6 Text Indexing – Searching whole genomes

7 March 2022

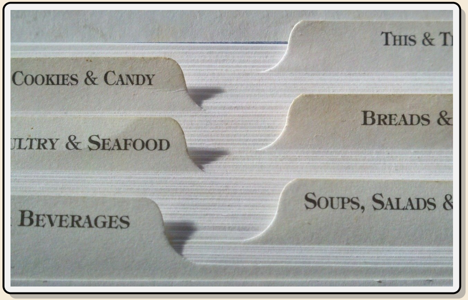
Sebastian Wild

COMP526 (Spring 2022)
University of Liverpool
version `2022-03-14 10:46`

Learning Outcomes

1. Know and understand methods for text indexing: *inverted indices*, *suffix trees*, *(enhanced) suffix arrays*
2. Know and understand *generalized suffix trees*
3. Know properties, in particular *performance characteristics*, and limitations of the above data structures.
4. Design (simple) *algorithms based on suffix trees*.
5. Understand *construction algorithms* for suffix arrays and LCP arrays.

Unit 6: ***Text Indexing***



Outline

6 Text Indexing

6.1 Motivation

Text indexing

- *Text indexing* (also: *offline text search*):
  - case of string matching: find $P[0..m)$ in $T[0..n)$
  - but with *fixed* text $\rightsquigarrow$ preprocess T (instead of P)
  - $\rightsquigarrow$ expect many queries P, answer them without looking at all of T
  - $\rightsquigarrow$ essentially a data structuring problem: "building an *index* of T"

  Latin: "one who points out"

- application areas
  - web search engines
  - online dictionaries
  - online encyclopedia
  - DNA/RNA data bases
  - . . . searching in any collection of text documents (that grows only moderately)

Inverted indices

same as "indexes"

- original indices in books: list of (key) words $\mapsto$ page numbers where they occur
- assumption: searches are only for **whole** (key) **words**

$\rightsquigarrow$ often reasonable for natural language text

Inverted indices

same as "indexes"

- original indices in books: list of (key) words $\mapsto$ page numbers where they occur
- assumption: searches are only for **whole** (key) **words**

$\rightsquigarrow$ often reasonable for natural language text

Inverted index:

- collect all words in T
  - can be as simple as splitting T at whitespace
  - actual implementations typically support *stemming* of words
    `goes` $\longrightarrow$ `go`, `cats` $\longrightarrow$ `cat`
- store mapping from words to a list of occurrences $\rightsquigarrow$ *how?*

could use BST
$\hookrightarrow$ $\Omega(\log n)$
too slow!

dictionary

keys = words

values = list of starting indices of occurrences

Clicker Question

Do you know what a *trie* is?

- **A** A what? No!
- **B** I have heard the term, but don't quite remember.
- **C** I remember hearing about it in a module.
- **D** Sure.

`sli.do/comp526`

Tries

- efficient dictionary data structure for strings
- name from re**trie**val, but pronounced “try”
- tree based on symbol comparisons

- **Assumption:** stored strings are *prefix-free* (no string is a prefix of another)
  - strings of same length
  - strings have “end-of-string” marker \$

some character $\notin \Sigma$

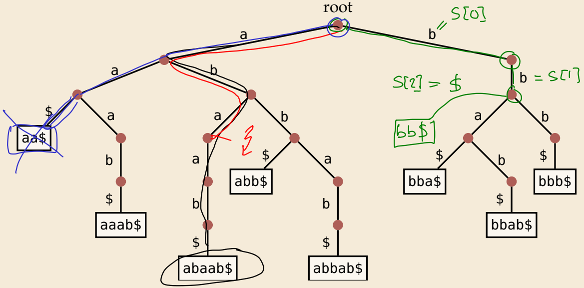
- **Example**: $\Sigma = \{a, b\}$

  {aa\$, aaab\$, abaab\$, abb\$, abbab\$, bba\$, bbab\$, bbb\$}

insert S=bb\$

aba\$ $\notin$ ST

delete aa\$



Clicker Question

does the string occur?

Suppose we have a trie that stores n strings over $\Sigma = \{\texttt{A}, \ldots, \texttt{Z}\}$. Each stored string consists of m characters.
We now search for a query string Q with $|Q| = q$ (with $q \leq m$).
How many **nodes** in the trie are **visited** during this **query**?

A $\Theta(\log n)$

B $\Theta(\log(nm))$

C $\Theta(m \cdot \log n)$

D $\Theta(m + \log n)$

E $\Theta(m)$

F $\Theta(\log m)$

G $\Theta(q)$

H $\Theta(\log q)$

I $\Theta(q \cdot \log n)$

J $\Theta(q + \log n)$

sli.do/comp526

Clicker Question

Suppose we have a trie that stores n strings over $\Sigma = \{\texttt{A}, \ldots, \texttt{Z}\}$. Each stored string consists of m characters.
We now search for a query string Q with $|Q| = q$ (with $q \le m$).
How many **nodes** in the trie are **visited** during this **query**?

A $\cancel{\Theta(\log n)}$

B $\cancel{\Theta(\log(nm))}$

C $\cancel{\Theta(m \cdot \log n)}$

D $\cancel{\Theta(m + \log n)}$

E $\cancel{\Theta(m)}$

F $\cancel{\Theta(\log m)}$

G $\Theta(q)$

H $\cancel{\Theta(\log q)}$

I $\cancel{\Theta(q \cdot \log n)}$

J $\cancel{\Theta(q + \log n)}$

`sli.do/comp526`

Clicker Question

Suppose we have a trie that stores n strings over $\Sigma = \{\mathtt{A}, \ldots, \mathtt{Z}\}$. Each stored string consists of m characters.
How many **nodes** does the trie have **in total** *in the worst case*?

A $\Theta(n)$

B $\Theta(n + m)$

C $\Theta(n \cdot m)$

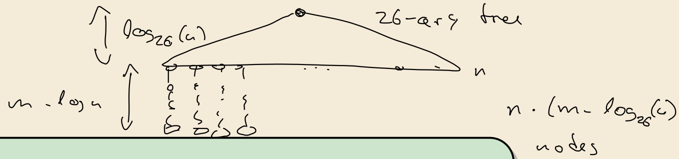
D $\Theta(n \log m)$

E $\Theta(m)$

F $\Theta(m \log n)$

`sli.do/comp526`

Clicker Question



Suppose we have a trie that stores n strings over $\Sigma = \{\mathtt{A}, \ldots, \mathtt{Z}\}$. Each stored string consists of m characters.
How many **nodes** does the trie have **in total** *in the worst case*?

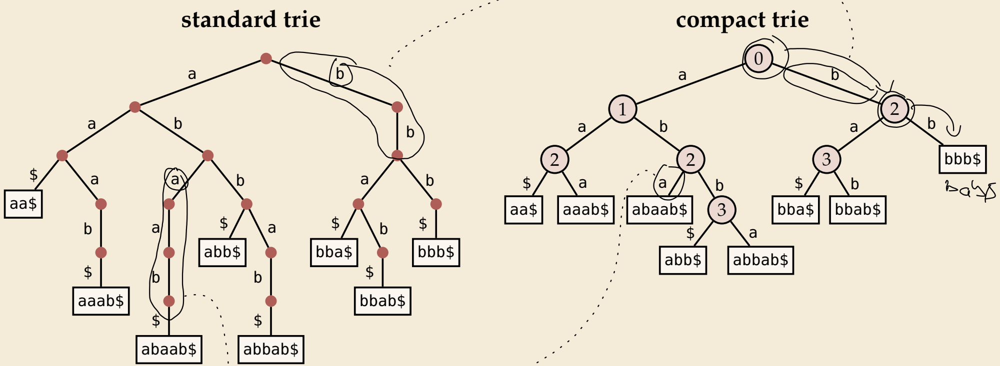
- **A** ~~$\Theta(n)$~~
- **B** ~~$\Theta(n + m)$~~
- **C** $\Theta(n \cdot m)$ ✓
- **D** ~~$\Theta(n \log m)$~~
- **E** ~~$\Theta(m)$~~
- **F** ~~$\Theta(m \log n)$~~

sli.do/comp526

Compact tries

- compress paths of unary nodes into single edge
  =1 child
- nodes store *index* of next character to check

bab$



⇝ searching slightly trickier, but same time complexity as in trie

- all nodes ≥ 2 children ⇝ #nodes $\leq$ #leaves = #strings ⇝ linear space $O(n)$

Tries as inverted index

- simple
- fast lookup

- cannot handle more general queries:
  - search part of a word
  - search phrase (sequence of words)

Tries as inverted index

- 👍 simple
- 👍 fast lookup

- 👎 cannot handle more general queries:
  - search part of a word
  - search phrase (sequence of words)

