8 Error-Correcting Codes

21 April 2020

Sebastian Wild

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Outline

8.1 Introduction

Noisy Communication

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  - humans: acoustic noise, unclear pronunciation, misunderstanding, foreign languages

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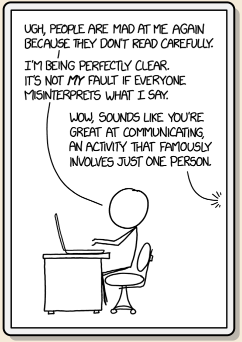
⇝ We can

1. **detect errors** "This sentence has aao pi dgsdho gioasghds."
2. **correct** (some) **errors** "Tiny errs ar corrrected automaticly."
   (sometimes too eagerly as in the Chinese Whispers / Telephone)



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- transmit a binary string
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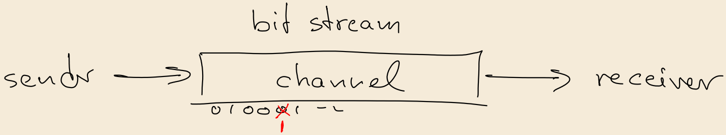

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  1. **error detection** ⇝ can request a re-transmit
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- This will require *redundancy*: sending *more* bits than plain message
  ⇝ **goal:** robust code with lowest redundancy

that's the opposite of compression!

Clicker Question

What do you think, how many extra bits (percentage of message) do we need to **detect** a **single bit error**?
(Answer 100 if you think we have to double the message length.)

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8.2 Lower Bounds

Block codes

- **model:**
  - want to send message $S \in \{0,1\}^\star$ (bitstream) across a *(communication) channel*
  - any bit transmitted through the channel might ***flip*** ($0 \to 1$ resp. $1 \to 0$)
    no other errors occur (no bits lost, duplicated, inserted, etc.)
  - instead of S, we send *encoded bitstream* $C \in \{0,1\}^\star$
    sender *encodes* S to C, receiver *decodes* C to S (hopefully)
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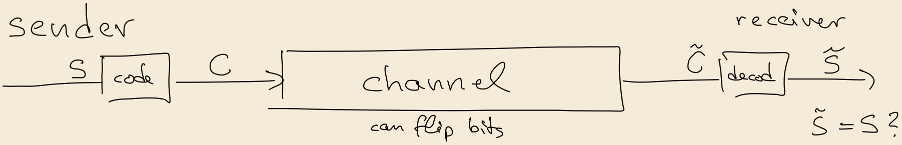

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- all codes discussed here are ***block codes***
  - divide S into *messages* $m \in \{0,1\}^k$ of k bits each (k = *message length*)
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$\frac{n-k}{k}$ = redundancy

- between 0 and n bits might be flipped
  - how many flipped bits can we definitely **detect**? (invalid code)
  - how many flipped bits can we **correct** without retransmit? (i. e. decoding m still possible)

Code distance

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- define $\mathcal{C}$ = set of all codewords = $C(\{0,1\}^k)$ = $\{ y \in \{0,1\}^n : \exists x \in \{0,1\}^k : C(x) = y \}$

$\rightsquigarrow \mathcal{C} \subseteq \{0,1\}^n$

$|\mathcal{C}| = 2^k$ out of 2^n n-bit strings are valid codewords

- decoding = finding closest valid codeword

receive block $C \notin \mathcal{C}$ → error has occurred

map C to $\arg\min_{m \in \{0,1\}^k} d_H(C(m), C)$

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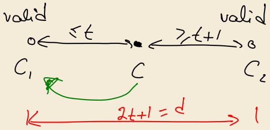
Implications for codes

1. need distance d to **detect** errors flipping up to $d-1$ bits
2. need distance d to **correct** errors flipping up to $\left\lfloor \frac{d-1}{2} \right\rfloor$ bits



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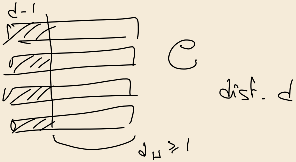
  - *proof sketch:* We have 2^k codeswords with distance d
    after deleting the first $d-1$ bits, all are still distinct
    but there are only $2^{n-(d-1)}$ such shorter bitstrings.



