3 Efficient Sorting

24 February 2021

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Outline

3 Efficient Sorting

Why study sorting?

- fundamental problem of computer science that is still not solved

  Algorithm with optimal #comparisons in worst case?

- building brick of many more advanced algorithms
  - for preprocessing
  - as subroutine
- playground of manageable complexity
  to practice algorithmic techniques

Here:

- "classic" fast sorting method
- **parallel** sorting

Part I

The Basics

Rules of the game

- **Given:**
  - array $A[0..n-1]$ of n objects
  - a total order relation $\leq$ among $A[0], \ldots, A[n-1]$
    (a comparison function) Comparator (Java) x.compareTo(y) <= 0

    $x \leq 0$
- **Goal:** rearrange (=permute) elements within A,
  so that A is *sorted*, i. e., $A[0] \leq A[1] \leq \cdots \leq A[n-1]$

- for now: A stored in main memory (*internal sorting*)
  single processor (*sequential sorting*)

Clicker Question

What is the complexity of sorting? Type you answer, e. g., as "`Theta(sqrt(n))`"

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3.1 Mergesort

Clicker Question

How does mergesort work?

- **A** Split elements around median, then recurse on small / large elements.
- **B** Recurse on left / right half, then combine sorted halves.
- **C** Grow sorted part on left, repeatedly add next element to sorted range.
- **D** Repeatedly choose 2 elements and swap them if they are out of order.
- **E** Don’t know.

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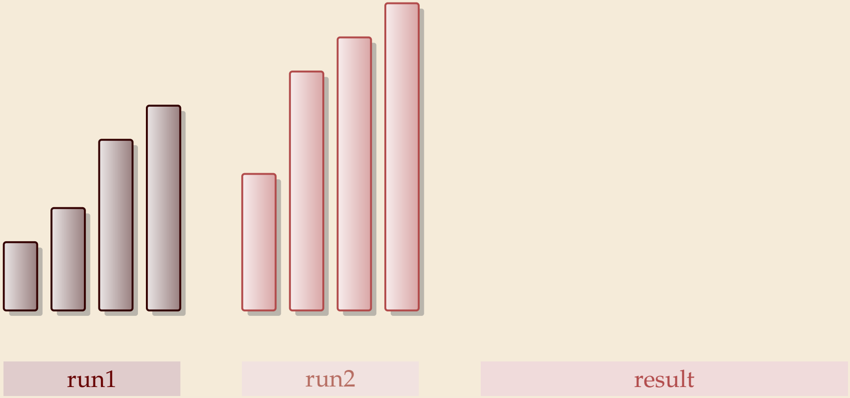
Merging sorted lists

Merging sorted lists



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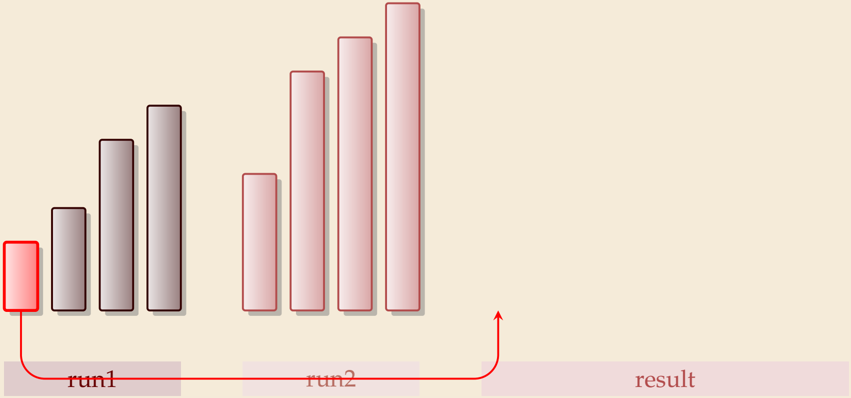

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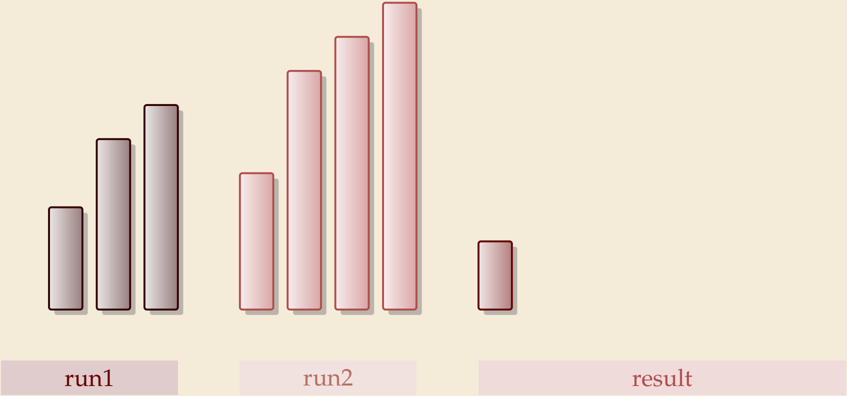

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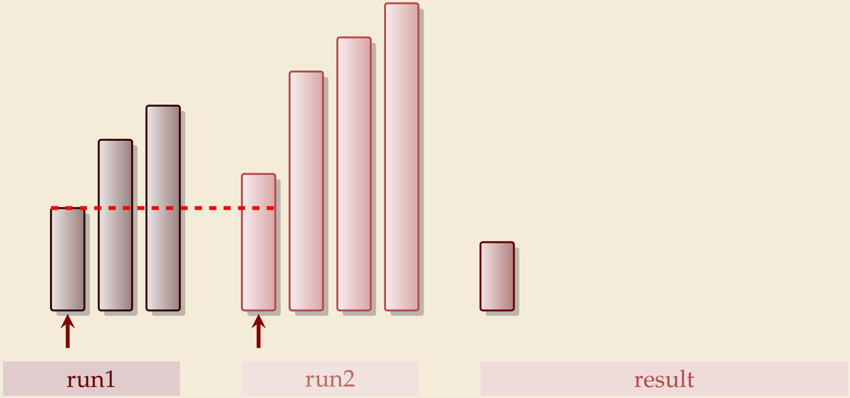

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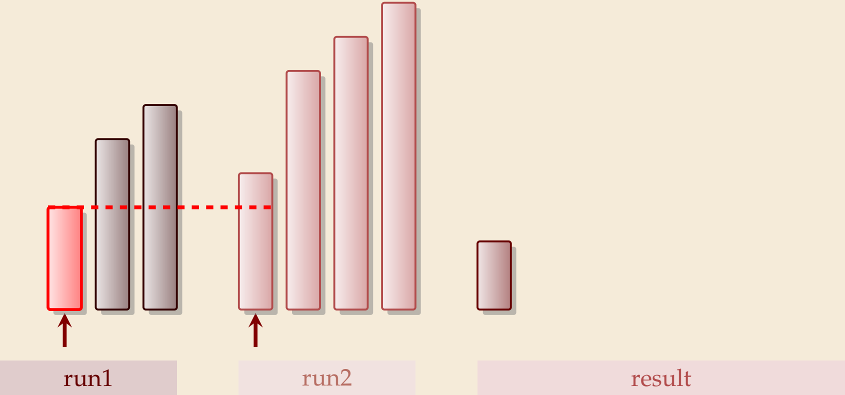

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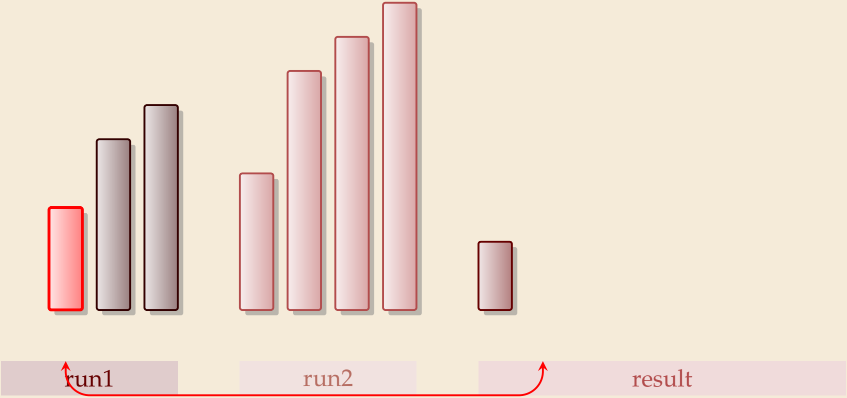

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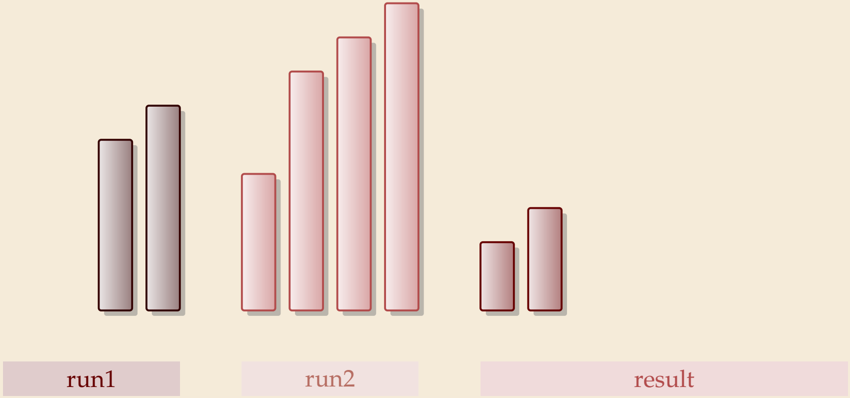

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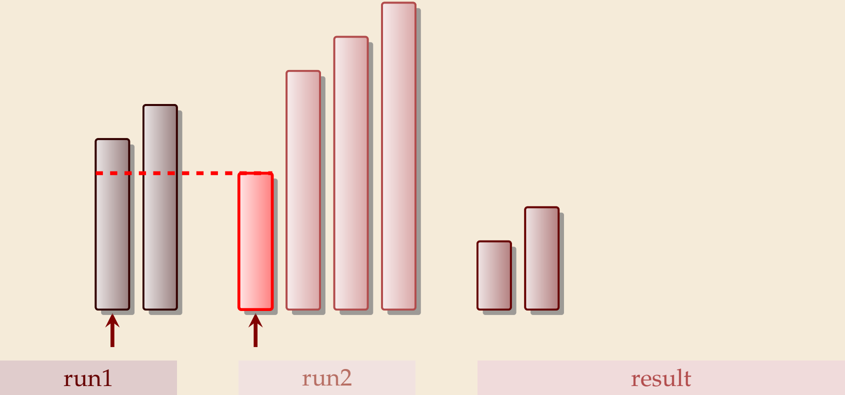