- 👎 what if the 'text' does not even have words to begin with?!
  - biological sequences

```
ACAAGATGCCATTGTCCCCCGGCCTCCTGCTGCTGCTGCTCTCCGGGGCCACGGCCACCGCTGCCCTGCCCCTGGAGGGTGGCCCCACCGGC
CGAGACAGCGAGCATATGCAGGAAGCGGCAGGAATAAGGAAAAGCAGCCTCCTGACTTTCCTCGCTTGGTGGTTTGAGTGGACCTCCCAGGC
CAGTGCCGGGCCCCTCATAGGAGAGGAAGCTCGGGAGGTGGCCAGGCGGCAGGAAGGCGCACCCCCCCAGCAATCCGCGCGCCGGGACAGAA
TGCCCTGCAGGAACTTCTTCTGGAAGACCTTCTCCTCCTGCAAATAAAACCTCACCCATGAATGCTCACGCAAGTTTAATTACAGACCTGAA
```

  - binary streams

```
000000101010011110101110000011111000111110111110011011010000111000100110111100000100011010 10
0110110000110101101000000010000000011101011000001000011110101110110010001100101101110111111 1
1100010100010110010100000011101010100110000000011011000011001111100001 01 0101011101111000011
101011100100101010101000001111101001100000011110011010100000000100100100000101100011000110111
```

⇝ need new ideas

6.2 Suffix Trees

Suffix trees – A 'magic' data structure

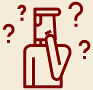
Appetizer: Longest common substring problem

- Given: strings $S_1, \ldots, S_k$ **Example:** $S_1 =$ `superiorcalifornialives`, $S_2 =$ `sealiver`
- Goal: find the longest substring that occurs in all k strings

Suffix trees – A 'magic' data structure

Appetizer: Longest common substring problem

- Given: strings $S_1, \ldots, S_k$ **Example:** $S_1 =$ `superiorcalifornialives`, $S_2 =$ `sealiver`
- Goal: find the longest substring that occurs in all k strings $\rightsquigarrow$ `alive`



Can we do this in time $O(|S_1| + \cdots + |S_k|)$? How??

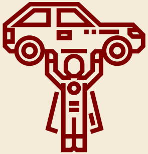
Suffix trees – A 'magic' data structure

Appetizer: Longest common substring problem

- Given: strings $S_1, \ldots, S_k$ **Example:** $S_1 =$ `superiorcalifornialives`, $S_2 =$ `sealiver`
- Goal: find the longest substring that occurs in all k strings $\rightsquigarrow$ `alive`

Can we do this in time $O(|S_1| + \cdots + |S_k|)$? How??

Enter: ***suffix trees***



- versatile data structure for index with full-text search
- linear time (for construction) and linear space
- allows efficient solutions for many advanced string problems

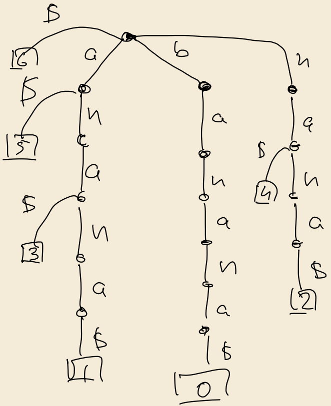
"Although the longest common substring problem looks trivial now, given our knowledge of suffix trees, it is very interesting to note that in 1970 Don Knuth conjectured that a linear-time algorithm for this problem would be impossible." *[Gusfield: Algorithms on Strings, Trees, and Sequences (1997)]*

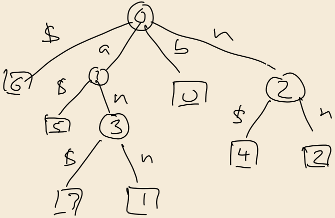
Suffix trees – Definition

- suffix tree $\mathcal{T}$ for text $T = T[0..n)$ $=$ compact trie of all suffixes of $T\$$ (set $T[n] := \$$)

$$\begin{array}{rccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ T = & b & a & n & a & n & a & \$ \end{array}$$

anana\$





Suffix trees – Definition

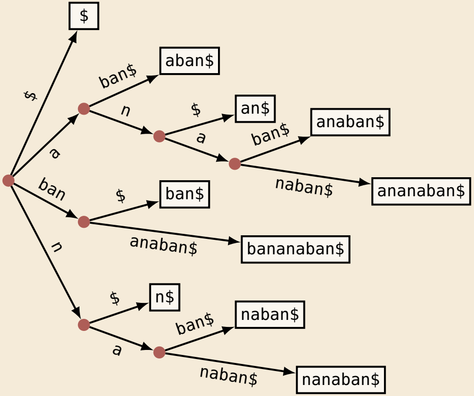
- suffix tree $\mathcal{T}$ for text $T = T[0..n)$ = compact trie of all suffixes of $T\$$ (set $T[n] := \$$)

Example:

T = bananaban$

suffixes: {bananaban$, ananaban$, nanaban$, anaban$, naban$, aban$, ban$, an$, n$, $}

0	1	2	3	4	5	6	7	8	9
b	a	n	a	n	a	b	a	n	$



Suffix trees – Definition

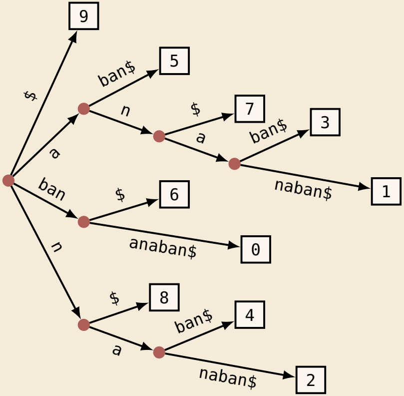
- $\boxed{\text{suffix tree } \mathcal{T} \text{ for text } T = T[0..n) \quad = \quad \text{compact trie of all suffixes of } T\$ \quad (\text{set } T[n] := \$)}$
- except: in leaves, store *start index* (instead of copy of actual string)

Example:

T = `bananaban$`

suffixes: {`bananaban$`, `ananaban$`, `nanaban$`, `anaban$`, `naban$`, `aban$`, `ban$`, `an$`, `n$`, `$`}

	0	1	2	3	4	5	6	7	8	9
$T =$	b	a	n	a	n	a	b	a	n	$



Suffix trees – Definition

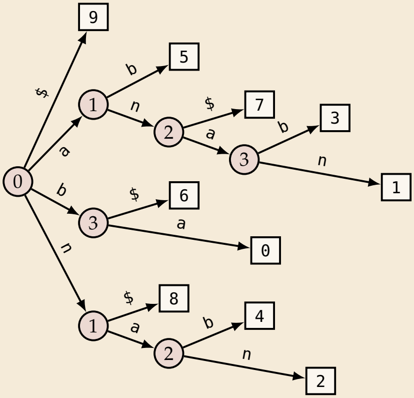
- suffix tree $\mathcal{T}$ for text $T = T[0..n)$ = compact trie of all suffixes of $T\$$ (set $T[n] := \$$)
- except: in leaves, store *start index* (instead of copy of actual string)

Example:

T = bananaban$

suffixes: {bananaban$, ananaban$, nanaban$, anaban$, naban$, aban$, ban$, an$, n$, $}

	0	1	2	3	4	5	6	7	8	9
T =	b	a	n	a	n	a	b	a	n	$



- also: edge labels like in compact trie
- (more readable form on slides to explain algorithms)

Suffix trees – Construction

- $T[0..n]$ has $n+1$ suffixes (starting at character $i \in [0..n]$)
- We can build the suffix tree by inserting each suffix of T into a compressed trie.
  But that takes time $\Theta(n^2)$. $\rightsquigarrow$ not interesting!

Suffix trees – Construction

- $T[0..n]$ has $n+1$ suffixes (starting at character $i \in [0..n]$)
- We can build the suffix tree by inserting each suffix of T into a compressed trie. But that takes time $\Theta(n^2)$. $\rightsquigarrow$ not interesting!

same order of growth as reading the text!

Amazing result: Can construct the suffix tree of T in $\Theta(n)$ time!

- algorithms are a bit tricky to understand
- but were a theoretical breakthrough
- and they are efficient in practice (and heavily used)!

$\rightsquigarrow$ for now, take linear-time construction for granted. What can we do with them?

Clicker Question

Recap: Check all correct statements about suffix tree $\mathcal{T}$ of $T[0..n)$.

- **A** We require T to end with $\$$.
- **B** The size of $\mathcal{T}$ can be $\Omega(n^2)$ in the worst case.
- **C** $\mathcal{T}$ is a standard trie of all suffixes of $T\$$.
- **D** $\mathcal{T}$ is a compact trie of all suffixes of $T\$$.
- **E** The leaves of $\mathcal{T}$ store (a copy of) a suffix of $T\$$.
- **F** Naive construction of $\mathcal{T}$ takes $\Omega(n^2)$ (worst case).
- **G** $\mathcal{T}$ can be computed in $O(n)$ time (worst case).
- **H** $\mathcal{T}$ has n leaves.

