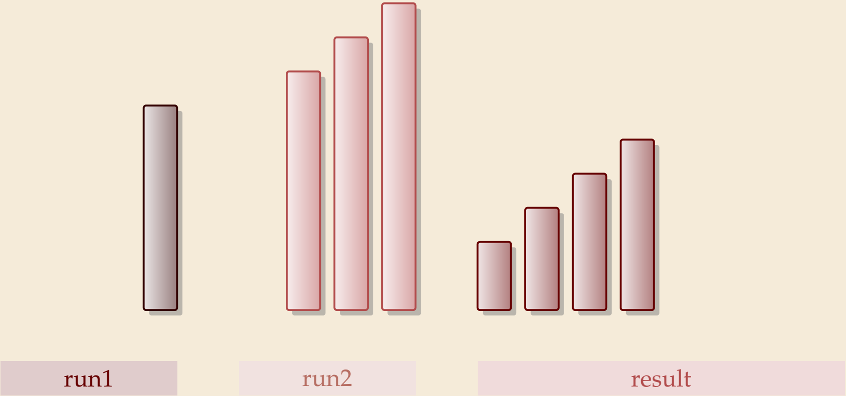

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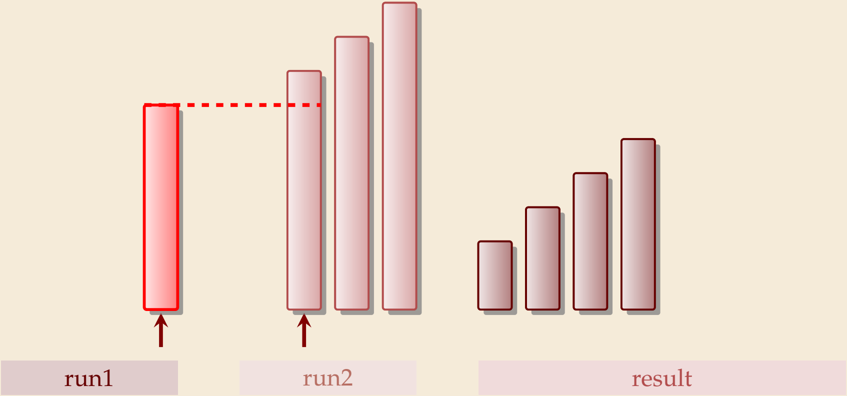

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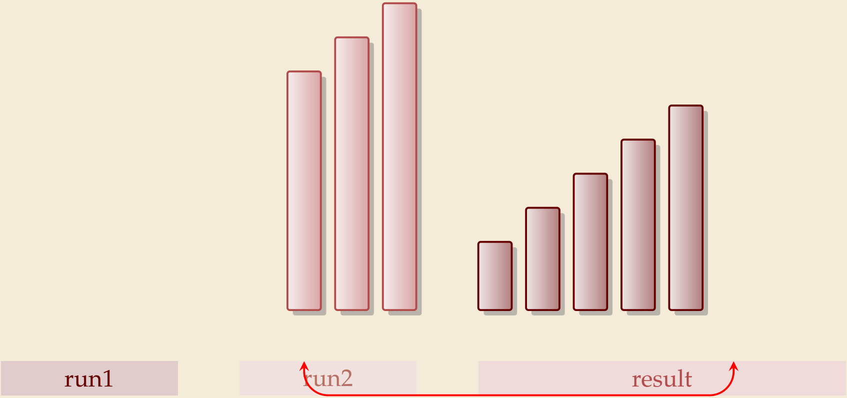

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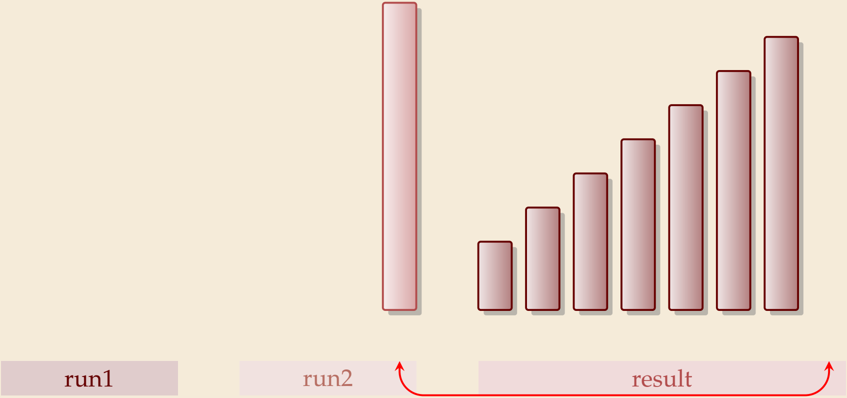

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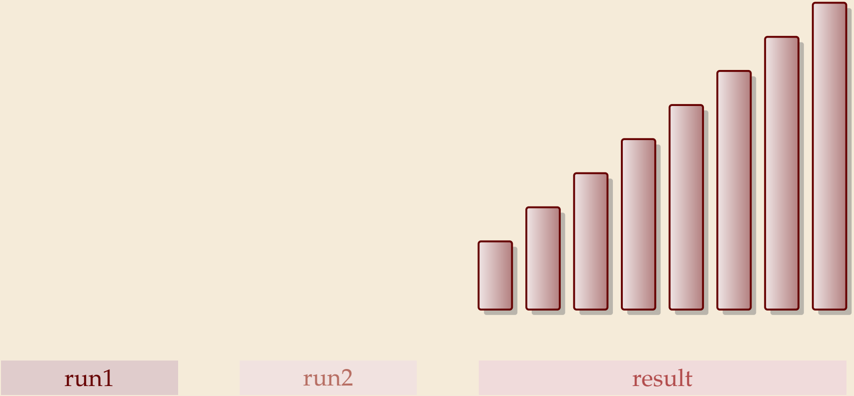

# Mergesort

**procedure** mergesort($A[l..r]$)
    $n := r - l + 1$
    **if** $n \geq 1$ **return**
    $m := l + \left\lfloor \frac{n}{2} \right\rfloor$
    mergesort($A[l..m-1]$)
    mergesort($A[m..r]$)
    merge($A[l..m-1]$, $A[m..r]$, *buf*)
    copy *buf* to $A[l..r]$

- recursive procedure; *divide & conquer*
- merging needs
  - temporary storage for result of same size as merged runs buf
  - to read and write each element twice (once for merging, once for copying back)

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- recursive procedure; *divide & conquer*
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2n

**Analysis:** count "element visits" (read and/or write)

$$C(n) = \begin{cases} 0 & n \leq 1 \\ C(\lfloor n/2 \rfloor) + C(\lceil n/2 \rceil) + 2n & n \geq 2 \end{cases}$$

same for best and worst case!

Simplification $\boxed{n = 2^k}$

$$C(2^k) = \begin{cases} 0 & k \leq 0 \\ 2 \cdot C(2^{k-1}) + 2 \cdot 2^k & k \geq 1 \end{cases} = 2 \cdot 2^k + 2^2 \cdot 2^{k-1} + 2^3 \cdot 2^{k-2} + \cdots + 2^k \cdot 2^1 = \underset{\lg n}{2k} \cdot \underset{n}{2^k}$$

$$C(n) = 2n \lg(n) = \Theta(n \log n)$$

# Mergesort – Discussion

- optimal time complexity of $\Theta(n \log n)$ in the worst case
- *stable* sorting method   i. e., retains relative order of equal-key items
- memory access is sequential (scans over arrays)
- requires $\Theta(n)$ extra space

there are in-place merging methods,
but they are substantially more complicated
and not (widely) used

2 3 1 2 2 5

→ 1 2 2 2 3 5 sorted

but not stably sorted

## 3.2 Quicksort

## Clicker Question

How does quicksort work?