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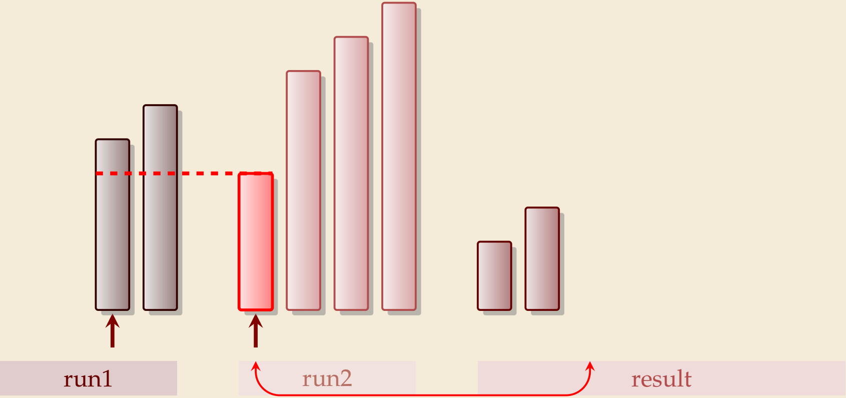

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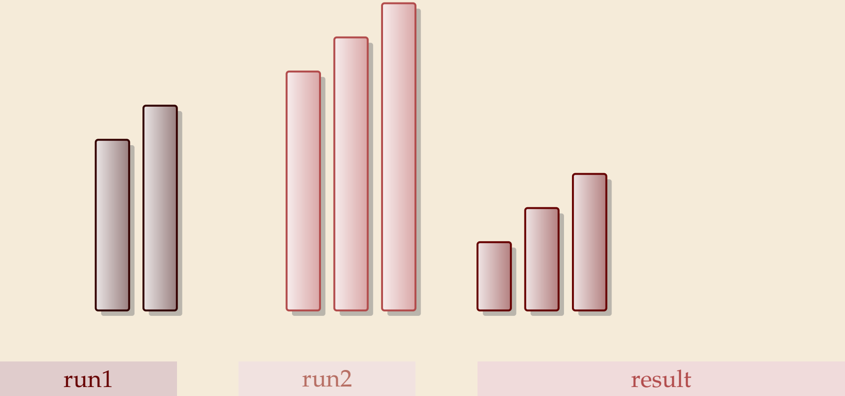

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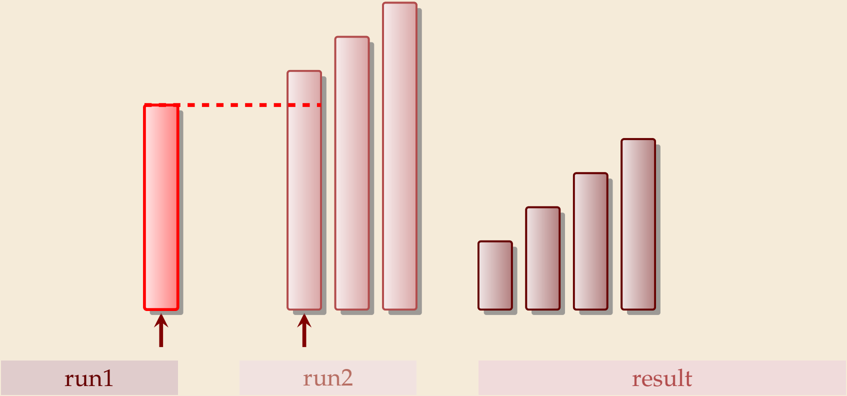

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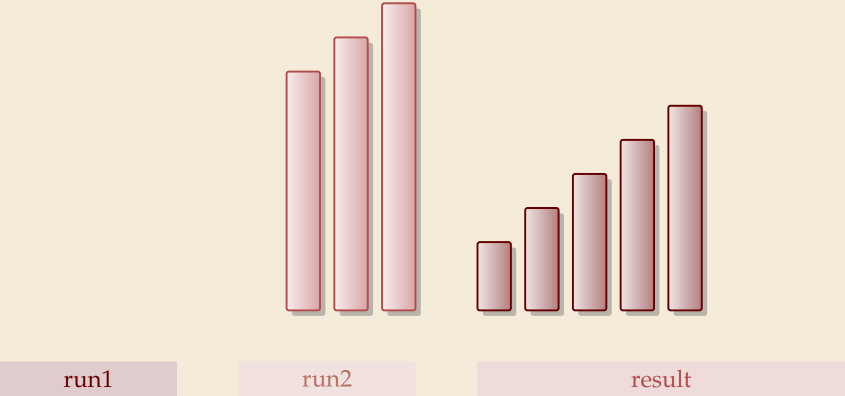

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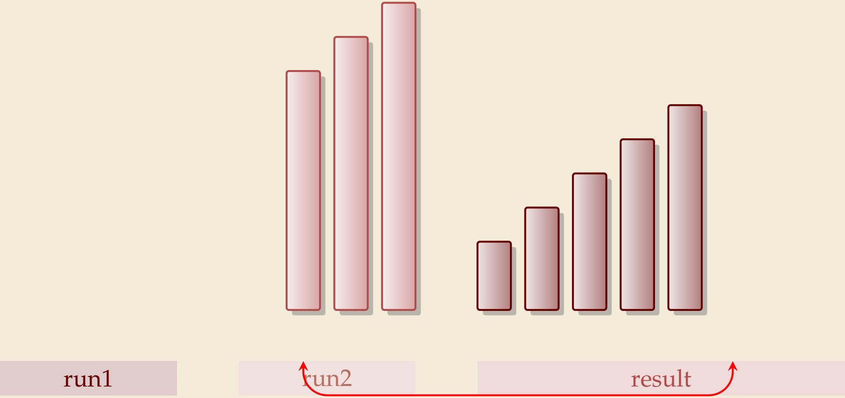

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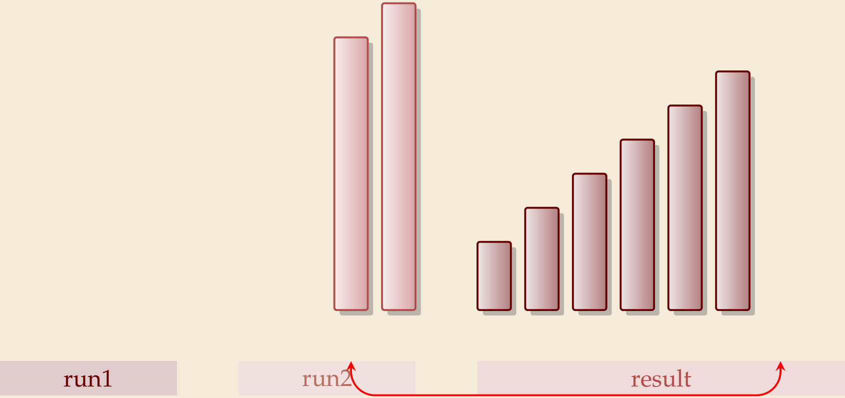

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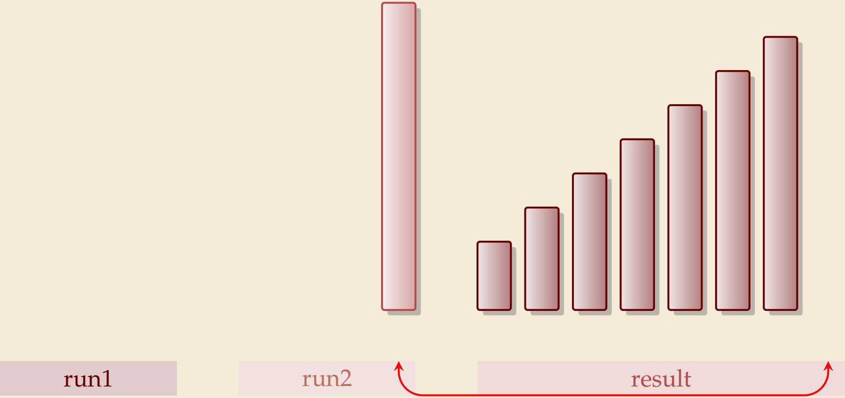

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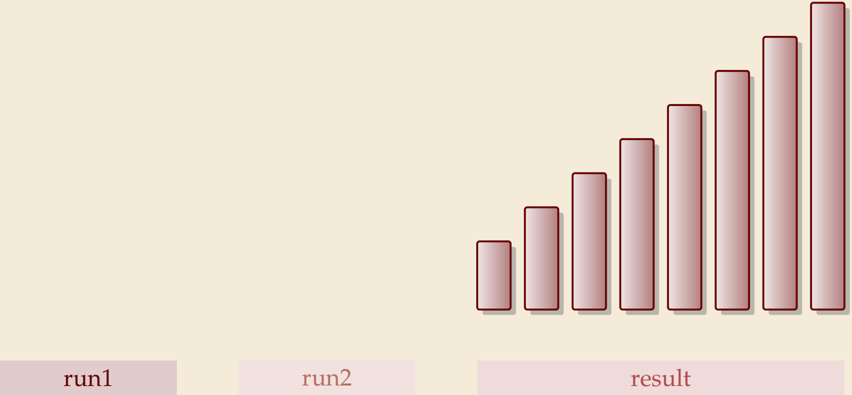

Clicker Question

What is the worst-case running time of mergesort?

A $\Theta(1)$

B $\Theta(\log n)$

C $\Theta(\log \log n)$

D $\Theta(\sqrt{n})$

E $\Theta(n)$

F $\Theta(n \log \log n)$

G $\Theta(n \log n)$

H $\Theta(n \log^2 n)$

I $\Theta(n^{1+\epsilon})$

J $\Theta(n^2)$

K $\Theta(n^3)$

L $\Theta(2^n)$

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Click on "Polls" tab

Clicker Question

What is the worst-case running time of mergesort?

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Mergesort

```
procedure mergesort(A[l..r])
    n := r − l + 1
    if n ≥ 1 return
    m := l + ⌊n/2⌋
    mergesort(A[l..m − 1])
    mergesort(A[m..r])
    merge(A[l..m − 1], A[m..r], buf)
    copy buf to A[l..r]
```

- recursive procedure; *divide & conquer*
- merging needs
  - temporary storage for result of same size as merged runs
  - to read and write each element twice (once for merging, once for copying back)

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- recursive procedure; *divide & conquer*
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Analysis: count "element visits" (read and/or write)

$$C(n) = \begin{cases} 0 & n \le 1 \\ C(\lfloor n/2 \rfloor) + C(\lceil n/2 \rceil) + 2n & n \ge 2 \end{cases}$$

same for best and worst case!