`sli.do/comp526`

Clicker Question

Recap: Check all correct statements about suffix tree $\mathcal{T}$ of $T[0..n)$.

- **A** We require T to end with $\$$.
- **B** ~~The size of $\mathcal{T}$ can be $\Omega(n^2)$ in the worst case.~~
- **C** ~~$\mathcal{T}$ is a standard trie of all suffixes of $T\$$.~~
- **D** $\mathcal{T}$ is a compact trie of all suffixes of $T\$$.
- **E** ~~The leaves of $\mathcal{T}$ store (a copy of) a suffix of $T\$$.~~
- **F** Naive construction of $\mathcal{T}$ takes $\Omega(n^2)$ (worst case).
- **G** $\mathcal{T}$ can be computed in $O(n)$ time (worst case).
- **H** ~~$\mathcal{T}$ has n leaves.~~

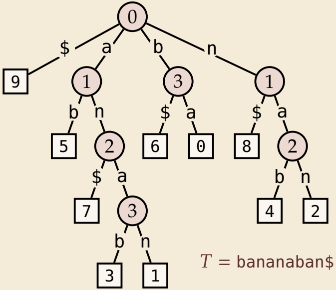
sli.do/comp526

6.3 Applications

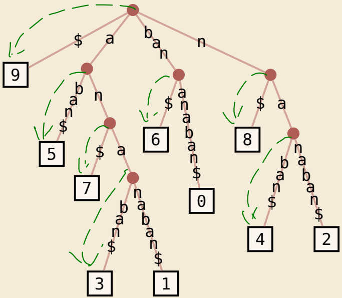
Applications of suffix trees

- In this section, always assume suffix tree $\mathcal{T}$ for T given.

Recall: $\mathcal{T}$ stored like this: but think about this:

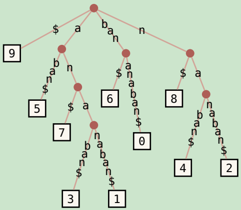


T = bananaban$



- Moreover: assume internal nodes store pointer to *leftmost leaf in subtree.*
- Notation: $T_i = T[i..n]$ (including $)

Clicker Question



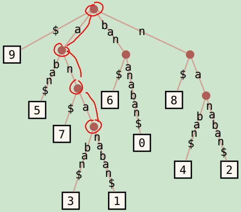
What does T's suffix tree (on the left) tell you about the question whether T contains the pattern P = `ana`?

Check all that apply to this example.

- **A** Nothing.
- **B** P occurs in T.
- **C** P does not occur in T.
- **D** P occurs once in T.
- **E** P occurs twice in T.
- **F** P starts at index 0.
- **G** P starts at index 1.
- **H** P starts at index 2.
- **I** P starts at index 3.
- **J** P starts at index 4.
- **K** P starts at index 7.

`sli.do/comp526`

Clicker Question



What does T's suffix tree (on the left) tell you about the question whether T contains the pattern P = ana?

Check all that apply to this example.

A ~~Nothing.~~

B P occurs in T. ✓

C ~~P does not occur in T.~~

D ~~P occurs once in T.~~

E P occurs twice in T. ✓

F ~~P starts at index 0.~~

G P starts at index 1. ✓

H ~~P starts at index 2.~~

I P starts at index 3. ✓

J ~~P starts at index 4.~~

K ~~P starts at index 7.~~

sli.do/comp526

Application 1: Text Indexing / String Matching

- $\boxed{P \text{ occurs in } T \iff P \text{ is a prefix of a suffix of } T}$
- we have all suffixes in $\mathcal{T}$!

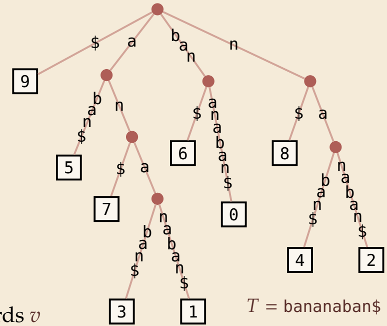
Application 1: Text Indexing / String Matching

- $\boxed{P \text{ occurs in } T \iff P \text{ is a prefix of a suffix of } T}$
- we have all suffixes in $\mathcal{T}$!

⇝ (try to) follow path with label P, until

1. **we get stuck**
   at internal node (no node with next character of P)
   or *inside edge* (mismatch of next characters)
   ⇝ P does not occur in T
2. **we run out of pattern**
   reach end of P at internal node v or inside edge towards v
   ⇝ P occurs at all leaves in subtree of v
3. **we run out of tree**
   reach a leaf ℓ with part of P left ⇝ compare P to ℓ.

   ⚠ This cannot happen when testing edge labels since $\$ \notin \Sigma$,
   but needs check(s) in compact trie implementation!



- Finding first match (or `NO_MATCH`) takes $O(|P|)$ time!

follow leftmost_leaf pointer

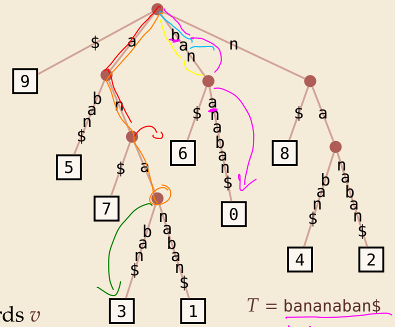
Application 1: Text Indexing / String Matching

- P occurs in T $\iff$ P is a prefix of a suffix of T
- we have all suffixes in $\mathcal{T}$!

⇝ (try to) follow path with label P, until

1. **we get stuck**
   at internal node (no node with next character of P)
   or *inside edge* (mismatch of next characters)
   ⇝ P does not occur in T
2. **we run out of pattern**
   reach end of P at internal node v or inside edge towards v
   ⇝ P occurs at all leaves in subtree of v
3. **we run out of tree**
   reach a leaf ℓ with part of P left ⇝ compare P to ℓ.

   ⚠ This cannot happen when testing edge labels since $\$ \notin \Sigma$, but needs check(s) in compact trie implementation!



T = bananaban\$

briar

Examples:

- P = ann
- P = baa
- P = ana
- P = ba
- P = briar

notice mismatch only in leaf

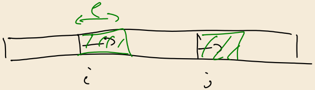
- Finding first match (or `NO_MATCH`) takes $O(|P|)$ time!

Application 2: Longest repeated substring

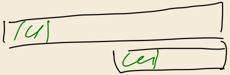
- **Goal:** Find longest substring $T[i..i+\ell)$ that occurs also at $j \neq i$: $T[j..j+\ell) = T[i..i+\ell)$.

e. g. for compression $\rightsquigarrow$ Unit 7

How can we efficiently check *all possible substrings?*



longest ℓ ?



Application 2: Longest repeated substring

- **Goal:** Find longest substring $T[i..i+\ell)$ that occurs also at $j \neq i$: $T[j..j+\ell) = T[i..i+\ell)$.

  e. g. for compression $\rightsquigarrow$ Unit 7

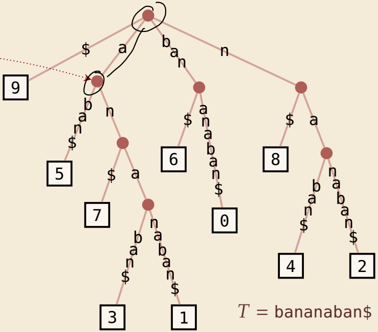
How can we efficiently check *all possible substrings?*

Repeated substrings $=$ shared paths in *suffix tree*

- T_5 = `aban$` and T_7 = `an$` have *longest common prefix* '`a`'

$\rightsquigarrow$ $\exists$ internal node with path label '`a`'

here single edge, can be longer path



T = `bananaban$`

Application 2: Longest repeated substring

- **Goal:** Find longest substring $T[i..i+\ell)$ that occurs also at $j \neq i$: $T[j..j+\ell) = T[i..i+\ell)$.

e. g. for compression $\rightsquigarrow$ Unit 7

How can we efficiently check *all possible substrings?*

Repeated substrings = shared paths in *suffix tree*

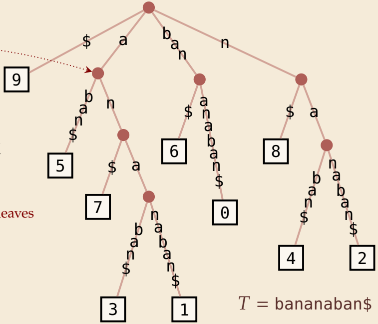
- T_5 = aban\$ and T_7 = an\$ have *longest common prefix* 'a'