- **A** split elements around median, then recurse on small / large elements.
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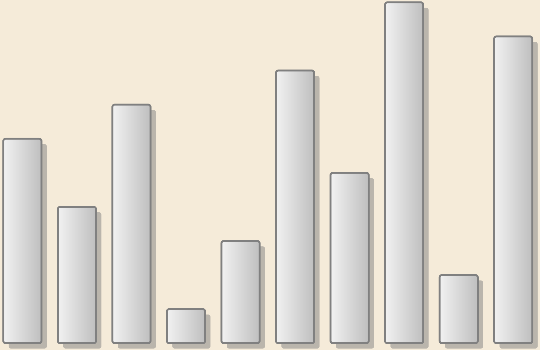

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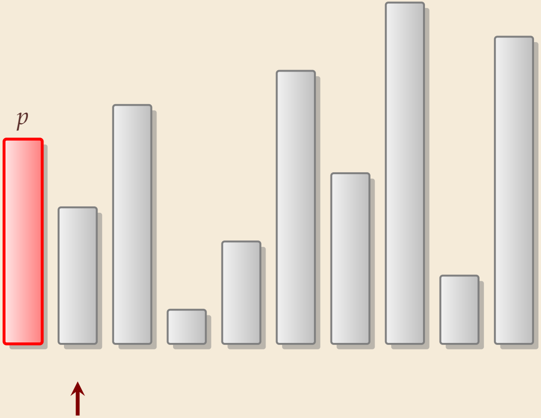

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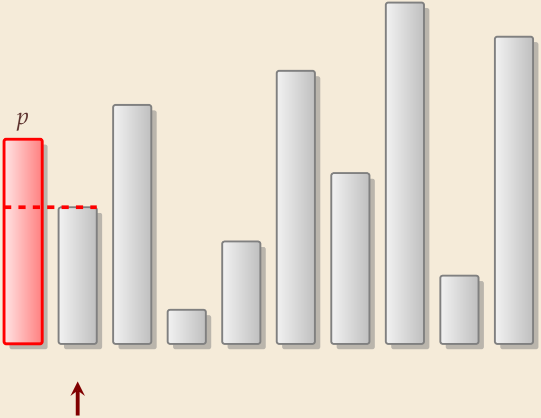

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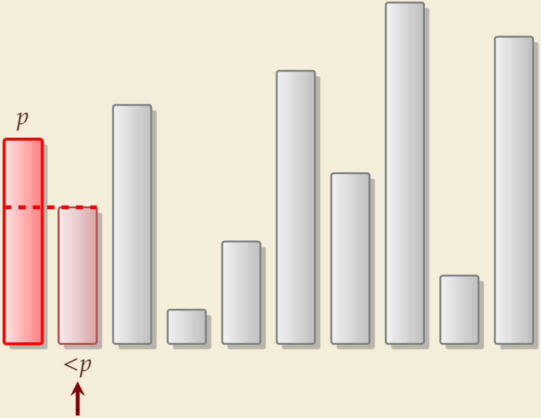

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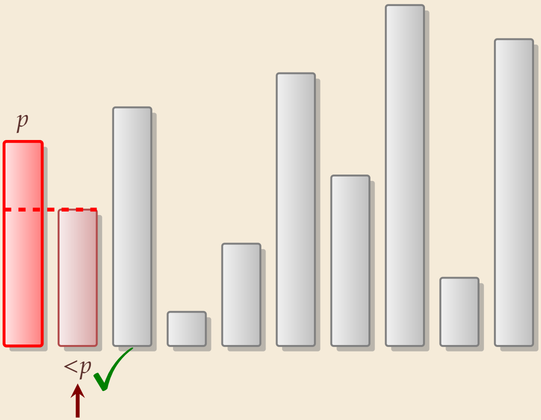

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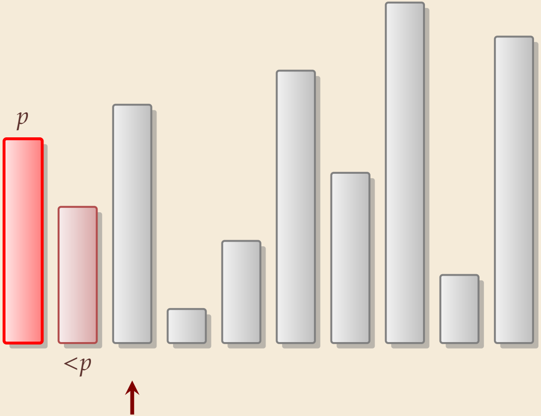

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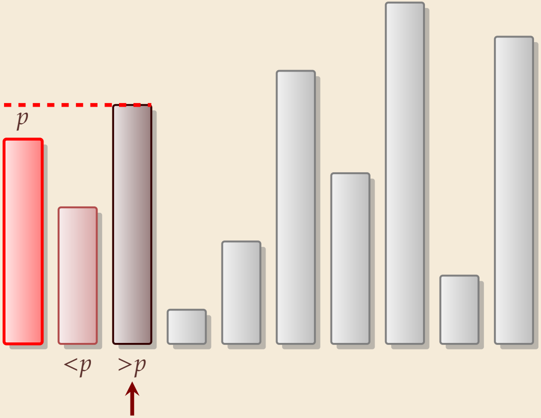

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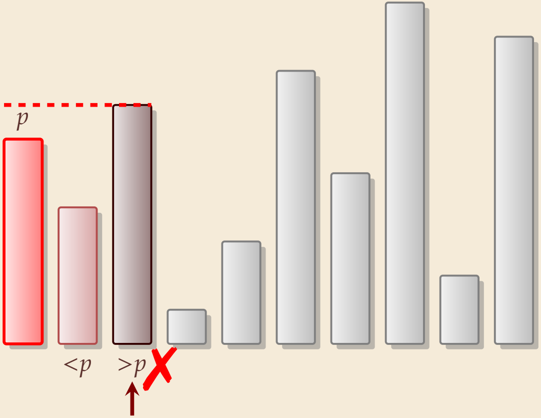

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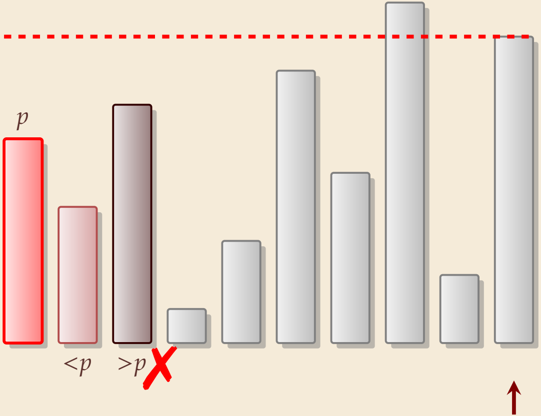

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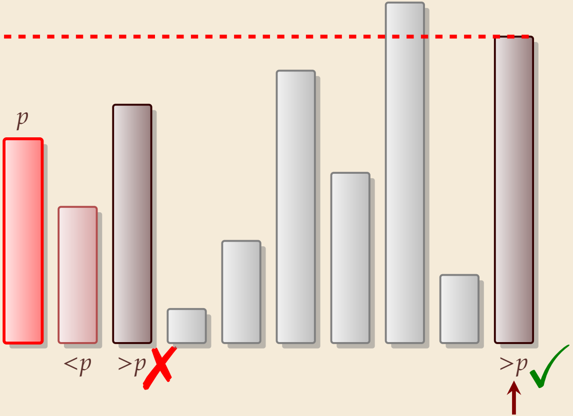

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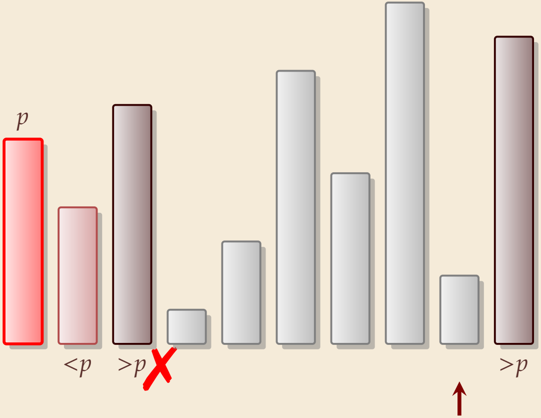

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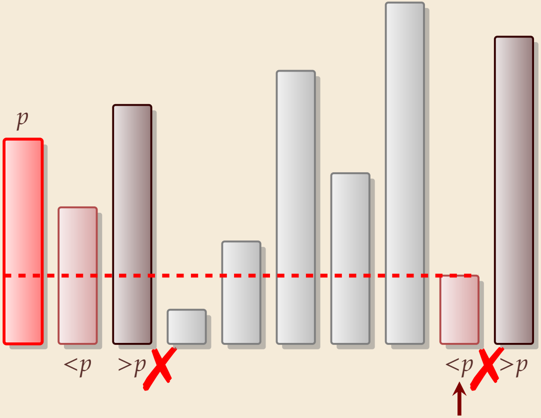

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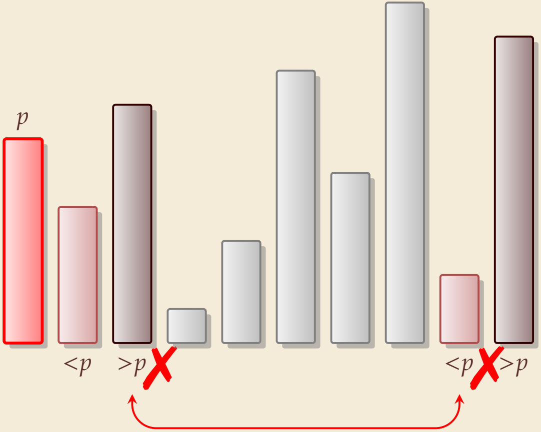