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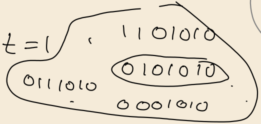
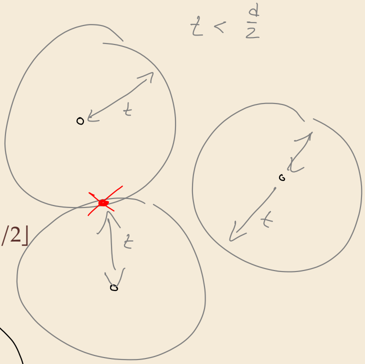
- ***Hamming bound:*** $2^k \le \dfrac{2^n}{\sum_{f=0}^{\lfloor (d-1)/2 \rfloor} \binom{n}{f}}$
  - *proof idea:* consider "balls" of bitstrings around codewords
    count bitstrings with Hamming-distance $\le t = \lfloor (d-1)/2 \rfloor$
    correcting t errors means all these balls are disjoint
    so $2^k \cdot$ ball size $\le 2^n$



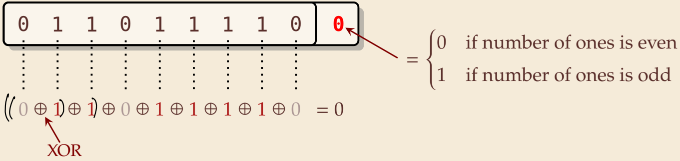


$\rightsquigarrow$ We will come back to these.

8.3 Hamming Codes

Parity Bit

- simplest possible error-detecting code: add a **parity bit**



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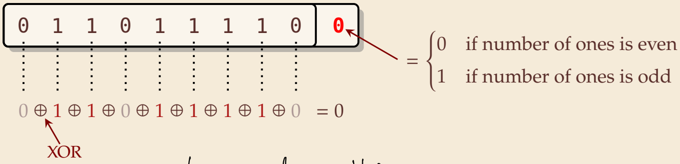

⇝ code distance 2 → cannot correct anything

- can detect any single-bit error (actually, any odd number of flipped bits)
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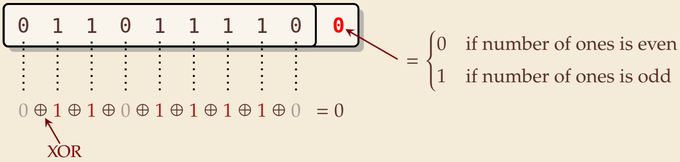

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- can detect any single-bit error (actually, any odd number of flipped bits)
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very simple and cheap

cannot correct any errors

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one extra bit

$$\frac{k+1}{k}$$

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Error-correcting codes

any downtime is expensive!

- typical application: heavy-duty server RAM
  - bits can randomly flip (e. g., by cosmic rays)
  - individually very unlikely,
    but in always-on server with lots of RAM, it happens!

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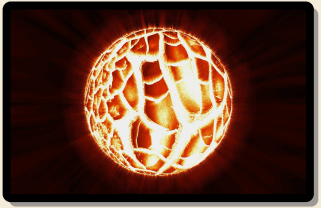

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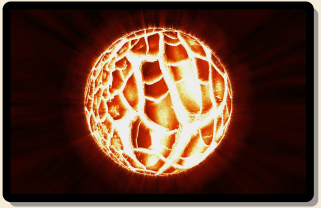

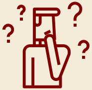

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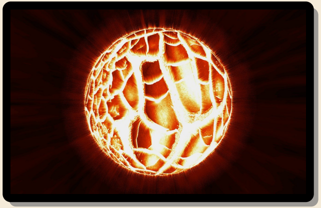