$\rightsquigarrow$ $\exists$ internal node with path label 'a'

here single edge, can be longer path

$\rightsquigarrow$ longest repeated substring = longest common prefix (LCP) of two suffixes

actually: adjacent leaves



T = bananaban\$

Application 2: Longest repeated substring

- **Goal:** Find longest substring $T[i..i+\ell)$ that occurs also at $j \neq i$: $T[j..j+\ell) = T[i..i+\ell)$.

e. g. for compression $\rightsquigarrow$ Unit 7

How can we efficiently check *all possible substrings?*

Repeated substrings = shared paths in *suffix tree*

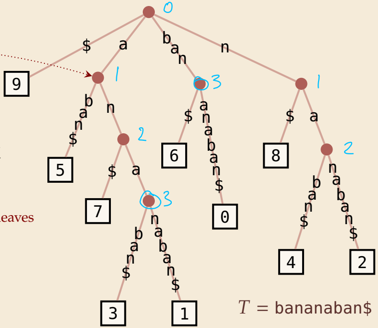
- T_5 = `aban$` and T_7 = `an$` have *longest common prefix* '`a`'

$\rightsquigarrow$ $\exists$ internal node with path label '`a`'

here single edge, can be longer path

$\rightsquigarrow$ longest repeated substring = longest common prefix (LCP) of two suffixes

actually: adjacent leaves



T = `bananaban$`

- Algorithm:
  1. Compute *string depth* (=length of path label) of nodes
  2. Find internal nodes with maximal string depth

- Both can be done in depth-first traversal $\rightsquigarrow$ $\Theta(n)$ time

Generalized suffix trees

- longest *repeated* substring (of one string) feels very similar to longest *common* substring of several strings $T^{(1)}, \ldots, T^{(k)}$ with $T^{(j)} \in \Sigma^{n_j}$
- can we solve that in the same way?
- could build the suffix tree for each $T^{(j)}$ … but doesn't seem to help

Generalized suffix trees

- longest *repeated* substring (of one string) feels very similar to longest *common* substring of several strings $T^{(1)}, \ldots, T^{(k)}$ with $T^{(j)} \in \Sigma^{n_j}$
- can we solve that in the same way?
- could build the suffix tree for each $T^{(j)}$ ... but doesn't seem to help

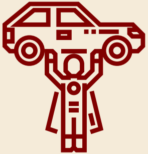
$\rightsquigarrow$ need a *single/joint* suffix tree for *several* texts

Generalized suffix trees

- longest *repeated* substring (of one string) feels very similar to longest *common* substring of several strings $T^{(1)}, \ldots, T^{(k)}$ with $T^{(j)} \in \Sigma^{n_j}$
- can we solve that in the same way?
- could build the suffix tree for each $T^{(j)}$ ... but doesn't seem to help

$\rightsquigarrow$ need a *single/joint* suffix tree for *several* texts

Enter: ***generalized suffix tree***



- Define $T := T^{(1)}\$_1 T^{(2)}\$_2 \cdots T^{(k)}\$_k$ for k new end-of-word symbols
- Construct suffix tree $\mathcal{T}$ for T

$\rightsquigarrow$ $\$_j$-edges always leads to leaves $\rightsquigarrow$ $\exists$ leaf (j, i) for each suffix $T_i^{(j)} = T^{(j)}[i..n_j]$

Clicker Question

What is the longest common substring of the strings `bcabcac`, `aabca` and `bcaa`?

`sli.do/comp526`

Application 3: Longest common substring

- With that new idea, we can find longest common substrings:

  1. Compute generalized suffix tree $\mathcal{T}$.
  2. Store with each node the *subset of strings* that contain its path label:
     2.1. Traverse $\mathcal{T}$ bottom-up.
     2.2. For a leaf (j, i), the subset is $\{j\}$.
     2.3. For an internal node, the subset is the union of its children.

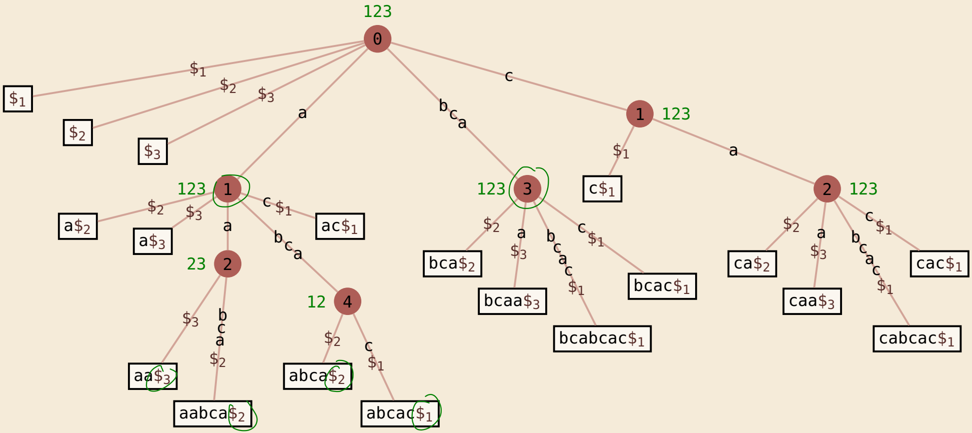
     $O(n)$ time
  3. In top-down traversal, compute *string depths* of nodes. (as above)
  4. Report deepest node (by string depth) whose subset is $\{1, \ldots, k\}$.

- Each step takes time $\Theta(n)$ for $n = n_1 + \cdots + n_k$ the total length of all texts.

"Although the longest common substring problem looks trivial now, given our knowledge of suffix trees, it is very interesting to note that in 1970 Don Knuth conjectured that a linear-time algorithm for this problem would be impossible." *[Gusfield: Algorithms on Strings, Trees, and Sequences (1997)]*

Longest common substring – Example

$T^{(1)}$ = bcabcac, $T^{(2)}$ = aabca, $T^{(3)}$ = bcaa



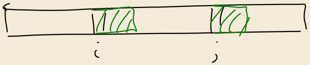
6.4 Longest Common Extensions

Application 4: Longest Common Extensions

- We implicitly used a special case of a more general, versatile idea:

  The ***longest common extension (LCE)*** data structure:

  - **Given:** String $T[0..n)$
  - **Goal:** Answer LCE queries, i. e.,
    given positions i, j in T,
    how far can we read the same text from there?
    formally: $\mathrm{LCE}(i,j) = \max\{\ell : T[i..i+\ell) = T[j..j+\ell)\}$



Application 4: Longest Common Extensions

- We implicitly used a special case of a more general, versatile idea:

  The *longest common extension (LCE)* ~~data structure:~~ ADT
  - **Given:** String $T[0..n)$
  - **Goal:** Answer LCE queries, i. e.,
    given positions i, j in T,
    how far can we read the same text from there?
    formally: $\mathrm{LCE}(i, j) = \max\{\ell : T[i..i+\ell) = T[j..j+\ell)\}$

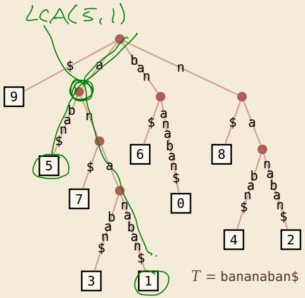
$\rightsquigarrow$ use suffix tree of T!

(length of) longest common prefix of ith and jth suffix

- In $\mathcal{T}$: $\mathrm{LCE}(i, j) = \mathrm{LCP}(T_i, T_j)$ $\rightsquigarrow$ same thing, different name!
  $=$ string depth of *lowest common ancester (LCA)* of leaves $\boxed{i}$ and $\boxed{j}$

- in short: $\boxed{\mathrm{LCE}(i, j) = \mathrm{LCP}(T_i, T_j) = \mathrm{stringDepth}(\mathrm{LCA}(\boxed{i}, \boxed{j}))}$

LCP(T_5, T_1)



T = bananaban$

Efficient LCA

How to find lowest common ancestors?

- Could walk up the tree to find LCA $\rightsquigarrow$ $\Theta(n)$ worst case 👎
- Could store all LCAs in big table $\rightsquigarrow$ $\Theta(n^2)$ space and preprocessing 👎

trivial: no preprocessing $\rightsquigarrow$ $\Theta(\ell)$ query

Efficient LCA

How to find lowest common ancestors?

- Could walk up the tree to find LCA $\rightsquigarrow$ $\Theta(n)$ worst case
- Could store all LCAs in big table $\rightsquigarrow$ $\Theta(n^2)$ space and preprocessing

Amazing result: Can compute data structure in $\Theta(n)$ time and space that finds any LCA is **constant(!) time**.