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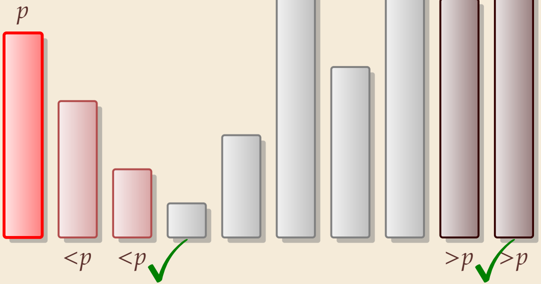

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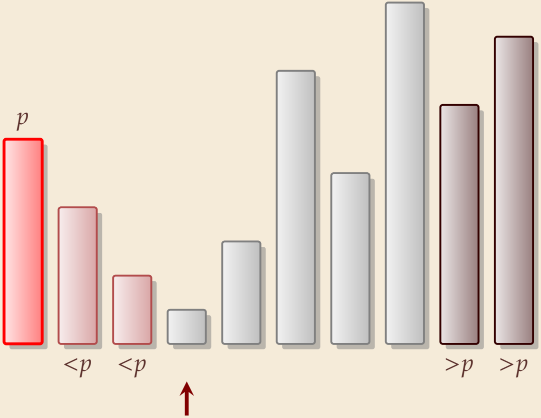

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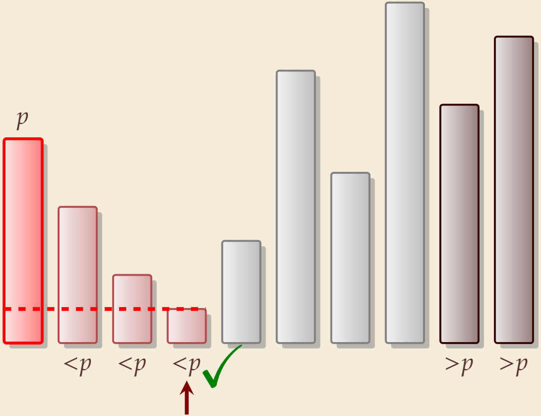

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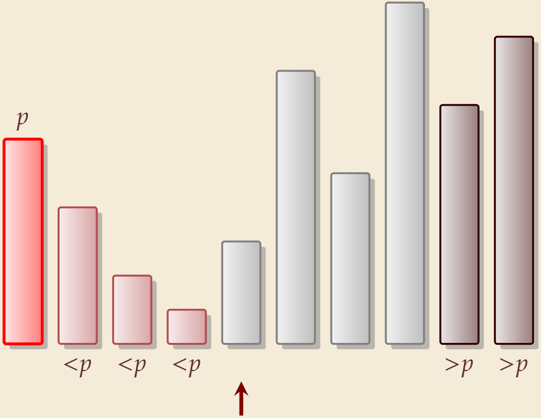

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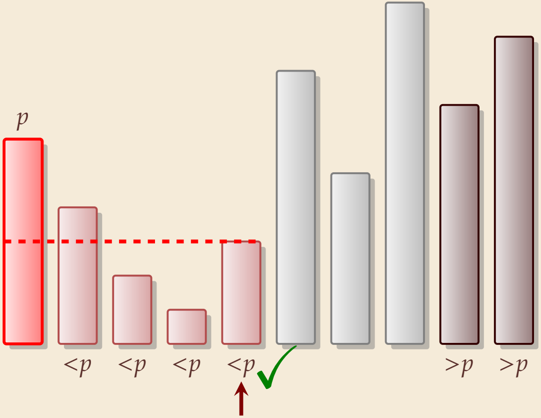

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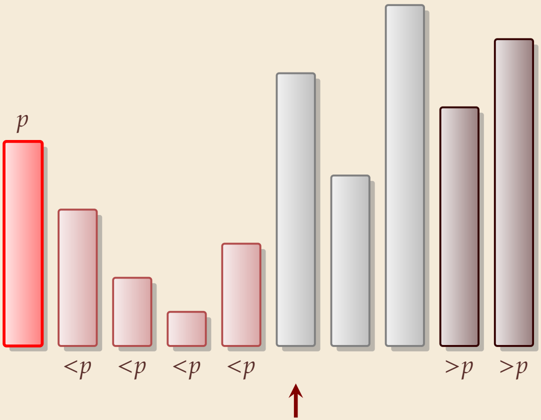

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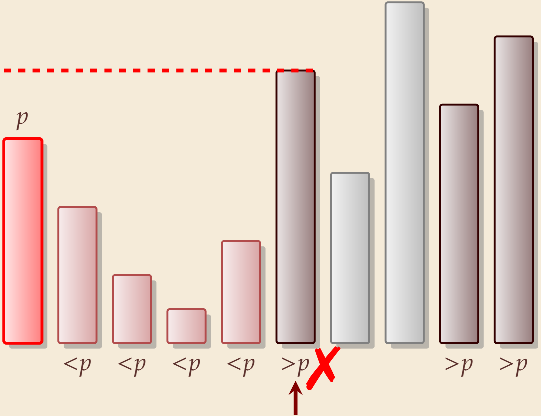

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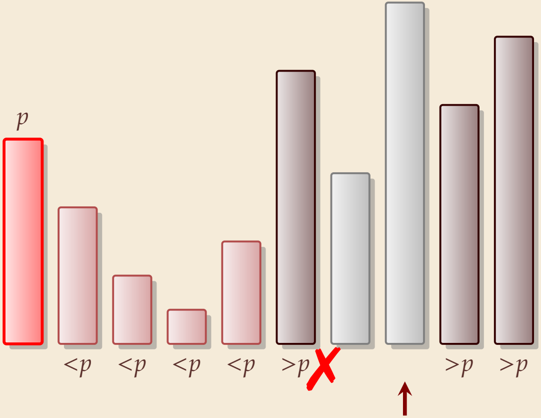

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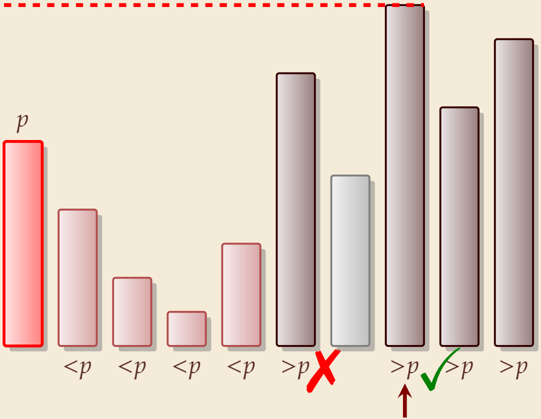

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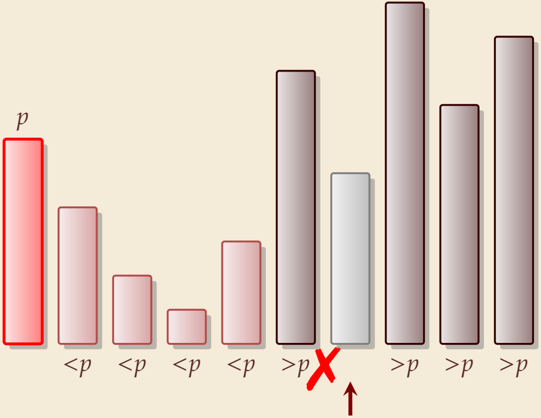

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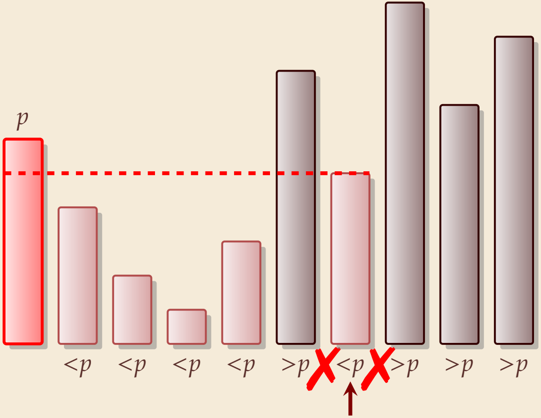

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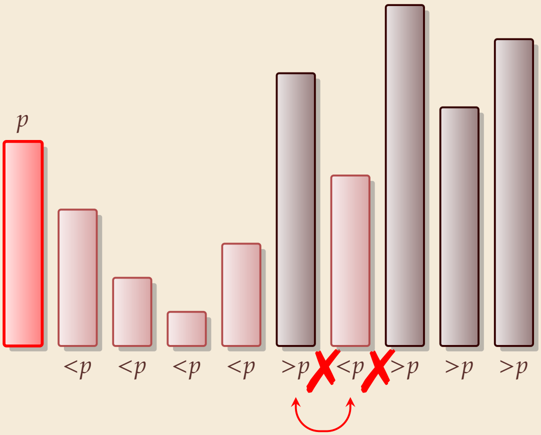