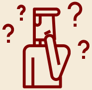

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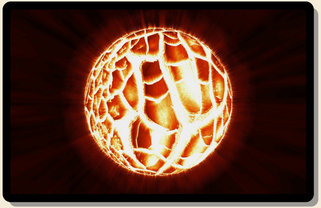

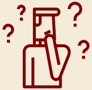

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redundancy 200%

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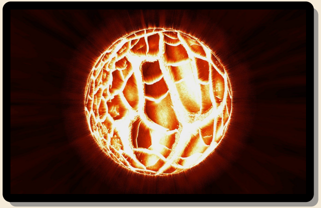

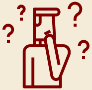

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Can do it with 11% extra memory! (instead of 200% (!))

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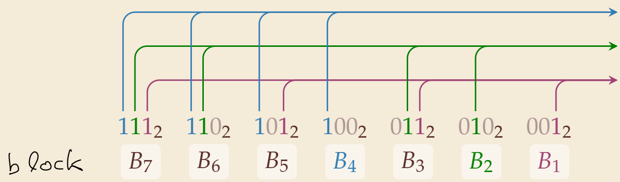

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B codeword iff $C = 0$

$B = B_7 \ldots B_1$, $C = C_2 C_1 C_0$

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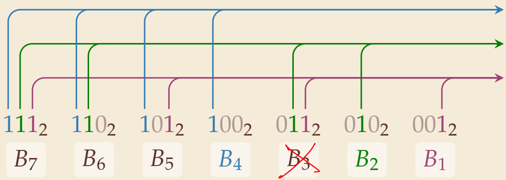

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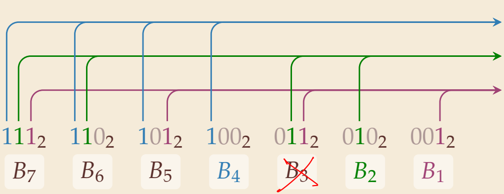

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$3 = 011_2$

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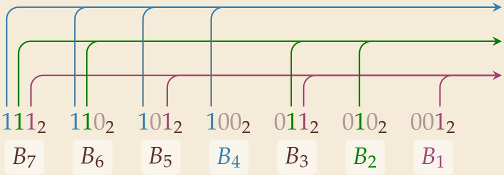

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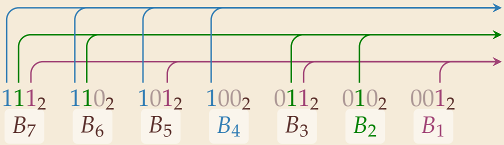
$C_j = 1$ iff jth bit in binary representation of i is 1 ⇝ *C encodes **position of error**!*

4+3 Hamming Code

- *How can we turn this into a code?*



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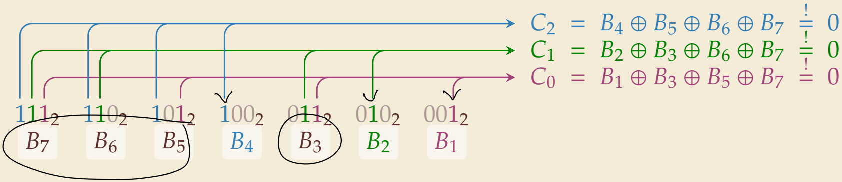

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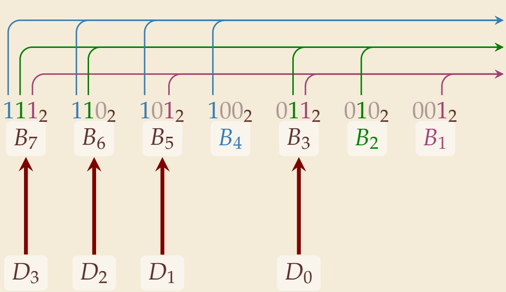