- a bit tricky to understand
- but a theoretical breakthrough
- and useful in practice

and suffix tree construction inside . . .

$\rightsquigarrow$ for now, use $O(1)$ LCA as black box.

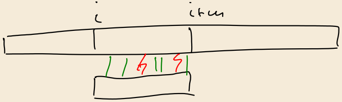
$\rightsquigarrow$ After linear preprocessing (time & space), we can find LCEs in $O(1)$ time.

Application 5: Approximate matching

***k*-mismatch matching**:

- **Input:** text $T[0..n)$, pattern $P[0..m)$, $k \in [0..m)$
- **Output:**
  - smallest i so that $T[i..i+m)$ are P differ in at most k characters
    "Hamming distance $\leq k$"
  - or `NO_MATCH` if there is no such i

$\rightsquigarrow$ searching with typos



- Adapted brute-force algorithm $\rightsquigarrow$ $O(n \cdot m)$

compare w/ exact string matching $O(n+m)$

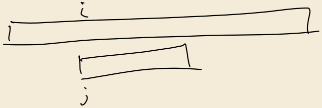
Application 5: Approximate matching

***k*-mismatch matching**:

- **Input:** text $T[0..n)$, pattern $P[0..m)$, $k \in [0..m)$
- **Output:**
  - smallest i so that $T[i..i+m)$ are P differ in at most k characters
  - or `NO_MATCH` if there is no such i

"Hamming distance $\le k$"

$\rightsquigarrow$ searching with typos



- Adapted brute-force algorithm $\rightsquigarrow$ $O(n \cdot m)$

- Assume longest common extensions in $T\$_1P\$_2$ can be found in $O(1)$
  - $\rightsquigarrow$ generalized suffix tree $\mathcal{T}$ has been built
  - $\rightsquigarrow$ string depths of all internal nodes have been computed
  - $\rightsquigarrow$ constant-time LCA data structure for $\mathcal{T}$ has been built

Clicker Question

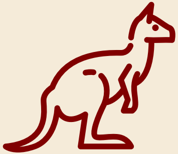
What is the Hamming distance between `heart` and `beard`?

beard

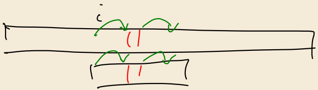
2

sli.do/comp526

Kangaroo Algorithm for approximate matching



```
procedure kMismatch(T[0..n − 1], P[0..m − 1])
    // build LCE data structure
    for i := 0, ..., n − m − 1 do
        mismatches := 0;  t := i;  p := 0
        while mismatches ≤ k ∧ p < m do
            ℓ := LCE(t, p) // jump over matching part
            t := t + ℓ + 1;  p := p + ℓ + 1
            mismatches := mismatches + 1
        if p == m then
            return i
```



- **Analysis:** $\Theta(n + m)$ preprocessing $+$ $O(n \cdot k)$ matching

$\rightsquigarrow$ very efficient for small k good if $k \ll m$

- State of the art
  - $O\left(n\frac{k^2 \log k}{m}\right)$ possible with complicated algorithms
  - extensions for edit distance $\leq k$ possible

Application 6: Matching with wildcards

- Allow a wildcard character in pattern

  stands for arbitrary (single) character

```
    unit*          P
in␣unit5␣we␣will   T
```

- similar algorithm as for k-mismatch $\rightsquigarrow$ $O(n \cdot k + m)$ when P has k wildcards

you can also use suffix trees to traverse P.

$\approx \sigma^k$ paths explored for k wildcards

matchings in $O(\sigma^k \cdot m + \text{output})$

Application 6: Matching with wildcards

- Allow a wildcard character in pattern

  stands for arbitrary (single) character

```
  unit*          P
in␣unit5␣we␣will T
```

- similar algorithm as for k-mismatch $\rightsquigarrow$ $O(n \cdot k + m)$ when P has k wildcards

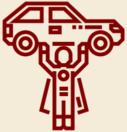
* * *

Many more applications, in particular for problems on biological sequences

20+ described in Gusfield, *Algorithms on strings, trees, and sequences* (1999)

Suffix trees – Discussion

- Suffix trees were a threshold invention

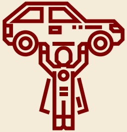


👍 linear time and space

👍 suddenly many questions efficiently solvable in theory

Suffix trees – Discussion

- Suffix trees were a threshold invention



- 👍 linear time and space
- 👍 suddenly many questions efficiently solvable in theory

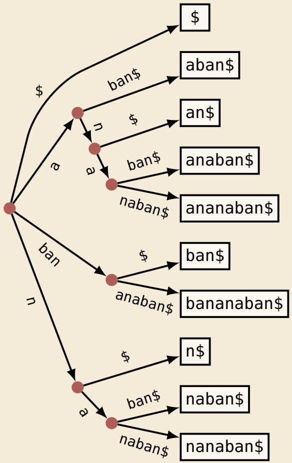
- 👎 construction of suffix trees:
  linear time, but significant overhead
- 👎 construction methods fairly complicated
- 👎 many pointers in tree incur large space overhead

$\Theta(n \cdot \sigma)$ if array of child pointers

$\Theta(n \lg \sigma)$ if we use dictionaries, but then search is slower

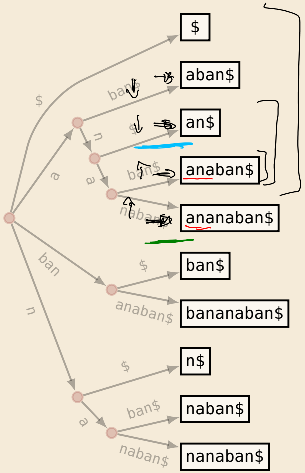
6.5 Suffix Arrays

Putting suffix trees on a diet



- **Observation:** order of leaves in suffix tree
  = suffixes lexicographically *sorted*

Putting suffix trees on a diet



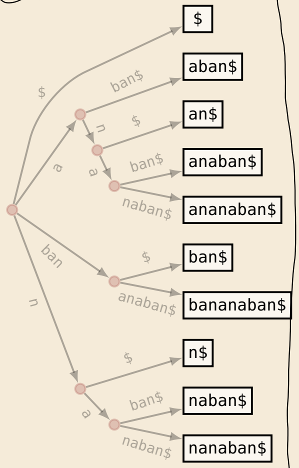
- **Observation:** order of leaves in suffix tree
  = suffixes lexicographically *sorted*

- Idea: only store list of leaves $L[0..n]$
- Enough to do efficient string matching!
  1. Use binary search for pattern P
  2. check if P is prefix of suffix after position found
- **Example:** P = ana\$

ana£

range contains all matches of P in T

Putting suffix trees on a diet



L[0..n]

9
5
7
3
1
6
0
8
4
2

- **Observation:** order of leaves in suffix tree = suffixes lexicographically *sorted*
- Idea: only store list of leaves $L[0..n]$
- Enough to do efficient string matching!
  1. Use binary search for pattern P
  2. check if P is prefix of suffix after position found
- **Example:** P = ana

⇝ $L[0..n]$ is called ***suffix array***:

$\boxed{L[r] = \text{(start index of) } r\text{th suffix in sorted order}}$

- using L, can do string matching with $\le (\lg n + 2) \cdot m$ character comparisons

SM in $O(m \cdot \log n)$ time

Clicker Question

Check all correct statements about *suffix array* $L[0..n]$ and *suffix tree* $\mathcal{T}$ of text $T[0..n)$ (for $\sigma = O(1)$)

A $L[0..n]$ lists the start indices of leaves of $\mathcal{T}$ in left-to-right order.

B $T\big[L[r]..n\big]$ is the path label in $\mathcal{T}$ to the leaf storing r.

C $T\big[L[r]..n\big]$ is the path label to the rth leaf in $\mathcal{T}$.

D $T_{L[r]}$ is the rth smallest suffix of T (lexicographic order).

E In terms of Θ-classes, $\mathcal{T}$ needs more space than L.

F L (and T) suffice to solve the text indexing problem.

`sli.do/comp526`

Clicker Question

Check all correct statements about *suffix array* $L[0..n]$ and *suffix tree* $\mathcal{T}$ of text $T[0..n)$ (for $\sigma = O(1)$)

A $L[0..n]$ lists the start indices of leaves of $\mathcal{T}$ in left-to-right order. ✓

B ~~$T\big[L[r]..n\big]$ is the path label in $\mathcal{T}$ to the leaf storing r.~~

C $T\big[L[r]..n\big]$ is the path label to the rth leaf in $\mathcal{T}$. ✓

D $T_{L[r]}$ is the rth smallest suffix of T (lexicographic order). ✓

E ~~In terms of Θ-classes, $\mathcal{T}$ needs more space than L.~~

F L (and T) suffice to solve the text indexing problem. ✓

`sli.do/comp526`

Suffix arrays – Construction

How to compute $L[0..n]$?