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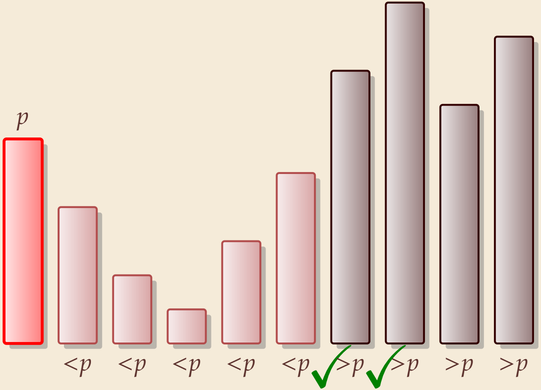

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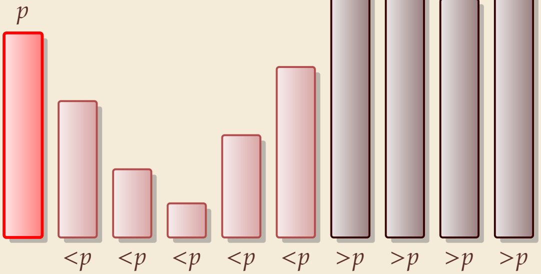

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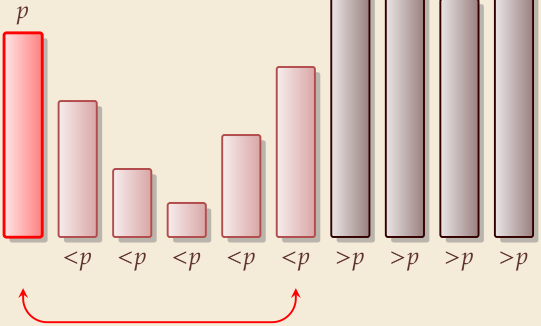

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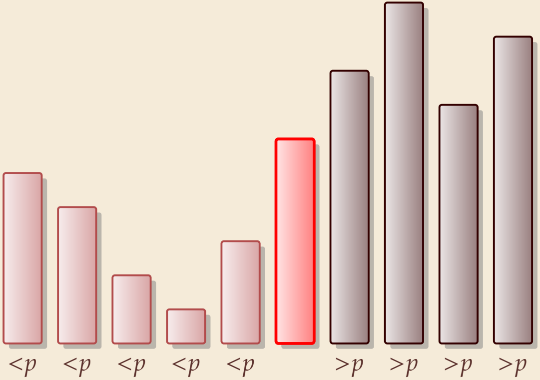

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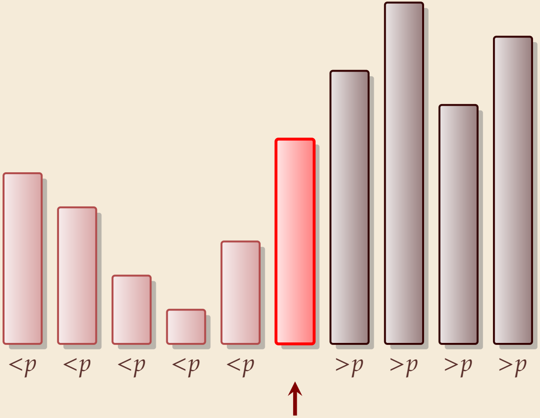


- no extra space needed
- visits each element once
- returns rank/position of pivot

# Partitioning – Detailed code

Beware: details easy to get wrong; use this code!

1 **procedure** partition($A$, $b$)<br>
2     *// input: array* $A[0..n-1]$*, position of pivot* $b \in [0..n-1]$<br>
3     swap($A[0]$, $A[b]$)<br>
4     $i := 0, \quad j := n$<br>
5     **while** *true* **do**<br>
6         **do** $i := i + 1$ **while** $i < n$ and $A[i] < A[0]$<br>
7         **do** $j := j - 1$ **while** $j \geq 1$ and $A[j] > A[0]$<br>
8         **if** $i \geq j$ **then break**    (goto 8)<br>
9         **else** swap($A[i]$, $A[j]$)<br>
10     **end while**<br>
11     swap($A[0]$, $A[j]$)<br>
12     **return** $j$

**Loop invariant (5–10):** $A$

| $p$ | $\leq p$ | ? | $\geq p$ |
|---|---|---|---|
| | | $i$ | $j$ |

# Quicksort

```
procedure quicksort(A[l..r])
    if l ≥ r then return
    b := choosePivot(A[l..r])
    j := partition(A[l..r], b)
    quicksort(A[l..j − 1])
    quicksort(A[j + 1..r])
```

- recursive procedure; *divide & conquer*
- choice of pivot can be
  - fixed position $\rightsquigarrow$ dangerous!
  - random
  - more sophisticated, e. g., median of 3

# Quicksort & Binary Search Trees

## Quicksort

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 7 | 4 | 2 | 9 | 1 | 3 | 8 | 5 | 6 |

# Quicksort & Binary Search Trees

**Quicksort**

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|---|---|

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| | | | | | | | | |
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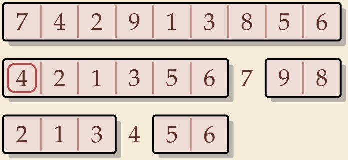

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|---|---|---|---|---|---|---|---|---|

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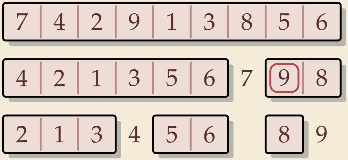
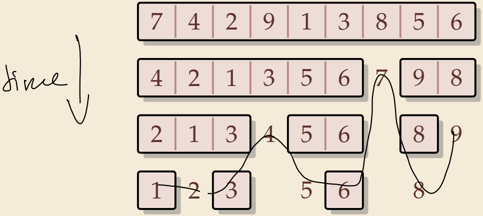


# Quicksort & Binary Search Trees

## Quicksort



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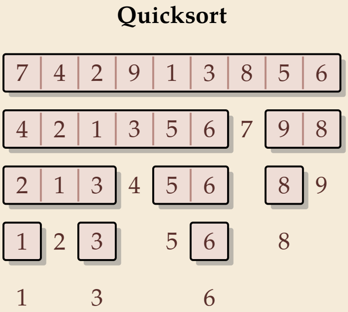

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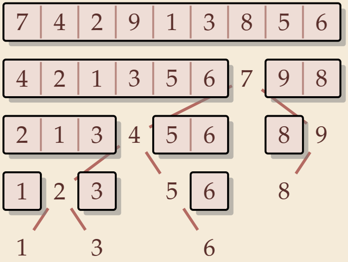

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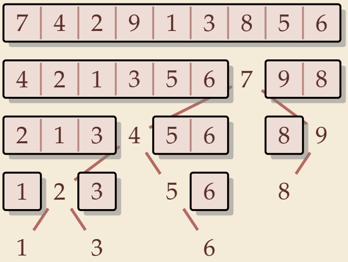


### Binary Search Tree (BST)

7 4 2 9 1 3 8 5 6

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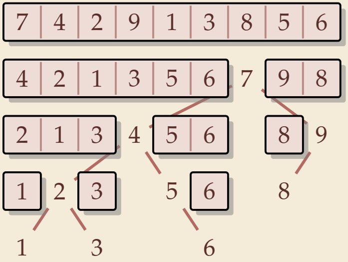


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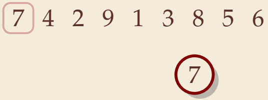

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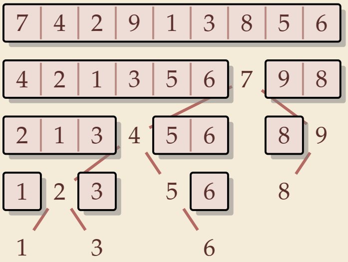


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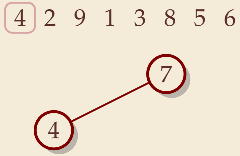

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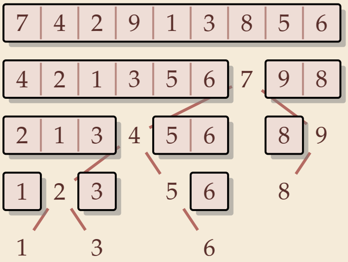


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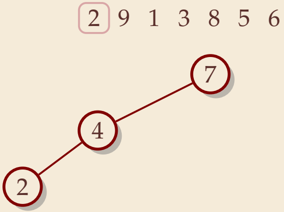

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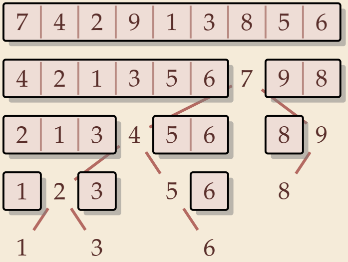


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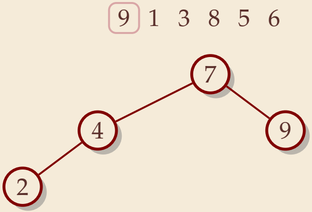