Simplification $\boxed{n = 2^k}$ $k \in \mathbb{N}$ $k = \lg n$

telescoping

$$C(2^k) = \begin{cases} 0 & k \le 0 \\ 2 \cdot C(2^{k-1}) + 2 \cdot 2^k & k \ge 1 \end{cases}$$

$k-1$

$$= \underbrace{2 \cdot 2^k + 2^2 \cdot 2^{k-1} + 2^3 \cdot 2^{k-2} + \cdots + 2^k \cdot 2^1}_{k \text{ summands}} = 2k \cdot 2^k$$

2^{k+1}

$$C(n) = 2n \lg(n) = \Theta(n \log n)$$

Mergesort – Discussion

- optimal time complexity of $\Theta(n \log n)$ in the worst case
- *stable* sorting method i. e., retains relative order of equal-key items
- memory access is sequential (scans over arrays)
- requires $\Theta(n)$ extra space

  there are in-place merging methods,
  but they are substantially more complicated
  and not (widely) used

3.2 Quicksort

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Partitioning around a pivot

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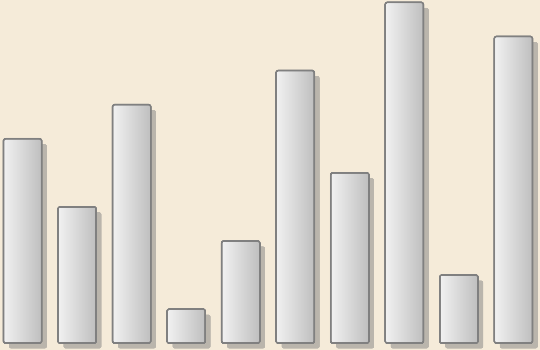

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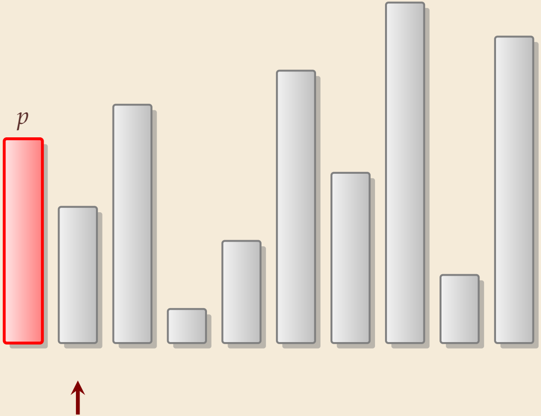

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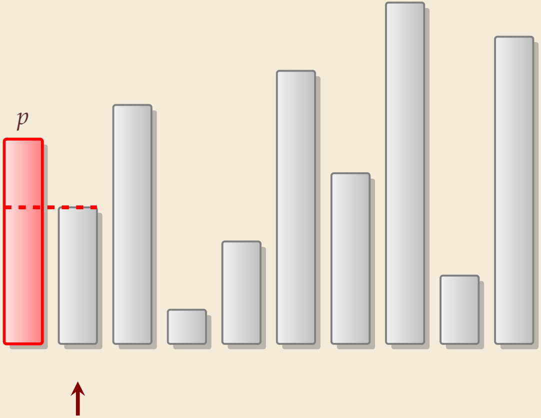

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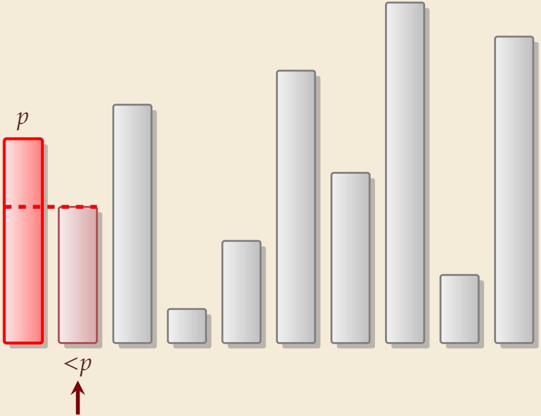

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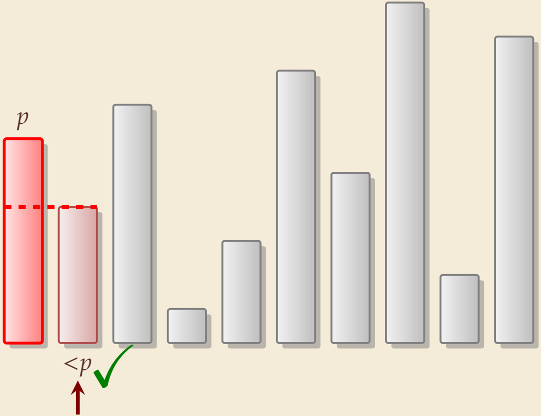

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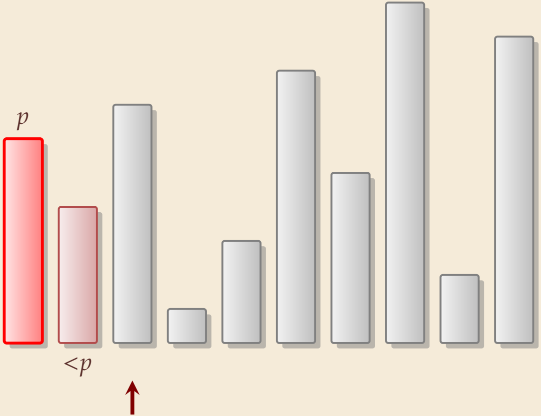

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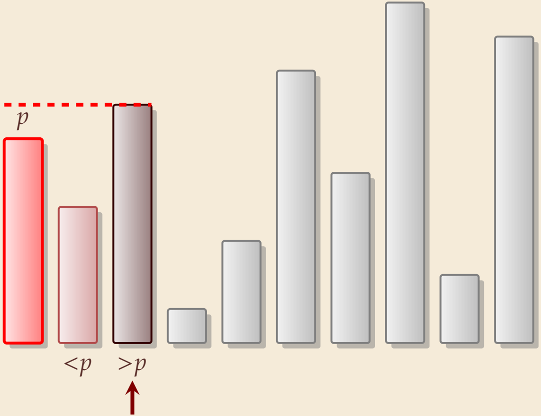

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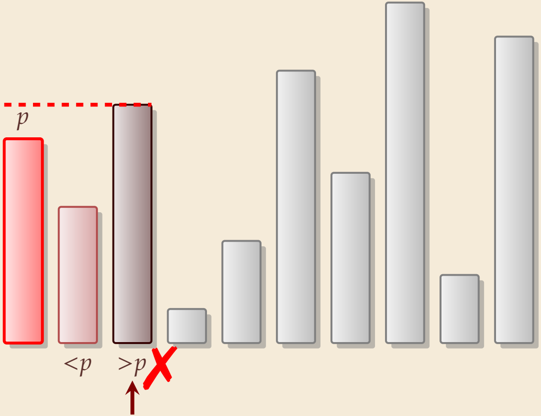

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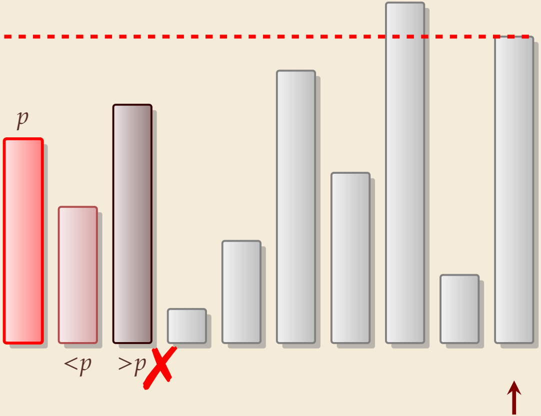

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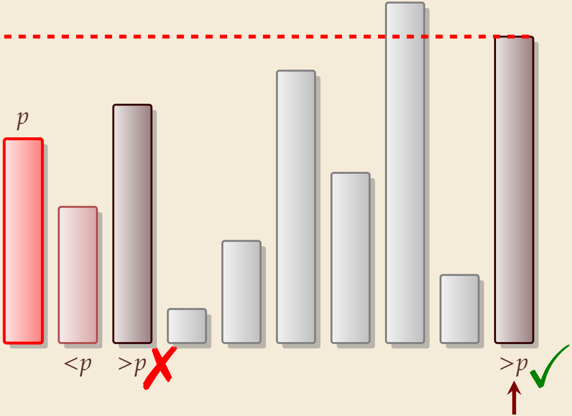

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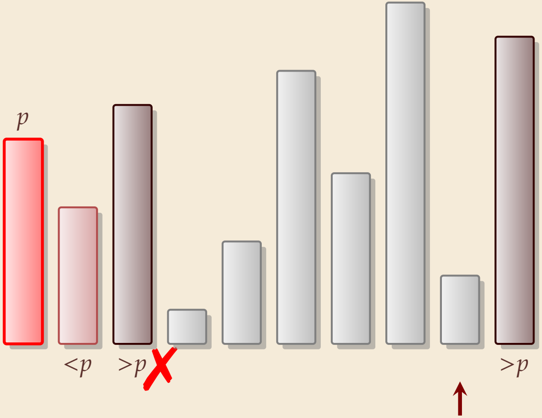

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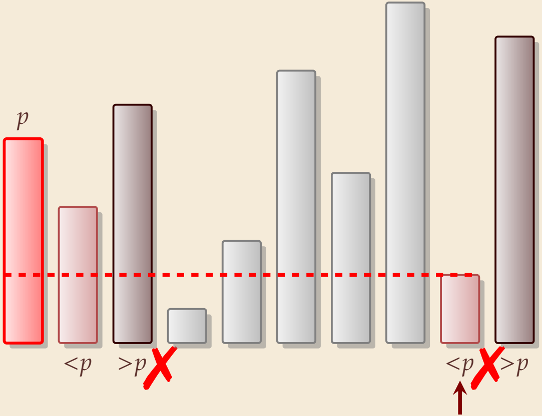

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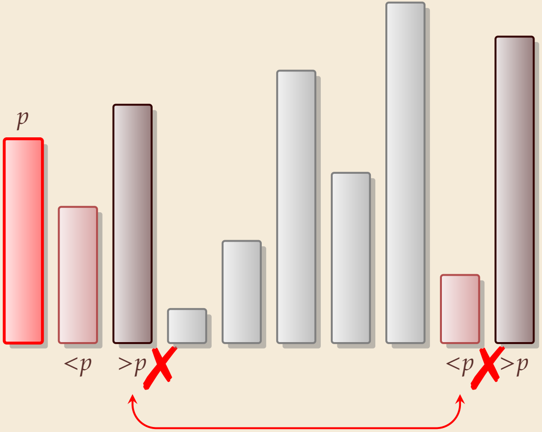

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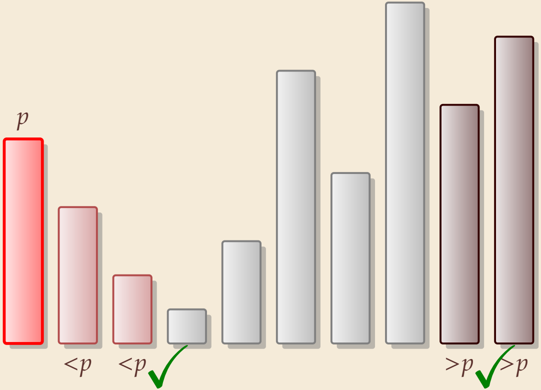

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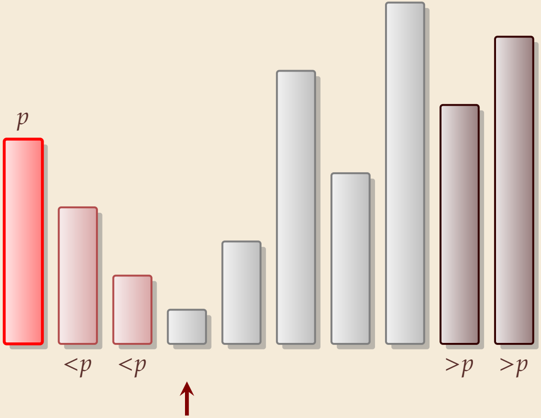