- from suffix tree
  - possible with traversal . . .
  - 👎 but we are trying to avoid constructing suffix trees!

- sorting the suffixes of T using general purpose sorting method
  - 👍 trivial to code!
  - but: comparing two suffixes can take $\Theta(n)$ character comparisons
  - 👎 $\Theta(n^2 \log n)$ time in worst case

$T = aa\ldots a\$$

Suffix arrays – Construction

How to compute $L[0..n]$?

- from suffix tree
  - possible with traversal . . .
  - but we are trying to avoid constructing suffix trees!

- sorting the suffixes of T using general purpose sorting method
  - trivial to code!
  - but: comparing two suffixes can take $\Theta(n)$ character comparisons
  - $\Theta(n^2 \log n)$ time in worst case

- We can do better!

① better string sorting

② suffix sorting

Fat-pivot radix quicksort – Example

she

sells

seashells

by

the

sea

shore

the

shells

she

sells

are

surely

seashells

Fat-pivot radix quicksort – Example

she
sells
seashells
by
the
sea
shore
the
shells
she
sells
are
surely
seashells

Fat-pivot radix quicksort – Example

she	by
sells	are
seashells	she
by	sells
the	seashells
sea	sea
shore	shore
the	shells
shells	she
she	sells
sells	surely
are	seashells
surely	the
seashells	the

Fat-pivot radix quicksort – Example

she	by	are
sells	are	by
seashells	she	
by	sells	
the	seashells	
sea	sea	
shore	shore	
the	shells	
shells	she	
she	sells	
sells	surely	
are	seashells	
surely	the	
seashells	the	

Fat-pivot radix quicksort – Example

she	by	are
sells	are	by
seashells	she	sells
by	sells	seashells
the	seashells	sea
sea	sea	sells
shore	shore	seashells
the	shells	she
shells	she	shore
she	sells	shells
sells	surely	she
are	seashells	surely
surely	the	
seashells	the	

Fat-pivot radix quicksort – Example

she	by	are
sells	are	by
seashells	she	sells
by	sells	seashells
the	seashells	sea
sea	sea	sells
shore	shore	seashells
the	shells	she
shells	she	shore
she	sells	shells
sells	surely	she
are	seashells	surely
surely	the	the
seashells	the	the

Fat-pivot radix quicksort – Example

she	by	are	
sells	are	by	
seashells	she	sells	sells
by	sells	seashells	seashells
the	seashells	sea	sea
sea	sea	sells	sells
shore	shore	seashells	seashells
the	shells	she	
shells	she	shore	
she	sells	shells	
sells	surely	she	
are	seashells	surely	
surely	the	the	
seashells	the	the	

Fat-pivot radix quicksort – Example

she	by	are	
sells	are	by	
seashells	she	sells	sells
by	sells	seashells	seashells
the	seashells	sea	sea
sea	sea	sells	sells
shore	shore	seashells	seashells
the	shells	she	she$
shells	she	shore	shells
she	sells	shells	she$
sells	surely	she	shore
are	seashells	surely	
surely	the	the	
seashells	the	the	

Fat-pivot radix quicksort – Example

she	by	are	
sells	are	by	
seashells	she	sells	sells
by	sells	seashells	seashells
the	seashells	sea	sea
sea	sea	sells	sells
shore	shore	seashells	seashells
the	shells	she	she$
shells	she	shore	shells
she	sells	shells	she$
sells	surely	she	shore
are	seashells	surely	
surely	the	the	the
seashells	the	the	the

Fat-pivot radix quicksort – Example

she	by	are		
sells	are	by		
seashells	she	sells	sells	seashells
by	sells	seashells	seashells	sea
the	seashells	sea	sea	seashells
sea	sea	sells	sells	sells
shore	shore	seashells	seashells	sells
the	shells	she	she$	
shells	she	shore	shells	
she	sells	shells	she$	
sells	surely	she	shore	
are	seashells	surely		
surely	the	the	the	
seashells	the	the	the	

Fat-pivot radix quicksort – Example

she	by	are		
sells	are	by		
seashells	she	sells	sells	seashells
by	sells	seashells	seashells	sea
the	seashells	sea	sea	seashells
sea	sea	sells	sells	sells
shore	shore	seashells	seashells	sells
the	shells	she	she$	she
shells	she	shore	shells	she
she	sells	shells	she$	shells
sells	surely	she	shore	
are	seashells	surely		
surely	the	the	the	
seashells	the	the	the	

Fat-pivot radix quicksort – Example

she	by	are		
sells	are	by		
seashells	she	sells	sells	seashells
by	sells	seashells	seashells	sea
the	seashells	sea	sea	seashells
sea	sea	sells	sells	sells
shore	shore	seashells	seashells	sells
the	shells	she	she$	she
shells	she	shore	shells	she
she	sells	shells	she$	shells
sells	surely	she	shore	
are	seashells	surely		
surely	the	the	the	the
seashells	the	the	the	the

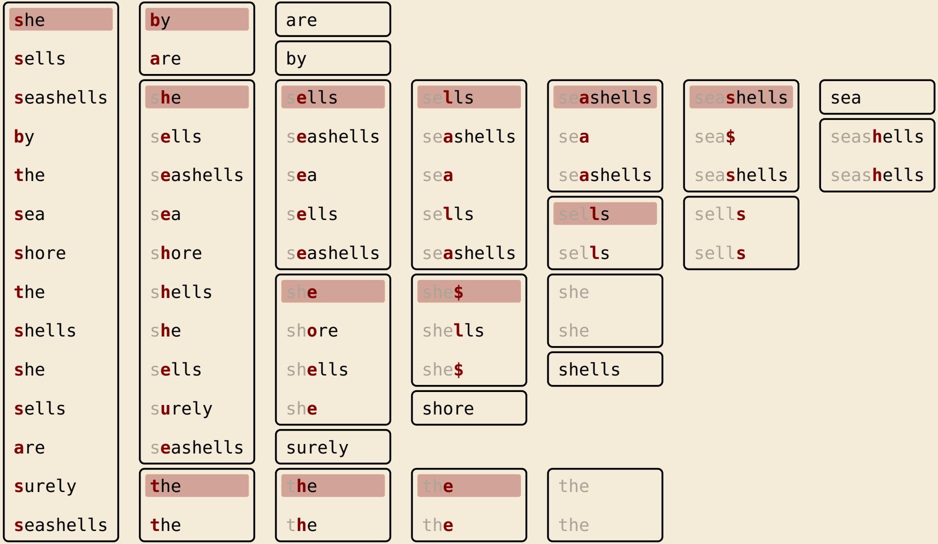
Fat-pivot radix quicksort – Example

she	by	are			
sells	are	by			
seashells	she	sells	sells	seashells	seashells
by	sells	seashells	seashells	sea	sea$
the	seashells	sea	sea	seashells	seashells
sea	sea	sells	sells	sells	
shore	shore	seashells	seashells	sells	
the	shells	she	she$	she	
shells	she	shore	shells	she	
she	sells	shells	she$	shells	
sells	surely	she	shore		
are	seashells	surely			
surely	the	the	the	the	
seashells	the	the	the	the	

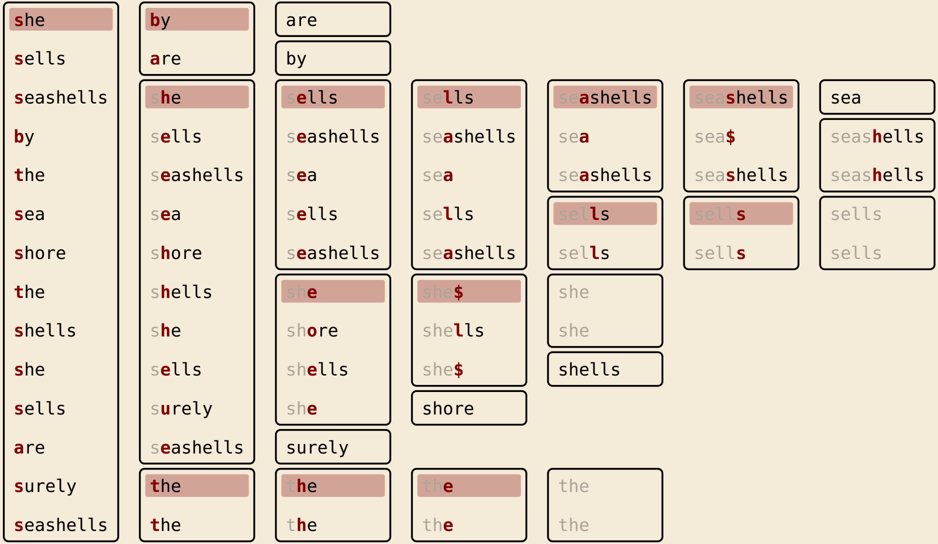
Fat-pivot radix quicksort – Example

she	by	are			
sells	are	by			
seashells	she	sells	sells	seashells	seashells
by	sells	seashells	seashells	sea	sea$
the	seashells	sea	sea	seashells	seashells
sea	sea	sells	sells	sells	sells
shore	shore	seashells	seashells	sells	sells
the	shells	she	she$	she	
shells	she	shore	shells	she	
she	sells	shells	she$	shells	
sells	surely	she	shore		
are	seashells	surely			
surely	the	the	the	the	
seashells	the	the	the	the	