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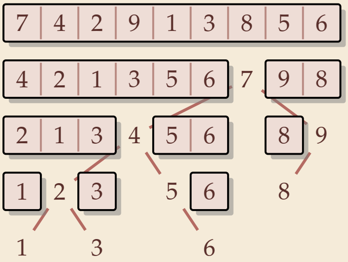


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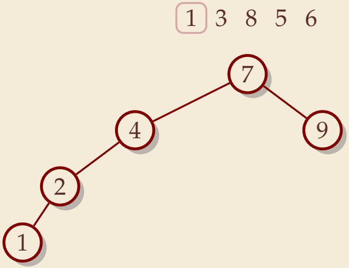

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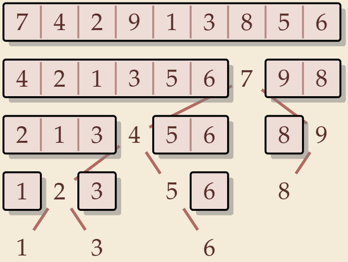


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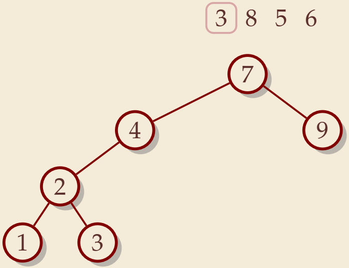

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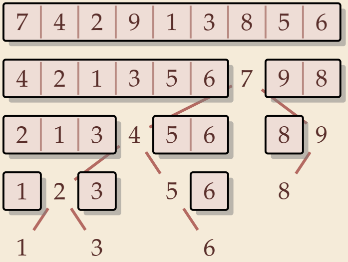


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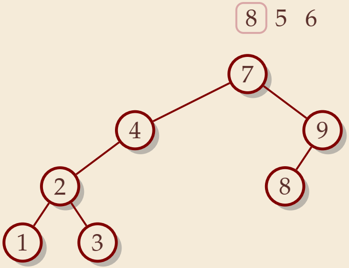

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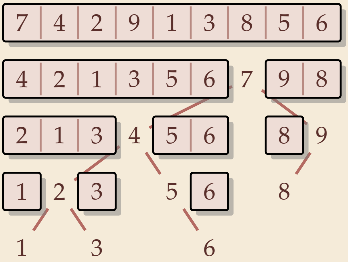


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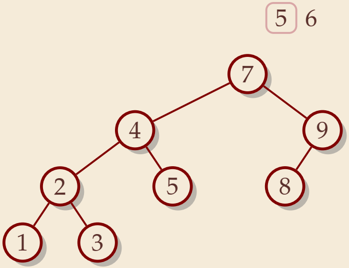

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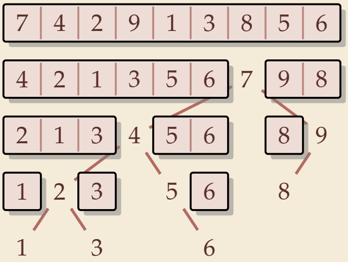


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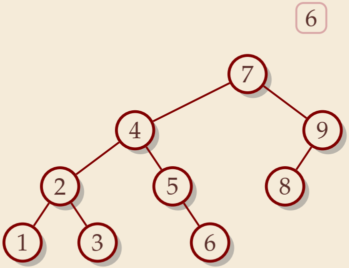

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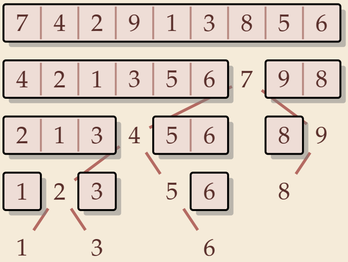


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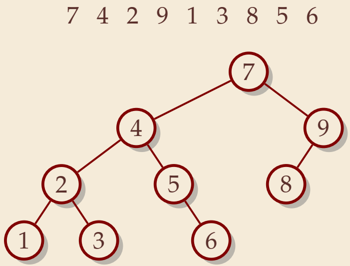



- recursion tree of quicksort $=$ binary search tree from successive insertion
- comparisons in quicksort $=$ comparisons to built BST
- comparisons in quicksort $\approx$ comparisons to search each element in BST

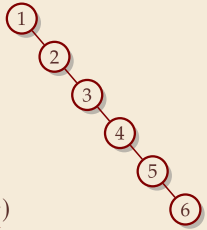
# Quicksort – Worst Case

- Problem: BSTs can degenerate
- Cost to search for $k$ is $k-1$

$\rightsquigarrow$ Total cost $\sum_{k=1}^{n}(k-1) = \dfrac{n(n-1)}{2} \sim \frac{1}{2}n^2$

$\rightsquigarrow$ quicksort worst-case running time is in $\Theta(n^2)$

terribly slow!




But, we can fix this:

**Randomized quicksort:**

- choose a *random pivot* in each step

$\rightsquigarrow$ same as randomly *shuffling* input before sorting

# Randomized Quicksort – Analysis

- $C(n)$ = element visits (as for mergesort)

$\rightsquigarrow$ quicksort needs $\sim 2\ln(2) \cdot n \lg n \approx \underbrace{1.39n}\lg n$ *in expectation*

- also: very unlikely to be much worse:
  e. g., one can prove: $\Pr[\text{cost} > 10n\lg n] \;=\; O(n^{-2.5})$
  distribution of costs is "concentrated around mean"
- intuition: have to be constantly unlucky with pivot choice

# Quicksort – Discussion

- 👍 fastest general-purpose method
- 👍 $\Theta(n \log n)$ average case
- 👍 works *in-place* (no extra space required)
- 👍 memory access is sequential (scans over arrays)
- 👎 $\Theta(n^2)$ worst case (although extremely unlikely)
- 👎 not a *stable* sorting method

Open problem: Simple algorithm that is fast, stable and in-place.

## 3.3 Comparison-Based Lower Bound

# Lower Bounds

- **Lower bound:** mathematical proof that no algorithm can do better.
  - very powerful concept: bulletproof *impossibility* result
    $\approx$ *conservation of energy* in physics
  - **(unique?) feature of computer science:**
    for many problems, solutions are known that (asymptotically) *achieve the lower bound*

  $\rightsquigarrow$ can speak of "*optimal* algorithms"

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    for many problems, solutions are known that (asymptotically) *achieve the lower bound*
  
  $\rightsquigarrow$ can speak of "*optimal* algorithms"

- To prove a statement about *all algorithms*, we must precisely define what that is!

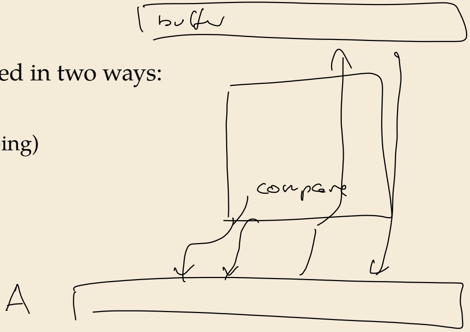
- already know one option: the word-RAM model

- Here: use a simpler, more restricted model.

# The Comparison Model

- In the *comparison model* data can only be accessed in two ways:
  - comparing two elements
  - moving elements around (e. g. copying, swapping)

  - Cost: number of these operations.



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That's good!
Keeps algorithms general!

- This makes very few assumptions on the kind of objects we are sorting.
- Mergesort and Quicksort work in the comparison model.

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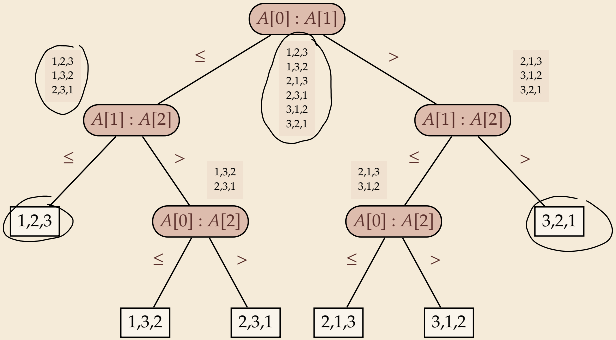
That's good!
Keeps algorithms general!

- This makes very few assumptions on the kind of objects we are sorting.
- Mergesort and Quicksort work in the comparison model.

$\rightsquigarrow$ Every comparison-based sorting algorithm corresponds to a ***decision tree***.
  - only model comparisons $\rightsquigarrow$ ignore data movement
  - nodes = comparisons the algorithm does
  - next comparisons can depend on outcomes $\rightsquigarrow$ different subtrees
  - child links = outcomes of comparison
  - leaf = unique initial input permutation compatible with comparison outcomes

# Comparison Lower Bound

**Example:** Comparison tree for a sorting method for $A[0..2]$:



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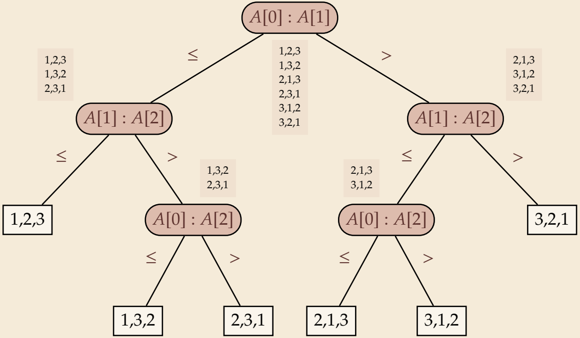


- Execution = follow a path in comparison tree.