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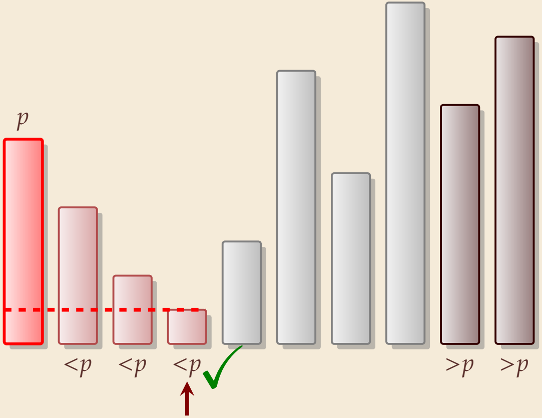

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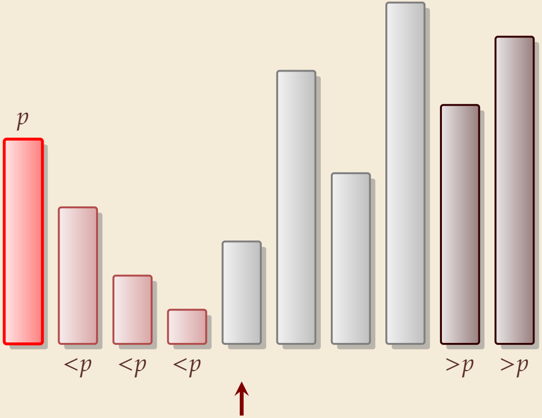

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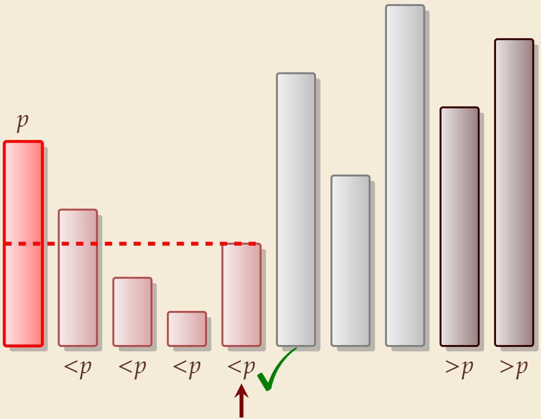

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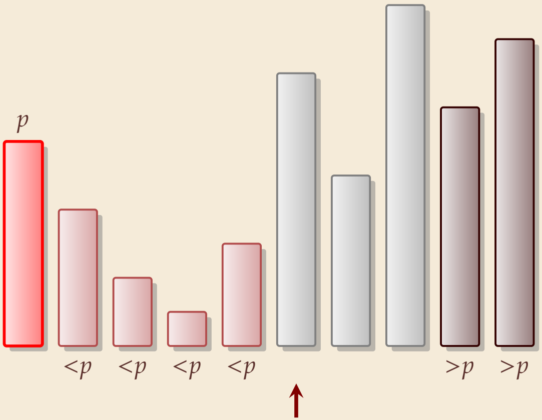

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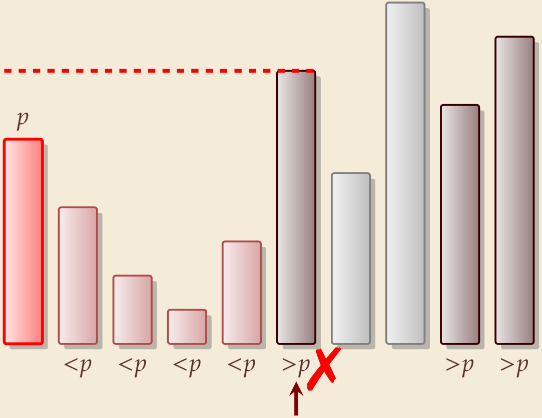

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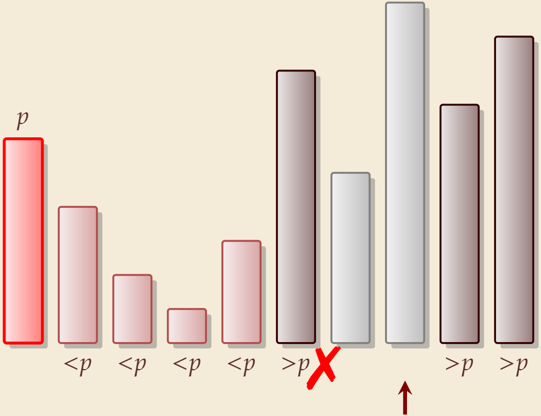

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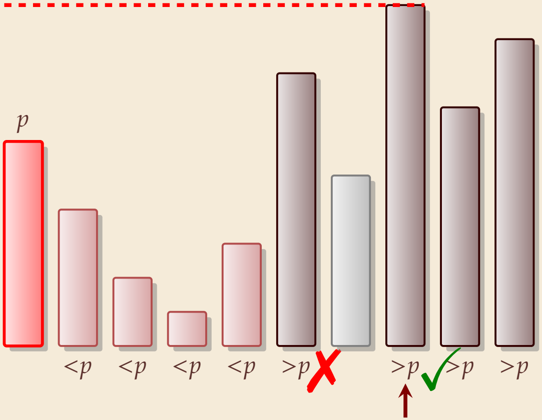

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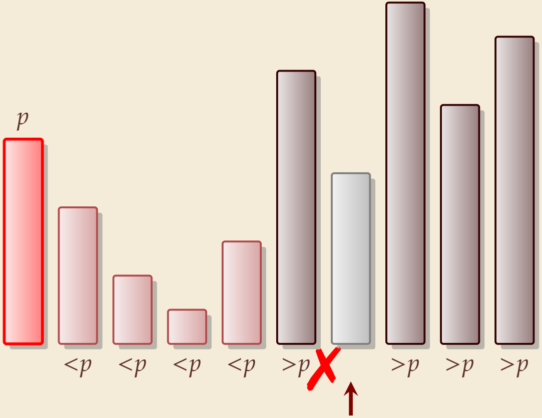

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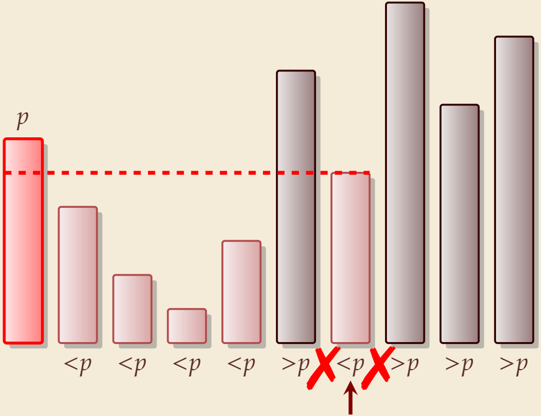

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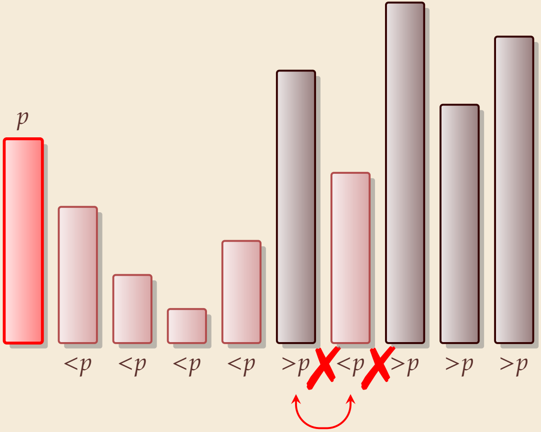

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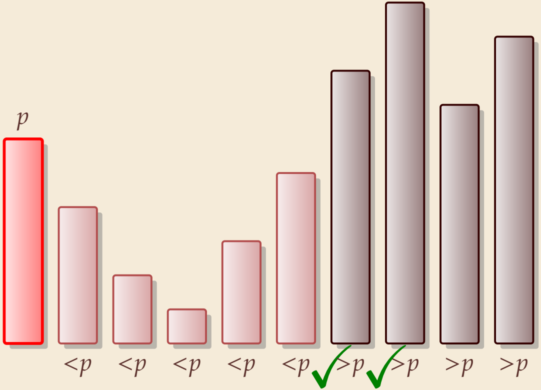

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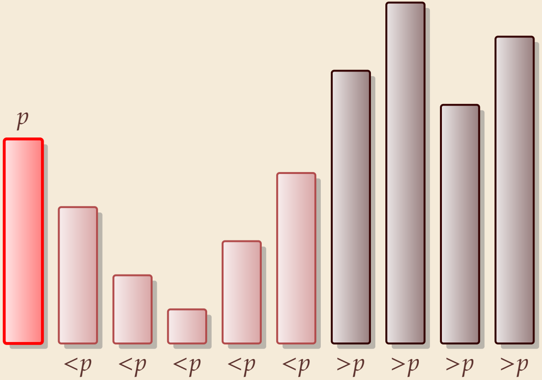

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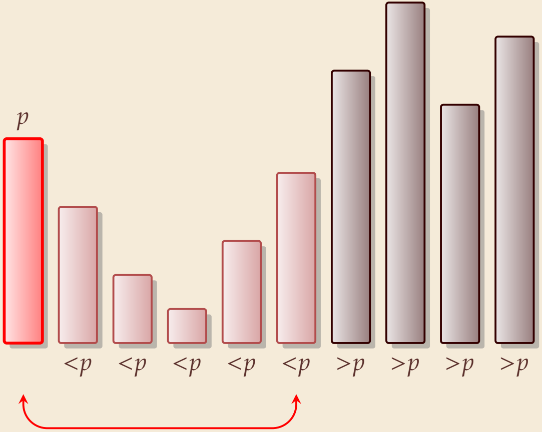

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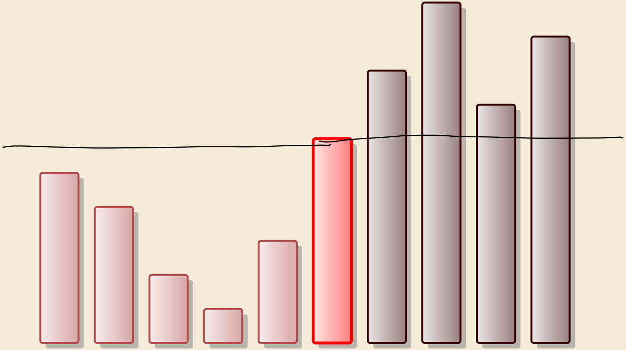

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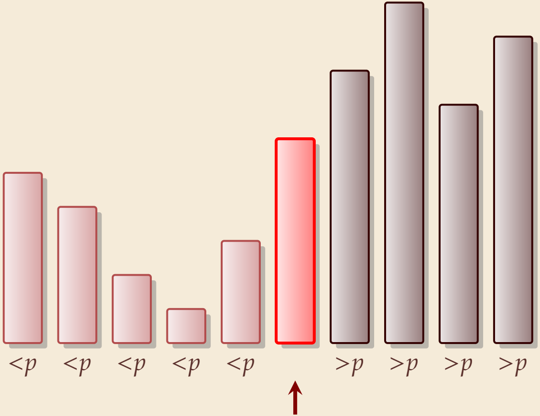

- no extra space needed
- visits each element once
- returns rank/position of pivot

Partitioning – Detailed code

> Beware: details easy to get wrong; use this code!

```
1  procedure partition(A, b)
2      // input: array A[0..n − 1], position of pivot b ∈ [0..n − 1]
3      swap(A[0], A[b])
4      i := 0,    j := n
5      while true do
6          do i := i + 1 while i < n and A[i] < A[0]
7          do j := j − 1 while j ≥ 1 and A[j] > A[0]
8          if i ≥ j then break      (goto 8)
9          else swap(A[i], A[j])
10     end while
11     swap(A[0], A[j])
12     return j
```

Loop invariant (5–10): A

p	$\le p$	?	$\ge p$
	i	j	

Quicksort

```
procedure quicksort(A[l..r])
    if l ≥ r then return
    b := choosePivot(A[l..r])
    j := partition(A[l..r], b)
    quicksort(A[l..j − 1])
    quicksort(A[j + 1..r])
```

- recursive procedure; *divide & conquer*
- choice of pivot can be
  - fixed position $\rightsquigarrow$ dangerous!
  - random
  - more sophisticated, e. g., median of 3

Clicker Question

What is the worst-case running time of quicksort?