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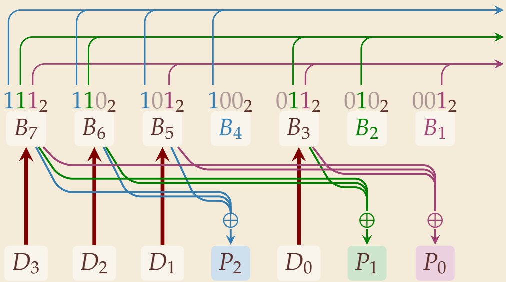
  2. copy $D_3 D_2 D_1 D_0$ to $B_7 B_6 B_5 B_3$

4+3 Hamming Code

- *How can we turn this into a code?*



$$\left.\begin{aligned} C_2 &= B_4 \oplus B_5 \oplus B_6 \oplus B_7 \overset{!}{=} 0 \\ C_1 &= B_2 \oplus B_3 \oplus B_6 \oplus B_7 \overset{!}{=} 0 \\ C_0 &= B_1 \oplus B_3 \oplus B_5 \oplus B_7 \overset{!}{=} 0 \end{aligned}\right)$$

$$\begin{aligned} P_2 &= D_3 \oplus D_2 \oplus D_1 \\ P_1 &= D_3 \oplus D_2 \oplus D_0 \\ P_0 &= D_3 \oplus D_1 \oplus D_0 \end{aligned}$$

- B_4, B_2 and B_1 occur only in one constraint each $\rightsquigarrow$ **define** them based on rest!
- **4 + 3 *Hamming Code* – Encoding**
  1. **Given:** message $D_3D_2D_1D_0$ of length $k = 4$
  2. copy $D_3D_2D_1D_0$ to $B_7B_6B_5B_3$
  3. compute $P_2P_1P_0 = B_4B_2B_1$ so that $C = 0$

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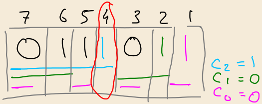
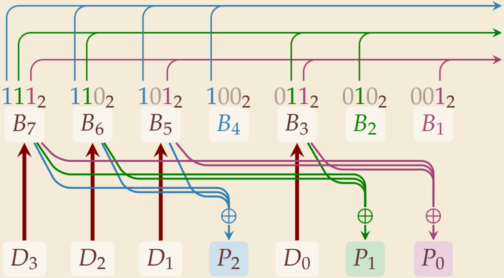


$C = 100_2 = 4$

$D = 0110$



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$P_2 = D_3 \oplus D_2 \oplus D_1$ $0 \oplus 1 \oplus 1 = 0$

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  4. send $D_3D_2D_1P_2D_0P_1P_0$

$D = 0110$

0110011

4+3 Hamming Code – Decoding

- **4 + 3 *Hamming Code* – Decoding**
  1. **Given:** block $B_7 B_6 B_5 B_4 B_3 B_2 B_1$ of length $n = 7$
  2. compute C (as above)
  3. if $C = 0$ no (detectable) error occurred
     otherwise, flip B_C (the Cth bit was twisted)
  4. return 4-bit message $B_7 B_6 B_5 B_3$

0 1 1 1 0 1 1

→ 0 1 1 0

4+3 Hamming Code – Decoding

- **4 + 3 *Hamming Code* – Decoding**
  1. **Given:** block $B_7B_6B_5B_4B_3B_2B_1$ of length $n = 7$
  2. compute C (as above)
  3. if $C = 0$ no (detectable) error occurred
     otherwise, flip B_C (the Cth bit was twisted)
  4. return 4-bit message $B_7B_6B_5B_3$

- **Properties**
  - distance $d = 3$
  - can *correct* any 1-bit error
  - How about 2-bit errors?
    - We can *detect* that *something* went wrong.
    - **But:** above decoder mistakes it for a (different!) 1-bit error and "corrects" that

Hamming Codes – General recipe

- construction can be generalized:
  - Start with $n = 2^\ell - 1$ bits for $\ell \in \mathbb{N}$ (we had $\ell = 3$)
  - use the ℓ bits whose index is a power of 2 as parity bits
  - the other $n - \ell$ are data bits

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👍 simple and efficient coding / decoding

👍 fairly space-efficient