$\rightsquigarrow$ height of comparison tree = worst-case # comparisons

- comparison trees are *binary* trees

$\rightsquigarrow$ $\ell$ leaves $\rightsquigarrow$ height $\ge \lceil \lg(\ell) \rceil$

- comparison trees for sorting method must have $\ge n!$ leaves

$\rightsquigarrow$ height $\ge \lg(n!) \sim n \lg n$

more precisely: $\lg(n!) = n \lg n - \lg(e)n + O(\log n)$

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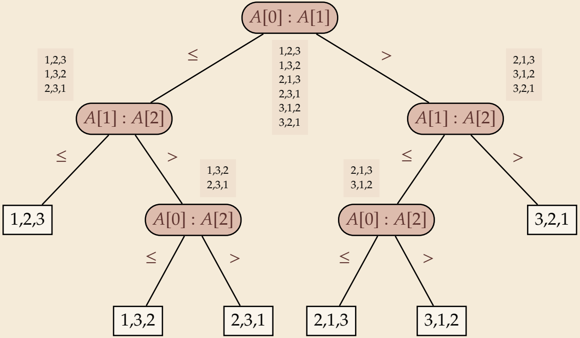


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more precisely: $\lg(n!) = n \lg n - \lg(e)n + O(\log n)$

- Mergesort achieves $\sim n \lg n$ comparisons $\rightsquigarrow$ asymptotically comparison-optimal!
- Open (theory) problem: Can we sort with $n \lg n - \lg(e)n + o(n)$ comparisons?

$\approx 1.4427$

# Clicker Question

Does the comparison-tree from the previous slide correspond to a worst-case optimal sorting method?

**A** Yes

**B** No

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# Clicker Question

Does the comparison-tree from the previous slide correspond to a worst-case optimal sorting method?

**A** Yes ✓

**B** ~~No~~

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## 3.4 Integer Sorting

## How to beat a lower bound

- Does the above lower bound mean, sorting always takes time $\Omega(n \log n)$?

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- Here: sort $n$ **integers**
  - can do *a lot* with integers: add them up, compute averages, ... (full power of word-RAM)

  $\rightsquigarrow$ we are **not** working in the comparison model
  $\rightsquigarrow$ *above lower bound does not apply!*

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  - $\rightsquigarrow$ we are **not** working in the comparison model
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  - but: a priori unclear how much arithmetic helps for sorting . . .

# Counting sort

- Important parameter: size/range of numbers
  - numbers in range $[0..U) = \{0, \ldots, U-1\}$ typically $U = 2^b$ $\rightsquigarrow$ $b$-bit binary numbers
- We can sort $n$ integers in $\Theta(n+U)$ time and $\Theta(U)$ space when $\boxed{b \le w}$

word size

**Counting sort**

```
procedure countingSort(A[0..n − 1])
    // A contains integers in range [0..U).
    C[0..U − 1] := new integer array, initialized to 0
    // Count occurrences
    for i := 0, . . . , n − 1
        C[A[i]] := C[A[i]] + 1
    i := 0 // Produce sorted list
    for k := 0, . . . U − 1
        for j := 1, . . . C[k]
            A[i] := k; i := i + 1
```

count #occurrences of all $i \in [0..U)$

Java uses this sort(byte[])

- *count* how often each *possible* value occurs
- produce sorted result directly from counts
- circumvents lower bound by using integers as array index / pointer offset

$\rightsquigarrow$ Can sort $n$ integers in range $[0..U)$ with $U = O(n)$ in time and space $\Theta(n)$.

# Integer Sorting – State of the art

- $O(n)$ time sorting also possible for numbers in range $U = O(n^c)$ for constant $c$.
  - *radix sort* with radix $2^w$

Unit 1 : $w = \Theta(\log n)$

- **algorithm theory**
  - suppose $U = 2^w$, but $w$ can be arbitrary function of $n$
  - how fast can we sort $n$ such $w$-bit integers on a $w$-bit word-RAM?
    - for $w = O(\log n)$: linear time (*radix/counting sort*)
    - for $w = \Omega(\log^{2+\varepsilon} n)$: linear time (*signature sort*)
    - for $w$ in between: can do $O(n\sqrt{\lg\lg n})$ (very complicated algorithm)
      don't know if that is best possible!

outside of exam

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      don't know if that is best possible!

* * *

- for the rest of this unit: back to the comparisons model!

# Part II

*Sorting with of many processors*

## 3.5 Parallel computation

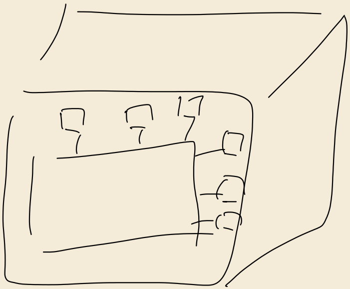
# Types of parallel computation

£££ can't buy you more time, but more computers!

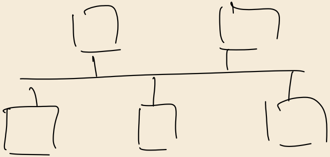
⇝ Challenge: Algorithms for parallel computation.

There are two main forms of parallelism

1. **shared-memory parallel computer** ← *focus of today*
   - $p$ *processing elements* (PEs, processors) working in parallel
   - single big memory, accessible from every PE
   - communication via shared memory
   - think: a big server, 128 CPU cores, terabyte of main memory



2. **distributed computing**
   - $p$ PEs working in parallel
   - each PE has private memory
   - communication by sending messages via a network
   - think: a cluster of individual machines

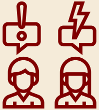


# PRAM – Parallel RAM

- extension of the RAM model (recall Unit 1)
- the $p$ PEs are identified by ids $0, \ldots, p-1$
  - like $w$ (the word size), $p$ is a parameter of the model that can grow with $n$
  - $p = \Theta(n)$ is not unusual     maaany processors!
- the PEs all **independently** run a RAM-style program (they can use their id there)
- each PE has its own registers, but `MEM` is shared among all PEs
- computation runs in **synchronous** steps: in each time step, every PE executes one instruction

questionable assumption in practice ...

# PRAM – Conflict management



**Problem:** What if several PEs simultaneously overwrite a memory cell?

- **EREW**-PRAM (exclusive read, exclusive write)
  any **parallel access** to same memory cell is **forbidden** (crash if happens)
- **CREW**-PRAM (concurrent read, exclusive write)
  parallel **write** access to same memory cell is *forbidden*, but reading is fine
- **CRCW**-PRAM (concurrent read, concurrent write)
  concurrent access is allowed,
  need a rule for write conflicts:
  - common CRCW-PRAM:
    all concurrent writes to same cell must write *same* value
  - arbitrary CRCW-PRAM:
    some unspecified concurrent write wins
  - (more exist . . . )

- no single model is always adequate, but our default is CREW

# PRAM – Execution costs

Cost metrics in PRAMs

- **space:** total amount of accessed memory    same as for RAM
- **time:** number of steps till all PEs finish    assuming sufficiently many PEs!
  sometimes called *depth* or *span*
- **work:** total #instructions executed on **all** PEs

Holy grail of PRAM algorithms:

- minimal time
- work (asymptotically) no worse than running time of best sequential algorithm
  - *work-efficient* algorithm: work in same $\Theta$-class as best sequential

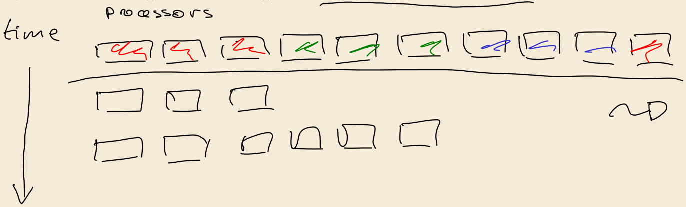
# The number of processors

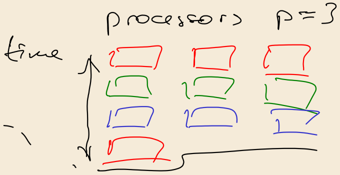
*Hold on, my computer does not have $\Theta(n)$ processors! Why should I care for span and work!?*

### Theorem 3.1 (Brent's Theorem:)

If an algorithm has span $T$ and work $W$ (for an arbitrarily large number of processors), it can be run on a PRAM with $p$ PEs in time $O(T + \frac{W}{p})$ (and using $O(W)$ work). ◂

*Proof:* schedule parallel steps in round-robin fashion on the $p$ PEs.