- **A** $\Theta(1)$
- **B** $\Theta(\log n)$
- **C** $\Theta(\log\log n)$
- **D** $\Theta(\sqrt{n})$
- **E** $\Theta(n)$
- **F** $\Theta(n \log\log n)$
- **G** $\Theta(n \log n)$
- **H** $\Theta(n \log^2 n)$
- **I** $\Theta(n^{1+\epsilon})$
- **J** $\Theta(n^2)$
- **K** $\Theta(n^3)$
- **L** $\Theta(2^n)$

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D $\cancel{\Theta(\sqrt{n})}$	**J** $\cancel{\Theta(n^2)}$
E $\cancel{\Theta(n)}$	**K** $\cancel{\Theta(n^3)}$
F $\cancel{\Theta(n \log \log n)}$	**L** $\cancel{\Theta(2^n)}$

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Click on “Polls” tab

Quicksort & Binary Search Trees

Quicksort

7	4	2	9	1	3	8	5	6

Quicksort & Binary Search Trees

Quicksort

7	4	2	9	1	3	8	5	6

Quicksort & Binary Search Trees

Quicksort

7	4	2	9	1	3	8	5	6

4	2	1	3	5	6

9	8

Quicksort & Binary Search Trees

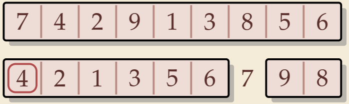
Quicksort

7	4	2	9	1	3	8	5	6

4	2	1	3	5	6	7	9	8

Quicksort & Binary Search Trees

Quicksort



Quicksort & Binary Search Trees

Quicksort

7	4	2	9	1	3	8	5	6

4	2	1	3	5	6		7		9	8

2	1	3		4		5	6

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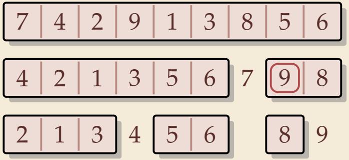

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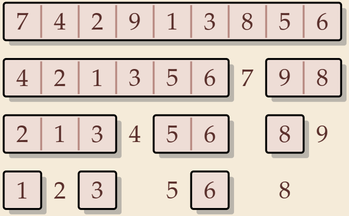

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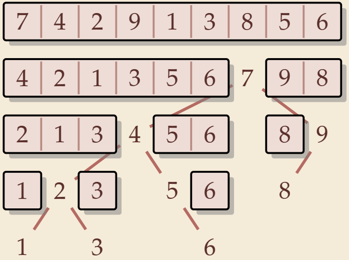

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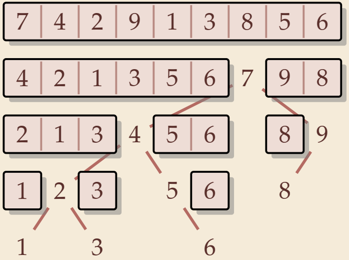
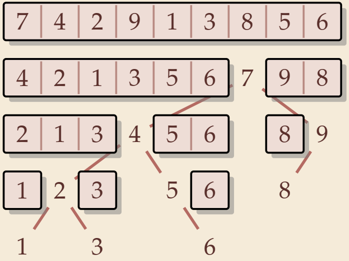


Binary Search Tree (BST)

7 4 2 9 1 3 8 5 6

Quicksort & Binary Search Trees

Quicksort



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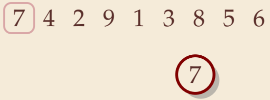

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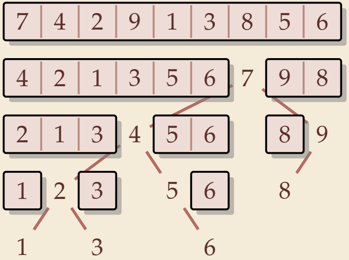

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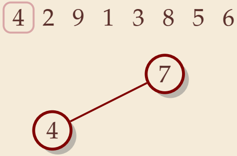

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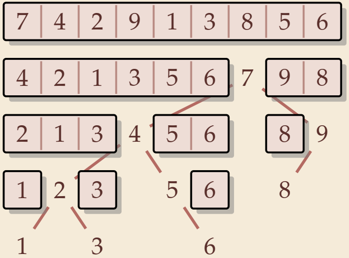

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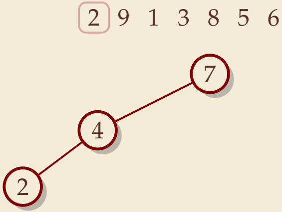

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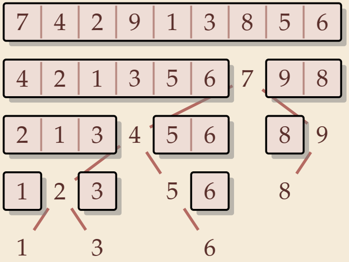

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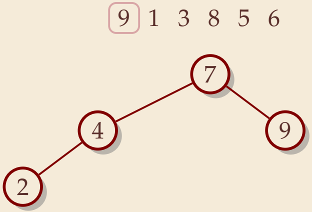

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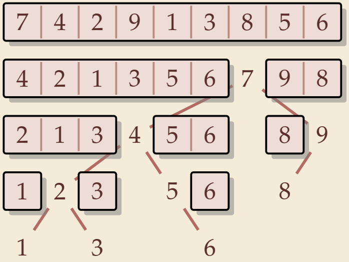

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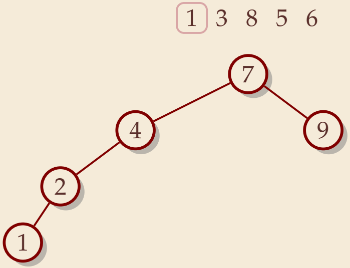

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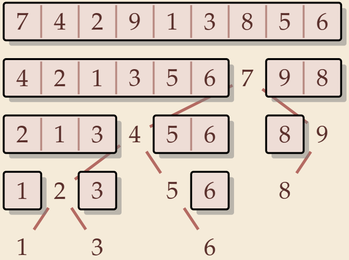

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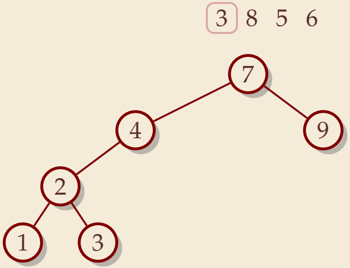

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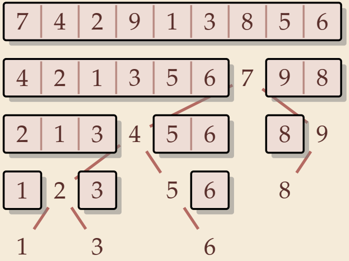

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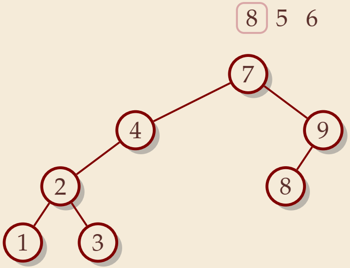

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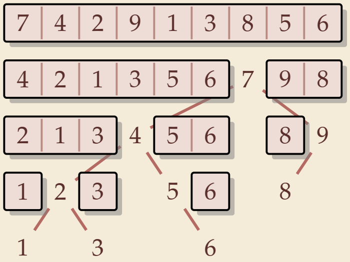

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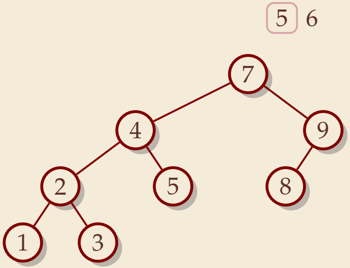

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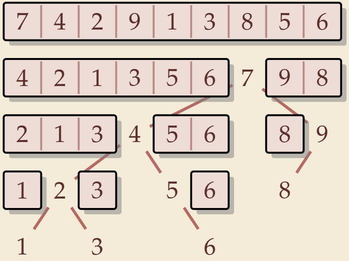

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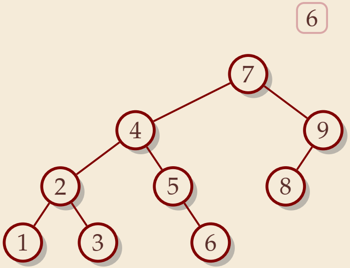

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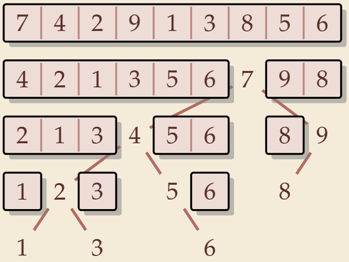

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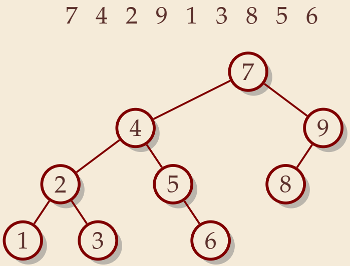
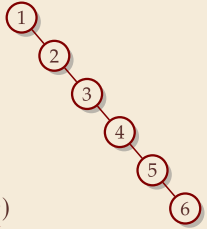


- recursion tree of quicksort = binary search tree from successive insertion
- comparisons in quicksort = comparisons to built BST
- comparisons in quicksort ≈ comparisons to search each element in BST

Quicksort – Worst Case

- Problem: BSTs can degenerate
- Cost to search for k is $k - 1$

$\leadsto$ Total cost $\sum_{k=1}^{n}(k-1) \;=\; \dfrac{n(n-1)}{2} \;\sim\; \frac{1}{2}n^2$



$\leadsto$ quicksort worst-case running time is in $\Theta(n^2)$

terribly slow!