Fat-pivot radix quicksort – Example



Fat-pivot radix quicksort – Example



Fat-pivot radix quicksort – Example

she	by	are				
sells	are	by				
seashells	she	sells	sells	seashells	seashells	sea
by	sells	seashells	seashells	sea	sea$	seashells
the	seashells	sea	sea	seashells	seashells	seashells
sea	sea	sells	sells	sells	sells	sells
shore	shore	seashells	seashells	sells	sells	sells
the	shells	she	she$	she		
shells	she	shore	shells	she		
she	sells	shells	she$	shells		
sells	surely	she	shore			
are	seashells	surely				
surely	the	the	the	the		
seashells	the	the	the	the		

. . .

Fat-pivot radix quicksort

details in §5.1 of Sedgewick, Wayne *Algorithms 4th ed.* (2011), Pearson

- **partition** based on d**th character** only (initially $d = 0$)

$\rightsquigarrow$ 3 segments: smaller, equal, or larger than dth symbol of pivot

- recurse on smaller and large with same d, on equal with $d + 1$
  - $\rightsquigarrow$ never compare equal prefixes twice

Fat-pivot radix quicksort

details in §5.1 of Sedgewick, Wayne *Algorithms 4th ed.* (2011), Pearson

- **partition** based on ***d*th character** only (initially $d = 0$)

$\rightsquigarrow$ 3 segments: smaller, equal, or larger than dth symbol of pivot

- recurse on smaller and large with same d, on equal with $d + 1$
  - $\rightsquigarrow$ never compare equal prefixes twice

for random strings

$\rightsquigarrow$ can show: $\sim 2\ln(2) \cdot n \lg n \approx 1.39 n \lg n$ character comparisons on average

👍 simple to code

👍 efficient for sorting many lists of strings

random string

- fat-pivot radix quicksort finds suffix array in $O(n \log n)$ expected time

Fat-pivot radix quicksort

details in §5.1 of Sedgewick, Wayne *Algorithms 4th ed.* (2011), Pearson

- **partition** based on d**th character** only (initially $d = 0$)

$\rightsquigarrow$ 3 segments: smaller, equal, or larger than dth symbol of pivot

- recurse on smaller and large with same d, on equal with $d + 1$
  - $\rightsquigarrow$ never compare equal prefixes twice

for random strings

$\rightsquigarrow$ can show: $\sim 2\ln(2) \cdot n \lg n \approx 1.39 n \lg n$ character comparisons on average

worst case for suffix sorting $\Theta(n^2)$

- simple to code
- efficient for sorting many lists of strings

random string

- fat-pivot radix quicksort finds suffix array in $O(n \log n)$ expected time

$T = a \dots a\$$

*but we can do $O(n)$ time **worst case**!*

6.6 Linear-Time Suffix Sorting: Overview

Inverse suffix array: going left & right

- to understand the fastest algorithm, it is helpful to define the *inverse suffix array*:
  - $R[i] = r \iff L[r] = i$ L = *leaf array*
    $\iff$ there are r suffixes that come before T_i in sorted order
    $\iff$ T_i has (0-based) *rank* r $\rightsquigarrow$ call $R[0..n]$ the ***rank array***

i	$R[i]$	T_i
0	6th	bananaban$
1	4th	ananaban$
2	9th	nanaban$
3	3th	anaban$
4	8th	naban$
5	1th	aban$
6	5th	ban$
7	2th	an$
8	7th	n$
9	0th	$

right

$R[0] = 6$

$L[8] = 4$

left

r	$L[r]$	$T_{L[r]}$
0	9	$
1	5	aban$
2	7	an$
3	3	anaban$
4	1	ananaban$
5	6	ban$
6	0	bananaban$
7	8	n$
8	4	naban$
9	2	nanaban$

sort suffixes

Clicker Question

Recap: Check all correct statements about suffix array $L[0..n]$, inverse suffix array $R[0..n]$, and suffix tree $\mathcal{T}$ of text T.

- **A** L lists the leaves of $\mathcal{T}$ in left-to-right order.
- **B** R lists the leaves of $\mathcal{T}$ in right-to-left order.
- **C** R lists starting indices of suffixes in lexciographic order.
- **D** L lists starting indices of suffixes in lexciographic order.
- **E** $L[r] = i$ iff $R[i] = r$
- **F** L stands for leaf
- **G** L stands for left
- **H** R stands for rank
- **I** R stands for right

sli.do/comp526

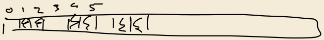
Clicker Question

Recap: Check all correct statements about suffix array $L[0..n]$, inverse suffix array $R[0..n]$, and suffix tree $\mathcal{T}$ of text T.

- **A** L lists the leaves of $\mathcal{T}$ in left-to-right order. ✓
- **B** ~~R lists the leaves of $\mathcal{T}$ in right-to-left order.~~
- **C** ~~R lists starting indices of suffixes in lexciographic order.~~
- **D** L lists starting indices of suffixes in lexciographic order. ✓
- **E** $L[r] = i$ iff $R[i] = r$ ✓
- **F** L stands for leaf ✓
- **G** L stands for left ✓
- **H** R stands for rank ✓
- **I** R stands for right ✓

sli.do/comp526

Linear-time suffix sorting



DOC3 / Skew algorithm

not a multiple of 3

1. Compute rank array $R_{1,2}$ for suffixes T_i starting at $i \not\equiv 0 \pmod 3$ *recursively*.

2. Induce rank array R_3 for suffixes $T_0, T_3, T_6, T_9, \ldots$ from $R_{1,2}$.

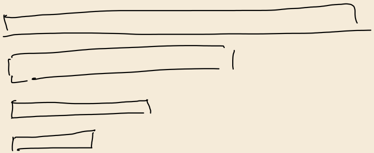
3. Merge $R_{1,2}$ and R_0 using $R_{1,2}$.
   $\rightsquigarrow$ rank array R for entire input

Linear-time suffix sorting

DC3 / Skew algorithm

not a multiple of 3

1. Compute rank array $R_{1,2}$ for suffixes T_i starting at $i \not\equiv 0 \pmod 3$ *recursively*.
2. Induce rank array R_3 for suffixes $T_0, T_3, T_6, T_9, \ldots$ from $R_{1,2}$.
3. Merge $R_{1,2}$ and R_0 using $R_{1,2}$.
   $\rightsquigarrow$ rank array R for entire input



top most recursive call

- We will show that steps 2. and 3. take $\Theta(n)$ time

$\rightsquigarrow$ Total complexity is $n + \frac{2}{3}n + \left(\frac{2}{3}\right)^2 n + \left(\frac{2}{3}\right)^3 n + \cdots \le n \cdot \sum_{i \ge 0} \left(\frac{2}{3}\right)^i = 3n = \Theta(n)$

Linear-time suffix sorting

DCB / Skew algorithm

not a multiple of 3

1. Compute rank array $R_{1,2}$ for suffixes T_i starting at $i \not\equiv 0 \pmod 3$ *recursively.*
2. Induce rank array R_3 for suffixes $T_0, T_3, T_6, T_9, \ldots$ from $R_{1,2}$.
3. Merge $R_{1,2}$ and R_0 using $R_{1,2}$.
   $\rightsquigarrow$ rank array R for entire input

- We will show that steps 2. and 3. take $\Theta(n)$ time

$\rightsquigarrow$ Total complexity is $n + \frac{2}{3}n + \left(\frac{2}{3}\right)^2 n + \left(\frac{2}{3}\right)^3 n + \cdots \;\le\; n \cdot \sum_{i \ge 0} \left(\frac{2}{3}\right)^i \;=\; 3n \;=\; \Theta(n)$

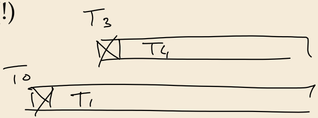
- **Note:** L can easily be computed from R in one pass, and vice versa.
  $\rightsquigarrow$ Can use whichever is more convenient.