⇝ span and work give guideline for *any* number of processors

## 3.6 Parallel primitives

# Prefix sums

Before we come to parallel sorting, we study some useful building blocks.

**Prefix-sum problem** (also: cumulative sums, running totals)

- Given: array $A[0..n-1]$ of numbers
- Goal: compute all prefix sums $A[0] + \cdots + A[i]$ for $i = 0, \ldots, n-1$
  may be done "in-place", i. e., by overwriting $A$

**Example:**

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| input: | 3 | 0 | 0 | 5 | 7 | 0 | 0 | 2 | 0 | 0 | 0 | 4 | 0 | 8 | 0 | 1 |
| output: | 3 | 3 | 3 | 8 | 15 | 15 | 15 | 17 | 17 | 17 | 17 | 21 | 21 | 29 | 29 | 30 |

# Clicker Question

What is the *sequential* running time achievable for prefix sums?

**A** $O(n^3)$

**B** $O(n^2)$

**C** $O(n \log n)$

**D** $O(n)$

**E** $O(\sqrt{n})$

**F** $O(\log n)$

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# Clicker Question

What is the *sequential* running time achievable for prefix sums?

**A** $\cancel{O(n^3)}$

**B** $\cancel{O(n^2)}$

**C** $\cancel{O(n \log n)}$

**D** $O(n)$ ✓

**E** $\cancel{O(\sqrt{n})}$

**F** $\cancel{O(\log n)}$

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# Prefix sums – Sequential

- sequential solution does $n-1$ additions
- but: cannot parallelize them
  data dependencies!

$\rightsquigarrow$ need a different approach

**procedure** prefixSum($A[0..n-1]$)
  **for** $i := 1, \ldots, n-1$ **do**
    $A[i] := A[i-1] + A[i]$

# Prefix sums – Sequential

- sequential solution does $n-1$ additions
- but: cannot parallelize them
  data dependencies!

$\rightsquigarrow$ need a different approach

```
procedure prefixSum(A[0..n − 1])
    for i := 1, ..., n − 1 do
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```

Let's try a simpler problem first.

**Excursion:** Sum

- Given: array $A[0..n-1]$ of numbers
- Goal: compute $A[0] + A[1] + \cdots + A[n-1]$
  (solved by prefix sums)

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$1+2+3$

$=(1+2)+3$

$=1+(2+3)$

**Excursion:** Sum

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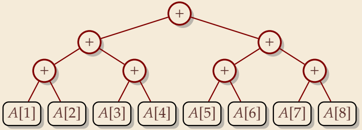
- Goal: compute $A[0]+A[1]+\cdots+A[n-1]$
  (solved by prefix sums)


Any algorithm must do $n-1$ binary additions

height

$\rightsquigarrow$ ~~Depth~~ of tree = parallel time!



# Parallel prefix sums

- Idea: Compute all prefix sums with balanced trees in parallel
  Remember partial results for reuse

input:

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 0 | 0 | 5 | 7 | 0 | 0 | 2 | 0 | 0 | 0 | 4 | 0 | 8 | 0 | 1 |

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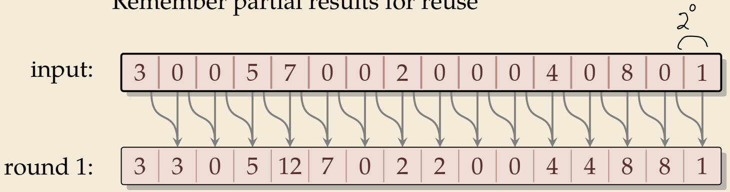

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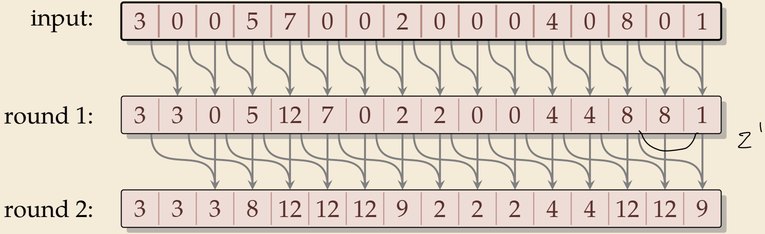

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| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| input: | 3 | 0 | 0 | 5 | 7 | 0 | 0 | 2 | 0 | 0 | 0 | 4 | 0 | 8 | 0 | 1 |
| round 1: | 3 | 3 | 0 | 5 | 12 | 7 | 0 | 2 | 2 | 0 | 0 | 4 | 4 | 8 | 8 | 1 |
| round 2: | 3 | 3 | 3 | 8 | 12 | 12 | 12 | 9 | 2 | 2 | 2 | 4 | 4 | 12 | 12 | 9 |
| round 3: | 3 | 3 | 3 | 8 | 15 | 15 | 15 | 17 | 14 | 14 | 14 | 13 | 6 | 14 | 14 | 13 |

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| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
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| round 1: | 3 | 3 | 0 | 5 | 12 | 7 | 0 | 2 | 2 | 0 | 0 | 4 | 4 | 8 | 8 | 1 |
| round 2: | 3 | 3 | 3 | 8 | 12 | 12 | 12 | 9 | 2 | 2 | 2 | 4 | 4 | 12 | 12 | 9 |
| round 3: | 3 | 3 | 3 | 8 | 15 | 15 | 15 | 17 | 14 | 14 | 14 | 13 | 6 | 14 | 14 | 13 |
| round 4: | 3 | 3 | 3 | 8 | 15 | 15 | 15 | 17 | 17 | 17 | 17 | 21 | 21 | 29 | 29 | 30 |

# Parallel prefix sums

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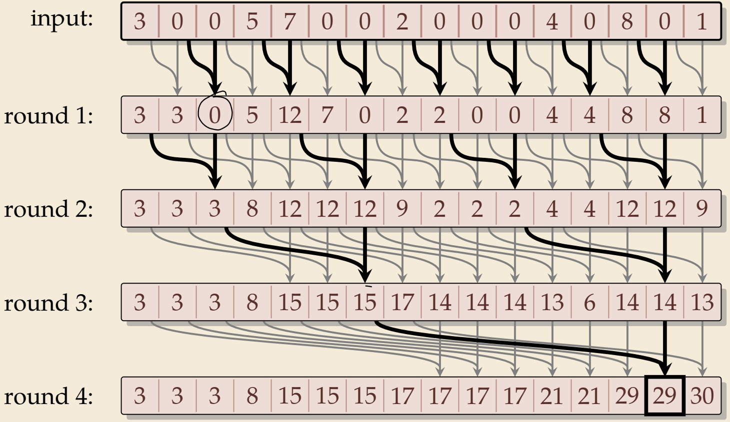
  Remember partial results for reuse



# Parallel prefix sums – Code

- can be realized in-place (overwriting $A$)
- assumption: in each parallel step, all reads precede all writes