But, we can fix this:

Randomized quicksort:

- choose a *random pivot* in each step

$\leadsto$ same as randomly *shuffling* input before sorting

Randomized Quicksort – Analysis

- $C(n)$ = element visits (as for mergesort)

$\rightsquigarrow$ quicksort needs $\sim \underline{2\ln(2) \cdot n \lg n} \approx 1.39 n \lg n$ *in expectation*

Mergesort $2n \lg n$

- also: very unlikely to be much worse:
  e. g., one can prove: $\Pr[\text{cost} > 10 n \lg n] = O(n^{-2.5})$
  distribution of costs is "concentrated around mean"
- intuition: have to be *constantly* unlucky with pivot choice

Quicksort – Discussion

- 👍 fastest general-purpose method
- 👍 $\Theta(n \log n)$ average case
- 👍 works *in-place* (no extra space required)
- 👍 memory access is sequential (scans over arrays)
- 👎 $\Theta(n^2)$ worst case (although extremely unlikely)
- 👎 not a *stable* sorting method

Open problem: Simple algorithm that is fast, stable and in-place.

3.3 Comparison-Based Lower Bound

Lower Bounds

- **Lower bound:** mathematical proof that *no algorithm* can do better.
  - very powerful concept: bulletproof *impossibility* result
    $\approx$ *conservation of energy* in physics
  - **(unique?) feature of computer science:**
    for many problems, solutions are known that (asymptotically) **achieve the lower bound**

$\rightsquigarrow$ can speak of "*optimal* algorithms"

Lower Bounds

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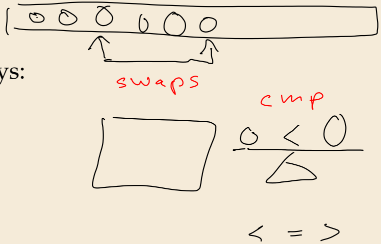
  $\rightsquigarrow$ can speak of "*optimal* algorithms"

- To prove a statement about *all algorithms*, we must precisely define what that is!
- already know one option: the word-RAM model
- Here: use a simpler, more restricted model.

The Comparison Model

- In the *comparison model* data can only be accessed in two ways:
  - comparing two elements
  - moving elements around (e. g. copying, swapping)
  - Cost: number of these operations.



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That's good!
Keeps algorithms general!

- This makes very few assumptions on the kind of objects we are sorting.
- Mergesort and Quicksort work in the comparison model.

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That's good!
Keeps algorithms general!

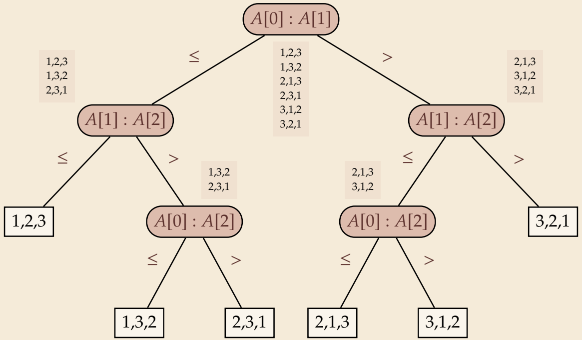
- This makes very few assumptions on the kind of objects we are sorting.
- Mergesort and Quicksort work in the comparison model.

$\leadsto$ Every comparison-based sorting algorithm corresponds to a ***decision tree***.

- only model comparisons $\leadsto$ ignore data movement
- nodes = comparisons the algorithm does   A[17] ? A[42]
- next comparisons can depend on outcomes $\leadsto$ different subtrees
- child links = outcomes of comparison   < (=) >
- leaf = unique initial input permutation compatible with comparison outcomes

Comparison Lower Bound

Example: Comparison tree for a sorting method for $A[0..2]$:



Comparison Lower Bound

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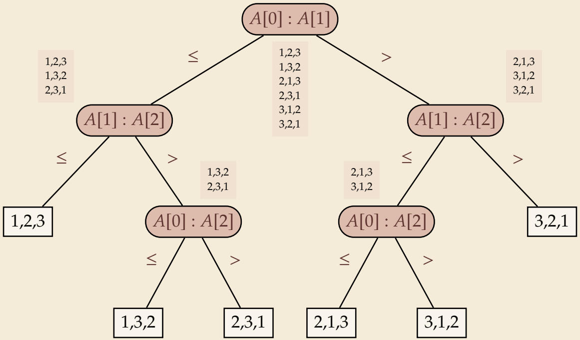


- Execution $=$ follow a path in comparison tree.
- $\rightsquigarrow$ height of comparison tree $=$ worst-case # comparisons
- comparison trees are *binary* trees
- $\rightsquigarrow$ ℓ leaves $\rightsquigarrow$ height $\geq \lceil \lg(\ell) \rceil$
  heaps
- comparison trees for sorting method must have $\geq n!$ leaves
- $\rightsquigarrow$ height $\geq \lg(n!) \sim n \lg n$

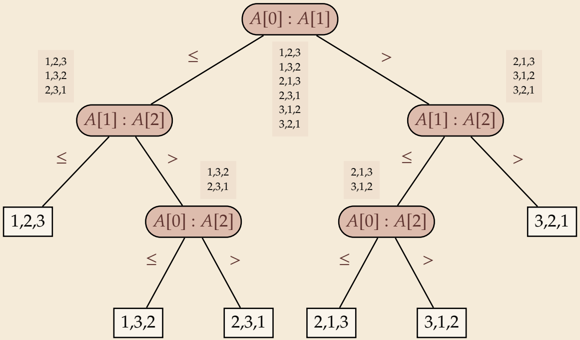
more precisely: $\lg(n!) = n \lg n - \lg(e) n + O(\log n)$

1 comparison tree $\hat{=}$ 1 algorithm and 1 input size

any cmp-based sorting algorithm needs $\geq \lg(n!) \sim n \lg n$ cmps in the worst case

Comparison Lower Bound

Example: Comparison tree for a sorting method for $A[0..2]$:



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$\rightsquigarrow$ height $\geq \lg(n!) \sim n \lg n$

more precisely: $\lg(n!) = n \lg n - \lg(e)n + O(\log n)$

- Mergesort achieves $\sim n \lg n$ comparisons $\rightsquigarrow$ asymptotically comparison-optimal!
- Open (theory) problem: Sorting algorithm with $n \lg n - \lg(e)n + o(n)$ comparisons?

≈ 1.4427

Clicker Question

Does the comparison-tree from the previous slide correspond to a worst-case optimal sorting method?

A Yes

B No

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Click on "Polls" tab

Clicker Question

Does the comparison-tree from the previous slide correspond to a worst-case optimal sorting method?

A Yes ✓

B ~~No~~

sli.do/comp526

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3.4 Integer Sorting

How to beat a lower bound

- Does the above lower bound mean, sorting always takes time $\Omega(n \log n)$?

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- Here: sort n **integers**
  - can do *a lot* with integers: add them up, compute averages, ... (full power of word-RAM)

  $\rightsquigarrow$ we are **not** working in the comparison model
  $\rightsquigarrow$ *above lower bound does not apply!*

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  $\rightsquigarrow$ we are **not** working in the comparison model
  $\rightsquigarrow$ *above lower bound does not apply!*

  - but: a priori unclear how much arithmetic helps for sorting ...

Counting sort

encoded in binary

- Important parameter: size/range of numbers
  - numbers in range $[0..U) = \{0, \ldots, U-1\}$ typically $U = 2^b$ $\leadsto$ b-bit binary numbers

Counting sort

- Important parameter: size/range of numbers
  - numbers in range $[0..U) = \{0, \ldots, U-1\}$ typically $U = 2^b$ $\rightsquigarrow$ b-bit binary numbers
- We can sort n integers in $\Theta(n+U)$ time and $\Theta(U)$ space when $\boxed{b \le w}$:

word size

used in Arrays.sort(byte[])

Counting sort

```
procedure countingSort(A[0..n − 1])
    // A contains integers in range [0..U).
    C[0..U − 1] := new integer array, initialized to 0          Θ(U)
    // Count occurrences
    for i := 0, . . . , n − 1                                   Θ(n)
        C[A[i]] := C[A[i]] + 1                  Θ(1)            (1)
    i := 0 // Produce sorted list
    for k := 0, . . . U − 1                     U               Θ(n+U)
        for j := 1, . . . C[k]                  C[k]            (2)
            A[i] := k; i := i + 1               O(1)
```

$\sum_k C[k] = n$

- *count* how often each *possible* value occurs (1)
- produce sorted result directly from counts (2)
- circumvents lower bound by using integers as array index / pointer offset

$\rightsquigarrow$ Can sort n integers in range $[0..U)$ with $U = O(n)$ in time and space $\Theta(n)$.

Integer Sorting – State of the art

- $O(n)$ time sorting also possible for numbers in range $U = O(n^c)$ for constant c.
  - *radix sort* with radix 2^w

- **Algorithm theory**

(Unit 1 : $w = \Theta(\log n)$)

  - suppose $U = 2^w$, but w can be an arbitrary function of n
  - how fast can we sort n such w-bit integers on a w-bit word-RAM?
    - for $w = O(\log n)$: linear time (*radix/counting sort*)
    - for $w = \Omega(\log^{2+\varepsilon} n)$: linear time (*signature sort*)
    - for w in between: can do $O(n\sqrt{\lg\lg n})$ (very complicated algorithm)
      don't know if that is best possible!

$\notin$ exam

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      don't know if that is best possible!

* * *

- for the rest of this unit: back to the comparisons model!

Clicker Question

Which statements are correct? Select all that apply.

My computer has 64-bit words, so an `int` has 64 bits. Hence I can sort any `int[]` of length n ...

A in time proportional to n.

B in $O(n)$ time.

C in $O(n \log n)$ time.

D in constant time.

E some time, but not possible to say from given information.

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Click on "Polls" tab

Clicker Question

Which statements are correct? Select all that apply.

My computer has 64-bit words, so an `int` has 64 bits. Hence I can sort any `int[]` of length n . . .

OK

A ~~in time proportional to n.~~

B in $O(n)$ time. ✓ $\boxed{\Theta(n+U)}$ with $U = 2^{64}$, $= \Theta(n)$ counting sort

C in $O(n \log n)$ time. ✓ (mergesort)

D ~~in constant time.~~

$n = 10$

need $\Theta(U)$ space

E some time, but not possible to say from given information. ✓

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Click on "Polls" tab

Part II

Sorting with of many processors

3.5 Parallel computation

Clicker Question

Have you ever written a concurrent program (explicit threads, job pools library, or using a framework for distributed computing)?

- **A** Yes
- **B** No
- **C** Concur. . . what?

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Clicker Question

Have you ever written a concurrent program (explicit threads, job pools library, or using a framework for distributed computing)?

- **A** ~~Yes~~
- **B** ~~No~~
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Types of parallel computation

£££ can't buy you more time . . . but more computers!

⇝ Challenge: Algorithms for *parallel* computation.

Types of parallel computation

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⇝ Challenge: Algorithms for *parallel* computation.

There are two main forms of parallelism:

1. **shared-memory parallel computer** ← *focus of today*
   - p *processing elements* (PEs, processors) working in parallel
   - **single** big memory, **accessible from every PE**
   - communication via shared memory
   - think: a big server, 128 CPU cores, terabyte of main memory
2. **distributed computing**
   - p PEs working in parallel
   - each PE has **private** memory
   - communication by sending **messages** via a network
   - think: a cluster of individual machines

PRAM – Parallel RAM

- extension of the RAM model (recall Unit 1)
- the p PEs are identified by ids $0, \ldots, p-1$
  - like w (the word size), p is a parameter of the model that can grow with n
  - $p = \Theta(n)$ is not unusual   maaany processors!
- the PEs all **independently** run ~~a~~ the same RAM-style program
  (they can use their id there)
- each PE has its own registers, but `MEM` is shared among all PEs
- computation runs in **synchronous** steps:
  in each time step, every PE executes one instruction

PRAM – Conflict management

Problem: What if several PEs simultaneously overwrite a memory cell?

- **EREW**-PRAM (exclusive read, exclusive write)
  any **parallel access** to same memory cell is **forbidden** (crash if happens)
- **CREW**-PRAM (concurrent read, exclusive write)
  parallel **write** access to same memory cell is *forbidden*, but reading is fine
- **CRCW**-PRAM (concurrent read, concurrent write)
  concurrent access is allowed,
  need a rule for write conflicts:
  - common CRCW-PRAM:
    all concurrent writes to same cell must write *same* value
  - arbitrary CRCW-PRAM:
    some unspecified concurrent write wins | closest to CPUs
  - (more exist . . .)

    race conditions

- no single model is always adequate, but our default is CREW

PRAM – Execution costs

Cost metrics in PRAMs

- **space:** total amount of accessed memory
- **time:** number of steps till all PEs finish   assuming sufficiently many PEs!
  sometimes called *depth* or *span*
- **work:** total #instructions executed on **all** PEs

PRAM – Execution costs

Cost metrics in PRAMs

- **space:** total amount of accessed memory
- **time:** number of steps till all PEs finish   assuming sufficiently many PEs!
  sometimes called *depth* or *span*
- **work:** total #instructions executed on **all** PEs   sequential: work = time

Holy grail of PRAM algorithms:

- minimal time (and space)   often: want poly log time $\Theta(\log^c n)$
  c constant
- work (asymptotically) no worse than running time of best sequential algorithm
  ⇝ *"work-efficient"* algorithm: work in same Θ-class as best sequential

Clicker Question

Does every computational problem allow a work-efficient algorithm?

A Yes

B No

`sli.do/comp526`

Click on "Polls" tab

Clicker Question

Does every computational problem allow a work-efficient algorithm?

A Yes ✓

B ~~No~~

sli.do/comp526

Click on "Polls" tab

The number of processors

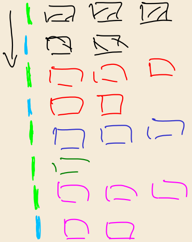
Hold on, my computer does not have $\Theta(n)$ processors! Why should I care for span and work!?

Theorem 3.1 (Brent's Theorem:)

If an algorithm has span T and work W (for an arbitrarily large number of processors), it can be run on a PRAM with p PEs in time $O(T + \frac{W}{p})$ (and using $O(W)$ work).

Proof: schedule parallel steps in round-robin fashion on the p PEs.

$p = 3$



\# steps : full up to last per PRAM step

$\leq \frac{W}{p} + T$

PRAM execution

$T = 5 \quad W = 19$

$\leadsto$ span and work give guideline for *any* number of processors

PRAM

$P = kp + 1$

$\square \; \cdots \; \square \quad p$

$W = T \cdot p$

$k \times \; | \; \square \; \square \; \square \quad p$

$| \; \square \qquad k-1$

$k = \frac{W}{p}$

3.6 Parallel primitives

Prefix sums

Before we come to parallel sorting, we study some useful building blocks.

Prefix-sum problem (also: cumulative sums, running totals)

- Given: array $A[0..n-1]$ of numbers
- Goal: compute all prefix sums $A[0] + \cdots + A[i]$ for $i = 0, \ldots, n-1$
  may be done "in-place", i. e., by overwriting A

Example:

input:	3	0	0	5	7	0	0	2	0	0	0	4	0	8	0	1

Σ

output:	3	3	3	8	15	15	15	17	17	17	17	21	21	29	29	30

Clicker Question

What is the *sequential* running time achievable for prefix sums?

A $O(n^3)$

B $O(n^2)$

C $O(n \log n)$

D $O(n)$

E $O(\sqrt{n})$

F $O(\log n)$

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Click on “Polls” tab

Clicker Question

What is the *sequential* running time achievable for prefix sums?

- **A** $\cancel{O(n^3)}$
- **B** $\cancel{O(n^2)}$
- **C** $\cancel{O(n \log n)}$
- **D** $O(n)$
- **E** $\cancel{O(\sqrt{n})}$
- **F** $\cancel{O(\log n)}$

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Prefix sums – Sequential

- sequential solution does $n-1$ additions
- but: cannot parallelize them!
  ↯ data dependencies!

$\rightsquigarrow$ need a different approach

```
procedure prefixSum(A[0..n − 1])
    for i := 1, . . . , n − 1 do
        A[i] := A[i − 1] + A[i]
```

Prefix sums – Sequential

- sequential solution does $n-1$ additions
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```
procedure prefixSum(A[0..n-1])
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```

Let's try a simpler problem first.

Excursion: Sum

- Given: array $A[0..n-1]$ of numbers
- Goal: compute $A[0] + A[1] + \cdots + A[n-1]$
  (solved by prefix sums)

Prefix sums – Sequential

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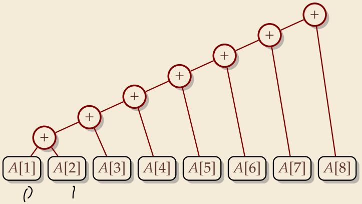
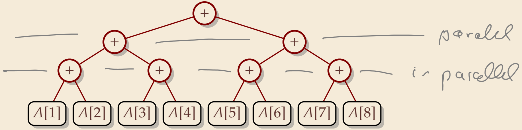


Excursion: Sum

- Given: array $A[0..n-1]$ of numbers
- Goal: compute $A[0] + A[1] + \cdots + A[n-1]$
  (solved by prefix sums)

Any algorithm *must* do $n-1$ binary additions

⇝ Height of tree = parallel time!



Parallel prefix sums

- Idea: Compute all prefix sums with balanced trees in parallel
  Remember partial results for reuse

input:

3	0	0	5	7	0	0	2	0	0	0	4	0	8	0	1

Parallel prefix sums

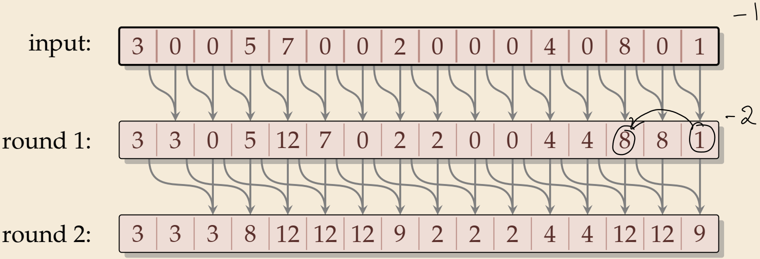
- Idea: Compute all prefix sums with balanced trees in parallel
  Remember partial results for reuse

−1

input:	3	0	0	5	7	0	0	2	0	0	0	4	0	8	0	1
round 1:	3	3	0	5	12	7	0	2	2	0	0	4	4	8	8	1

Parallel prefix sums

- Idea: Compute all prefix sums with balanced trees in parallel
  Remember partial results for reuse



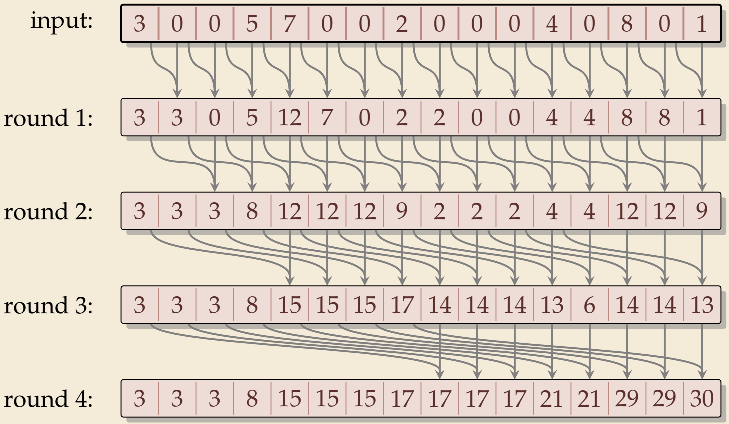
Parallel prefix sums

- Idea: Compute all prefix sums with balanced trees in parallel
  Remember partial results for reuse

input:	3	0	0	5	7	0	0	2	0	0	0	4	0	8	0	1
round 1:	3	3	0	5	12	7	0	2	2	0	0	4	4	8	8	1
round 2:	3	3	3	8	12	12	12	9	2	2	2	4	4	12	12	9
round 3:	3	3	3	8	15	15	15	17	14	14	14	13	6	14	14	13

Parallel prefix sums

- Idea: Compute all prefix sums with balanced trees in parallel
  Remember partial results for reuse



Parallel prefix sums

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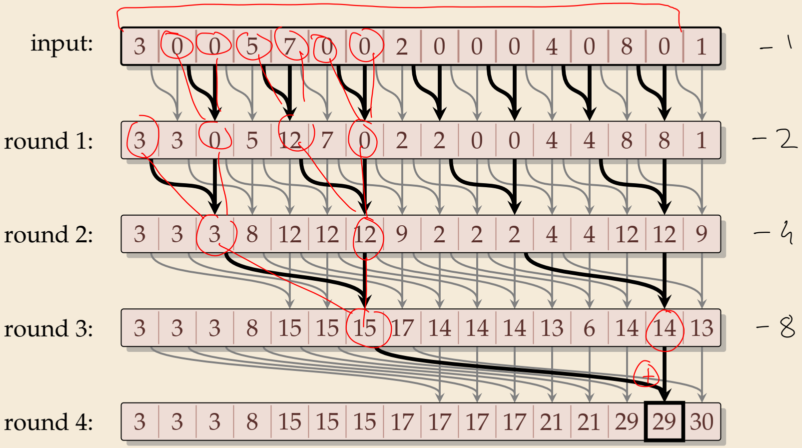

Parallel prefix sums – Code

- can be realized in-place (overwriting A)
- assumption: in each parallel step, all reads precede all writes

PRAM

= synchronous time

procedure parallelPrefixSums($A[0..n-1]$)
    for $r := 1, \ldots \lceil \lg n \rceil$ **do**
        $\mathit{step} := 2^{r-1}$
        for $i := \mathit{step}, \ldots n-1$ **do in parallel**
            $x := A[i] + A[i - \mathit{step}]$
            $A[i] := x$
        end parallel for
    end for

$O(1)$ $O(1)$ $O(1)$

$O(\log n)$

$O(n \log n)$

$O(n)$

assign PE i the ith iteration

(cannot have data dependencies)

Parallel prefix sums – Analysis

- **Time:**
  - all additions of one round run in parallel
  - $\lceil \lg n \rceil$ rounds
  - $\rightsquigarrow$ $\Theta(\log n)$ time best possible! (from sum)
- **Work:**
  - $\geq \frac{n}{2}$ additions in all rounds (except maybe last round)
  - $\rightsquigarrow$ $\Theta(n \log n)$ work
  - more than the $\Theta(n)$ sequential algorithm!