```
procedure parallelPrefixSums(A[0..n − 1])
    for r := 1, . . . ⌈lg n⌉ do
        step := 2^(r−1)
        for i := step, . . . n − 1 do in parallel
            A[i] := A[i] + A[i − step]      // assign to P_i
        end parallel for
    end for
```

$\Theta(n)$ work

$\Theta(1)$ time

PRAM: all synchronous
⇒ all reads happen before all writes

# Parallel prefix sums – Analysis

- **Time:**
  - all additions of one round run in parallel
  - $\lceil \lg n \rceil$ rounds
  - $\rightsquigarrow$ $\Theta(\log n)$ time    best possible!
- **Work:**
  - $\geq \frac{n}{2}$ additions in all rounds (except maybe last round)
  - $\rightsquigarrow$ $\Theta(n \log n)$ work
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  - ⇝ $\Theta(\log n)$ time best possible!
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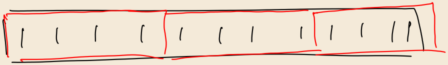
  - $\geq \frac{n}{2}$ additions in all rounds (except maybe last round)
  - ⇝ $\Theta(n \log n)$ work
  - more than the $\Theta(n)$ sequential algorithm!
- Typical trade-off: greater parallelism at the expense of more overall work
- For prefix sums:
  - can actually get $\Theta(n)$ work in *twice* that time!
  - ⇝ algorithm is slightly more complicated
  - instead here: linear work in *thrice* the time using "blocking trick"



$b = 4$

$\parallel$

$b = \Theta(\log n)$

work-inefficient

$\underbrace{1 \quad 2 \quad 3 \quad \boxed{4}}_{P_0} \quad \underbrace{1 \quad 2 \quad 3 \quad \boxed{4}}_{P_1} \quad \underbrace{1 \quad 2 \quad 3 \quad \boxed{4}}_{P_2}$

$O(b)$ time

$O(n)$ work

4 8 12

$O(\log n)$ time

$O(n' \log n')$

$n' = \frac{n}{b} = \frac{n}{\log n}$

$= O(n)$

$1 \quad 2 \quad 3 \quad 4 \quad \underbrace{5 \quad 6 \quad 7 \quad 8}_{P_1} \quad \underbrace{9 \quad 10 \quad 11 \quad 12}_{P_2}$

$O(b)$ time

$O(n)$ work

# Work-efficient parallel prefix sums

**standard trick to improve work:** compute small blocks sequentially

1. Set $b := \lceil \lg n \rceil$
2. For blocks of $b$ consecutive indices, i. e., $A[0..b)$, $A[b..2b)$, ... do in parallel: compute local prefix sums sequentially
3. Use previous work-inefficient algorithm only on rightmost elements of block, i. e., to compute prefix sums of $A[b-1]$, $A[2b-1]$, $A[3b-1]$, ...
4. For blocks $A[0..b)$, $A[b..2b)$, ... do in parallel: Add block-prefix sums to local prefix sums

**Analysis:**

- **Time:**
  - 2. & 4.: $\Theta(b) = \Theta(\log n)$ time
  - 3. $\Theta(\log(n/b)) = \Theta(\log n)$ times
- **Work:**
  - 2. & 4.: $\Theta(b)$ per block $\times$ $\lceil \frac{n}{b} \rceil$ blocks $\rightsquigarrow$ $\Theta(n)$
  - 3. $\Theta\left(\frac{n}{b} \log\left(\frac{n}{b}\right)\right) = \Theta(n)$

# Compacting subsequences

How do prefix sums help with sorting?   one more step to go . . .

**Goal:** *Compact* a subsequence of an array

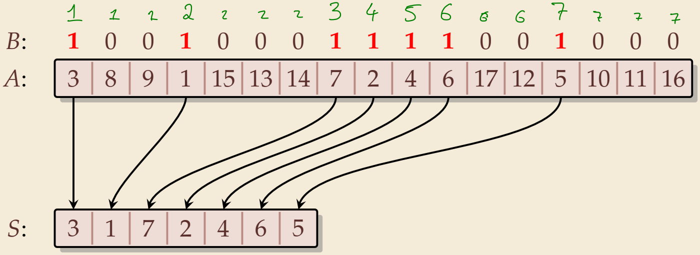
| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $B$: | **1** | 0 | 0 | **1** | 0 | 0 | 0 | **1** | **1** | **1** | **1** | 0 | 0 | **1** | 0 | 0 | 0 |
| $A$: | 3 | 8 | 9 | 1 | 15 | 13 | 14 | 7 | 2 | 4 | 6 | 17 | 12 | 5 | 10 | 11 | 16 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $S$: | 3 | 1 | 7 | 2 | 4 | 6 | 5 |

# Compacting subsequences

How do prefix sums help with sorting? one more step to go . . .

**Goal:** *Compact* a subsequence of an array




Use prefix sums on bitvector $B$

$\rightsquigarrow$ offset of selected cells in $S$

```
parallelPrefixSums(B)
for j := 0, ..., n − 1 do in parallel
    if B[j] == 1 then S[B[j] − 1] := A[j]
end parallel for
```

# Clicker Question

What is the parallel time and work achievable for *compacting* a subsequence of an array of size $n$?

**A** $O(1)$ time, $O(n)$ work

**B** $O(\log n)$ time, $O(n)$ work

**C** $O(\log n)$ time, $O(n \log n)$ work

**D** $O(\log^2 n)$ time, $O(n^2)$ work

**E** $O(n)$ time, $O(n)$ work

`pingo.upb.de/622222`

# Clicker Question

What is the parallel time and work achievable for *compacting* a subsequence of an array of size $n$?

- **A** ~~$O(1)$ time, $O(n)$ work~~
- **B** $O(\log n)$ time, $O(n)$ work ✓
- **C** ~~$O(\log n)$ time, $O(n \log n)$ work~~
- **D** ~~$O(\log^2 n)$ time, $O(n^2)$ work~~
- **E** ~~$O(n)$ time, $O(n)$ work~~

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## 3.7 Parallel sorting

# Parallel quicksort

Let's try to parallelize quicksort

- recursive calls can run in parallel (data independent)
- our sequential partitioning algorithm seems hard to parallelize

# Parallel quicksort

Let's try to parallelize quicksort

- recursive calls can run in parallel (data independent)
- our sequential partitioning algorithm seems hard to parallelize
- but can split partitioning into *rounds*:
  1. **comparisons:** compare all elements pivot (in parallel), store bitvector
  2. compute prefix sums of bit vectors (in parallel as above)
  3. **compact** subsequences of small and large elements (in parallel as above)

A 3 2 1 7 8 4 9 $\boxed{5}$

S 1 1 1 0 0 1 0

L 0 0 0 1 1 0 1

3 2 1 4 | 5 | 7 8 9

# Parallel quicksort – Code

```
procedure parQuicksort(A[l..r])
    b := choosePivot(A[l..r])
    j := parallelPartition(A[l..r], b)
    in parallel { parQuicksort(A[l..j − 1]), parQuicksort(A[j + 1..r]) }

procedure parallelPartition(A[l..r], b)
    swap(A[n − 1], A[b]);  p := A[n − 1]
    for i = 0, . . . , n − 2 do in parallel
        S[i] := [A[i] ≤ p]     // S[i] is 1 or 0
        L[i] := 1 − S[i]
    end parallel for
    in parallel { parallelPrefixSum(S[0..n − 2]);  parallelPrefixSum(L[0..n − 2]) }
    j := S[n − 2] + 1
    for i = 0, . . . , n − 2 do in parallel
        x := A[i]
        if x ≤ p then A[S[i] − 1] := x
        else A[j + L[i]] := x
    end parallel for
    A[j] := p
    return j
```

$$[\text{pred}] = \begin{cases} 1 & \text{pred true} \\ 0 & \text{else} \end{cases}$$

# Parallel quicksort – Analysis

- **Time:**
  - partition: all $O(1)$ time except prefix sums $\rightsquigarrow$ $\Theta(\log n)$ time
  - quicksort: expected depth of recursion tree is $\Theta(\log n)$
  
  $\rightsquigarrow$ total time $O(\log^2(n))$ in expectation

- **Work:**

  $O(n)$ with blocking

  - partition: $O(n)$ time except prefix sums $\rightsquigarrow$ $\Theta(n \log n)$ work
  
  $\rightsquigarrow$ quicksort $O(n \log^2(n))$ work in expectation
  - using a work-efficient prefix-sums algorithm yields (expected) work-efficient sorting!

# Parallel mergesort

- As for quicksort, recursive calls can run in parallel ✓

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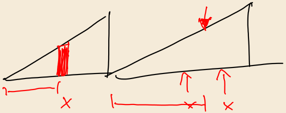
- how about merging sorted halves $A[l..m-1]$ and $A[m..r]$?
- Must treat elements independently.

# Parallel mergesort




- As for quicksort, recursive calls can run in parallel ✓
- how about merging sorted halves $A[l..m-1]$ and $A[m..r]$?
- Must treat elements independently.
  - correct position of $x$ in sorted output = *rank* of $x$ (#elements $\le x$) — breaking ties by position in $A$
  - # elements $\le x$ $=$ # elements from $A[l..m-1]$ that are $\le x$ + # elements from $A[m..r]$ that are $\le x$
  - Note: rank in own run is simply the index of $x$ in that run
  - find rank in *other* run by binary search

$\rightsquigarrow$ can move it to correct position

# Parallel mergesort – Analysis

- **Time:**
  - merge: $\Theta(\overset{\log n}{\cancel{n}})$ from binary search, rest $O(1)$ $\Theta(\log n)$
  - mergesort: depth of recursion tree is $\Theta(\log n)$
  
  $\rightsquigarrow$ total time $O(\log^2(n))$ w.c.

- **Work:**
  - merge: $n$ binary searches $\rightsquigarrow$ $\Theta(n \log n)$
  
  $\rightsquigarrow$ mergesort: $O(n \log^2(n))$ work

# Parallel mergesort – Analysis

- **Time:**
  - merge: $\Theta(n)$ from binary search, rest $O(1)$
  - mergesort: depth of recursion tree is $\Theta(\log n)$
  - $\rightsquigarrow$ total time $O(\log^2(n))$
- **Work:**
  - merge: $n$ binary searches $\rightsquigarrow$ $\Theta(n \log n)$
  - $\rightsquigarrow$ mergesort: $O(n \log^2(n))$ work
  - work can be reduced to $\Theta(n)$ for merge
    - do full binary searches only for regularly sampled elements
    - ranks of remaining elements are sandwiched between sampled ranks
    - use a sequential method for small blocks, treat blocks in parallel
    - (detailed omitted)

# Parallel sorting – State of the art

- more sophisticated methods can sort in $O(\log n)$ parallel time on CREW-RAM
- practical challenge: small units of work add overhead
- need a lot of PEs to see improvement from $O(\log n)$ parallel time

$\rightsquigarrow$ implementations tend to use simpler methods above

- check the Java library sources for interesting examples!
  `java.util.Arrays.parallelSort(int[])`