Parallel prefix sums – Analysis

- **Time:**
  - all additions of one round run in parallel
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- Typical trade-off: greater parallelism at the expense of more overall work

Parallel prefix sums – Analysis

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  - $\rightsquigarrow$ $\Theta(n \log n)$ work
  - more than the $\Theta(n)$ sequential algorithm!

- Typical trade-off: greater parallelism at the expense of more overall work

- For prefix sums:
  - can actually get $\Theta(n)$ work in *twice* that time!
  - $\rightsquigarrow$ algorithm is slightly more complicated
  - instead here: linear work in *thrice* the time using "blocking trick"

Work-efficient parallel prefix sums

standard trick to improve work: compute small blocks sequentially

1. Set $b := \lceil \lg n \rceil$
2. For blocks of b consecutive indices, i. e., $A[0..b)$, $A[b..2b)$, ... do in parallel: compute local prefix sums sequentially
3. Use previous work-inefficient algorithm only on rightmost elements of block, i. e., to compute prefix sums of $A[b-1]$, $A[2b-1]$, $A[3b-1]$, ...
4. For blocks $A[0..b)$, $A[b..2b)$, ... do in parallel: Add block-prefix sums to local prefix sums

Analysis:

- **Time:**
  - 2. & 4.: $\Theta(b) = \Theta(\log n)$ time
  - 3. $\Theta(\log(n/b)) = \Theta(\log n)$ times

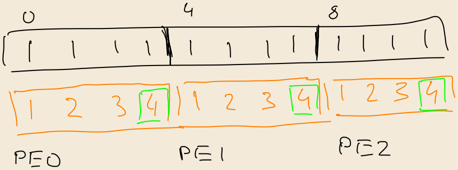
$O(\log n)$

- **Work:**
  - 2. & 4.: $\Theta(b)$ per block $\times$ $\lceil \frac{n}{b} \rceil$ blocks $\rightsquigarrow$ $\Theta(n)$
  - 3. $\Theta\left(\frac{n}{b} \log\left(\frac{n}{b}\right)\right) = \Theta(n)$

$O(n)$ work

$n = 12$

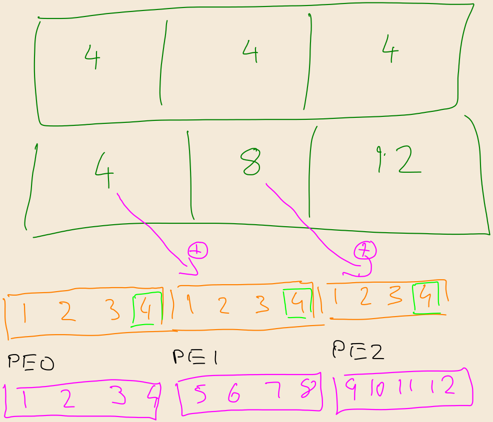
$b = 4$



$O(b)$ time $= O(\log n)$

$\frac{n}{b} \cdot O(b)$ work $= O(n)$

use work-inefficient algorithm



$n' = \frac{n}{b}$

$O(\log n')$ time $\leq O(\log n)$

$O(n' \log n')$ work $= O\left(\frac{n}{\log n} \log\left(\frac{n}{\log n}\right)\right) = O(n)$

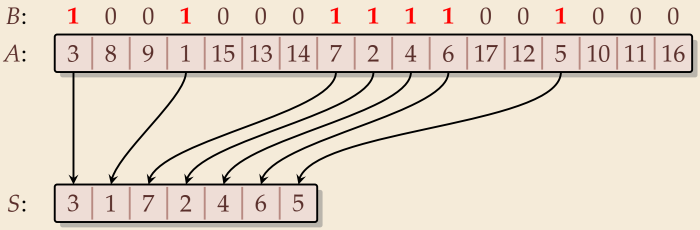
$O(b)$

$\frac{n}{b} O(b)$ work

Compacting subsequences

How do prefix sums help with sorting? one more step to go . . .

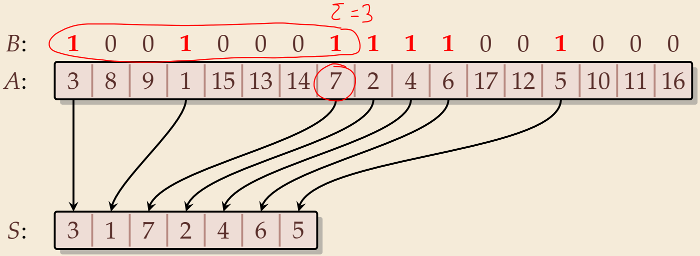
Goal: *Compact* a subsequence of an array



Compacting subsequences

How do prefix sums help with sorting?   one more step to go . . .

Goal: *Compact* a subsequence of an array



Use prefix sums on bitvector B

⇝ offset of selected cells in S

---

parallelPrefixSums(B)
for $j := 0, \ldots, n-1$ **do in parallel**
    if $B[j] == 1$ **then** $S[B[j]-1] := A[j]$
end parallel for

Clicker Question

What is the parallel time and work achievable for *compacting* a subsequence of an array of size n?

A $O(1)$ time, $O(n)$ work

B $O(\log n)$ time, $O(n)$ work

C $O(\log n)$ time, $O(n \log n)$ work

D $O(\log^2 n)$ time, $O(n^2)$ work

E $O(n)$ time, $O(n)$ work

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Click on "Polls" tab

Clicker Question

What is the parallel time and work achievable for *compacting* a subsequence of an array of size n?

- **A** ~~$O(1)$ time, $O(n)$ work~~
- **B** $O(\log n)$ time, $O(n)$ work
- **C** ~~$O(\log n)$ time, $O(n \log n)$ work~~
- **D** ~~$O(\log^2 n)$ time, $O(n^2)$ work~~
- **E** ~~$O(n)$ time, $O(n)$ work~~

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3.7 Parallel sorting

Parallel quicksort

Let's try to parallelize quicksort

- recursive calls can run in parallel (data independent)

  only this in parallel can only reduce time to $\Theta(n)$

- our sequential partitioning algorithm seems hard to parallelize

Parallel quicksort

Let's try to parallelize quicksort

- recursive calls can run in parallel (data independent)
- our sequential partitioning algorithm seems hard to parallelize
- but can split partitioning into *rounds*:
  1. **comparisons:** compare all elements pivot (in parallel), store bitvector
  2. compute prefix sums of bit vectors (in parallel as above)
  3. **compact** subsequences of small and large elements (in parallel as above)

	5	3	7	1	2	6	8
S		1	0	1	1	0	0
L		0	1	0	0	1	1

3 1 2 5 7 6 8

Parallel quicksort – Code

```
procedure parQuicksort(A[l..r])
    b := choosePivot(A[l..r])
    j := parallelPartition(A[l..r], b)
    in parallel { parQuicksort(A[l..j − 1]), parQuicksort(A[j + 1..r]) }

procedure parallelPartition(A[l..r], b)
    swap(A[n − 1], A[b]);  p := A[n − 1]
    for i = 0, . . . , n − 2 do in parallel
        S[i] := [A[i] ≤ p]    // S[i] is 1 or 0
        L[i] := 1 − S[i]
    end parallel for
    in parallel { parallelPrefixSum(S[0..n − 2]);  parallelPrefixSum(L[0..n − 2]) }
    j := S[n − 2] + 1
    for i = 0, . . . , n − 2 do in parallel
        x := A[i]
        if x ≤ p then A[S[i] − 1] := x
        else A[j + L[i]] := x
    end parallel for
    A[j] := p
    return j
```

$$[\text{pred}] = \begin{cases} 1 & \text{pred true} \\ 0 & \text{else} \end{cases}$$

Parallel quicksort – Analysis

- **Time:**
  - partition: all $O(1)$ time except prefix sums $\rightsquigarrow$ $\Theta(\log n)$ time
  - quicksort: expected depth of recursion tree is $\Theta(\log n)$

  $\rightsquigarrow$ total time $O(\log^2(n))$ in expectation
- **Work:**
  - partition: $O(n)$ time except prefix sums $\rightsquigarrow$ $\Theta(n \log n)$ work

    $O(n)$ with blocking

  $\rightsquigarrow$ quicksort $O(n \log^2(n))$ work in expectation
  - using a work-efficient prefix-sums algorithm yields (expected) work-efficient sorting!

Parallel mergesort

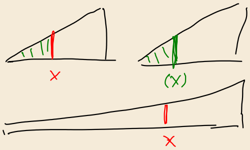
- As for quicksort, recursive calls can run in parallel ✓

Parallel mergesort

- As for quicksort, recursive calls can run in parallel $\checkmark$
- how about merging sorted halves $A[l..m-1]$ and $A[m..r]$?
- Must treat elements independently.

Parallel mergesort

- As for quicksort, recursive calls can run in parallel ✓



- how about merging sorted halves $A[l..m-1]$ and $A[m..r]$?
- Must treat elements independently.

  #elements $\leq x$

  - correct position of x in sorted output = *rank* of x

    breaking ties by position in A

  - # elements $\leq x$ = # elements from $A[l..m-1]$ that are $\leq x$
    + # elements from $A[m..r]$ that are $\leq x$

  - Note: rank in own run is simply the index of x in that run
  - find rank in *other* run by binary search

  $\rightsquigarrow$ can move it to correct position

Parallel mergesort – Analysis

- **Time:**
  - merge: $\Theta(\log n)$ from binary search, rest $O(1)$
  - mergesort: depth of recursion tree is $\Theta(\log n)$
  - $\rightsquigarrow$ total time $O(\log^2(n))$
- **Work:**
  - merge: n binary searches $\rightsquigarrow$ $\Theta(n \log n)$
  - $\rightsquigarrow$ mergesort: $O(n \log^2(n))$ work

Parallel mergesort – Analysis

- **Time:**
  - merge: $\Theta(\log n)$ from binary search, rest $O(1)$
  - mergesort: depth of recursion tree is $\Theta(\log n)$
  - $\rightsquigarrow$ total time $O(\log^2(n))$
- **Work:**
  - merge: n binary searches $\rightsquigarrow$ $\Theta(n \log n)$
  - $\rightsquigarrow$ mergesort: $O(n \log^2(n))$ work
  - work can be reduced to $\Theta(n)$ for merge
    - do full binary searches only for regularly sampled elements
    - ranks of remaining elements are sandwiched between sampled ranks
    - use a sequential method for small blocks, treat blocks in parallel
    - (detailed omitted)

Parallel sorting – State of the art

- more sophisticated methods can sort in $O(\log n)$ parallel time on CREW-RAM
- practical challenge: small units of work add overhead
- need a lot of PEs to see improvement from $O(\log n)$ parallel time

$\rightsquigarrow$ implementations tend to use simpler methods above

- check the Java library sources for interesting examples!
  `java.util.Arrays.parallelSort(int[])`