DC3 / Skew algorithm – Step 2: Inducing ranks

- **Assume:** rank array $R_{1,2}$ known:
  - $$R_{1,2}[i] \;=\; \begin{cases} \text{rank of } T_i \text{ among } T_1, T_2, T_4, T_5, T_7, T_8, \ldots & \text{for } i = 1, 2, 4, 5, 7, 8, \ldots \\ \text{undefined} & \text{for } i = 0, 3, 6, 9, \ldots \end{cases}$$
- **Task:** sort the suffixes $T_0, T_3, T_6, T_9, \ldots$ in linear time (!)

DC3 / Skew algorithm – Step 2: Inducing ranks

- **Assume:** rank array $R_{1,2}$ known:
  - $R_{1,2}[i] \;=\; \begin{cases} \text{rank of } T_i \text{ among } T_1, T_2, T_4, T_5, T_7, T_8, \ldots & \text{for } i = 1,2,4,5,7,8,\ldots \\ \text{undefined} & \text{for } i = 0,3,6,9,\ldots \end{cases}$
- **Task:** sort the suffixes $T_0, T_3, T_6, T_9, \ldots$ in linear time (!)



- Suppose we want to compare T_0 and T_3.
  - Characterwise comparisons too expensive
  - but: after removing first character, we obtain T_1 and T_4
  - these two can be compared in *constant time* by comparing $R_{1,2}[1]$ and $R_{1,2}[4]$!

$\rightsquigarrow$ T_0 comes before T_3 in lexicographic order
iff pair $(T[0], R_{1,2}[1])$ comes before pair $(T[3], R_{1,2}[4])$ in lexicographic order

DC3 / Skew algorithm – Inducing ranks example

T = `hannahbansbananasman$$$` (append 3 `$` markers)

T_0	`hannahbansbananasman$$$`
T_3	`nahbansbananasman$$$`
T_6	`bansbananasman$$$`
T_9	`sbananasman$$$`
T_{12}	`nanasman$$$`
T_{15}	`asman$$$`
T_{18}	`an$$$`
T_{21}	`$$`

T_1	`annahbansbananasman$$$`	$R_{1,2}[22] = 0$	T_{22}	`$`
T_2	`nnahbansbananasman$$$`	$R_{1,2}[20] = 1$	T_{20}	`$$$`
T_4	`ahbansbananasman$$$`	$R_{1,2}[\ 4] = 2$	T_4	`ahbansbananasman$$$`
T_5	`hbansbananasman$$$`	$R_{1,2}[11] = 3$	T_{11}	`ananasman$$$`
T_7	`ansbananasman$$$`	$R_{1,2}[13] = 4$	T_{13}	`anasman$$$`
T_8	`nsbananasman$$$`	$R_{1,2}[\ 1] = 5$	T_1	`annahbansbananasman$$$`
T_{10}	`bananasman$$$`	$R_{1,2}[\ 7] = 6$	T_7	`ansbananasman$$$`
T_{11}	`ananasman$$$`	$R_{1,2}[10] = 7$	T_{10}	`bananasman$$$`
T_{13}	`anasman$$$`	$R_{1,2}[\ 5] = 8$	T_5	`hbansbananasman$$$`
T_{14}	`nasman$$$`	$R_{1,2}[17] = 9$	T_{17}	`man$$$`
T_{16}	`sman$$$`	$R_{1,2}[19] = 10$	T_{19}	`n$$$`
T_{17}	`man$$$`	$R_{1,2}[14] = 11$	T_{14}	`nasman$$$`
T_{19}	`n$$$`	$R_{1,2}[\ 2] = 12$	T_2	`nnahbansbananasman$$$`
T_{20}	`$$$`	$R_{1,2}[\ 8] = 13$	T_8	`nsbananasman$$$`
T_{22}	`$`	$R_{1,2}[16] = 14$	T_{16}	`sman$$$`

$R_{1,2}$ (known)

DC3 / Skew algorithm – Inducing ranks example

T = `hannahbansbananasman$$$` (append 3 `$` markers)

T_0	`hannahbansbananasman$$$`
T_3	`nahbansbananasman$$$`
T_6	`bansbananasman$$$`
T_9	`sbananasman$$$`
T_{12}	`nanasman$$$`
T_{15}	`asman$$$`
T_{18}	`an$$$`
T_{21}	`$$`

`sman$$$` $= T_{16}$

T_1	`annahbansbananasman$$$`	$R_{1,2}[22] = 0$	T_{22}	`$`
T_2	`nnahbansbananasman$$$`	$R_{1,2}[20] = 1$	T_{20}	`$$$`
T_4	`ahbansbananasman$$$`	$R_{1,2}[\ 4] = 2$	T_4	`ahbansbananasman$$$`
T_5	`hbansbananasman$$$`	$R_{1,2}[11] = 3$	T_{11}	`ananasman$$$`
T_7	`ansbananasman$$$`	$R_{1,2}[13] = 4$	T_{13}	`anasman$$$`
T_8	`nsbananasman$$$`	$R_{1,2}[\ 1] = 5$	T_1	`annahbansbananasman$$$`
T_{10}	`bananasman$$$`	$R_{1,2}[\ 7] = 6$	T_7	`ansbananasman$$$`
T_{11}	`ananasman$$$`	$R_{1,2}[10] = 7$	T_{10}	`bananasman$$$`
T_{13}	`anasman$$$`	$R_{1,2}[\ 5] = 8$	T_5	`hbansbananasman$$$`
T_{14}	`nasman$$$`	$R_{1,2}[17] = 9$	T_{17}	`man$$$`
T_{16}	`sman$$$`	$R_{1,2}[19] = 10$	T_{19}	`n$$$`
T_{17}	`man$$$`	$R_{1,2}[14] = 11$	T_{14}	`nasman$$$`
T_{19}	`n$$$`	$R_{1,2}[\ 2] = 12$	T_2	`nnahbansbananasman$$$`
T_{20}	`$$$`	$R_{1,2}[\ 8] = 13$	T_8	`nsbananasman$$$`
T_{22}	`$`	$R_{1,2}[16] = 14$	T_{16}	`sman$$$`

$R_{1,2}$ (known)

$R_{1,2}[16] = 14$

T_0	`h05`
T_3	`n02`
T_6	`b06`
T_9	`s07`
T_{12}	`n04`
T_{15}	`a14`
T_{18}	`a10`
T_{21}	`$00`

DC3 / Skew algorithm – Inducing ranks example

T = `hannahbansbananasman$$$` (append 3 `$` markers)

T_0	`hannahbansbananasman$$$`
T_3	`nahbansbananasman$$$`
T_6	`bansbananasman$$$`
T_9	`sbananasman$$$`
T_{12}	`nanasman$$$`
T_{15}	`asman$$$`
T_{18}	`an$$$`
T_{21}	`$$`

`sman$$$` = T_{16}

T_1	`annahbansbananasman$$$`	$R_{1,2}[22] = 0$	T_{22}	`$`
T_2	`nnahbansbananasman$$$`	$R_{1,2}[20] = 1$	T_{20}	`$$$`
T_4	`ahbansbananasman$$$`	$R_{1,2}[\ 4] = 2$	T_4	`ahbansbananasman$$$`
T_5	`hbansbananasman$$$`	$R_{1,2}[11] = 3$	T_{11}	`ananasman$$$`
T_7	`ansbananasman$$$`	$R_{1,2}[13] = 4$	T_{13}	`anasman$$$`
T_8	`nsbananasman$$$`	$R_{1,2}[\ 1] = 5$	T_1	`annahbansbananasman$$$`
T_{10}	`bananasman$$$`	$R_{1,2}[\ 7] = 6$	T_7	`ansbananasman$$$`
T_{11}	`ananasman$$$`	$R_{1,2}[10] = 7$	T_{10}	`bananasman$$$`
T_{13}	`anasman$$$`	$R_{1,2}[\ 5] = 8$	T_5	`hbansbananasman$$$`
T_{14}	`nasman$$$`	$R_{1,2}[17] = 9$	T_{17}	`man$$$`
T_{16}	`sman$$$`	$R_{1,2}[19] = 10$	T_{19}	`n$$$`
T_{17}	`man$$$`	$R_{1,2}[14] = 11$	T_{14}	`nasman$$$`
T_{19}	`n$$$`	$R_{1,2}[\ 2] = 12$	T_2	`nnahbansbananasman$$$`
T_{20}	`$$$`	$R_{1,2}[\ 8] = 13$	T_8	`nsbananasman$$$`
T_{22}	`$`	$R_{1,2}[16] = 14$	T_{16}	`sman$$$`

$R_{1,2}$ (known)

$R_{1,2}[16] = 14$

T_0	`h05`
T_3	`n02`
T_6	`b06`
T_9	`s07`
T_{12}	`n04`
T_{15}	`a14`
T_{18}	`a10`
T_{21}	`$00`

radix sort

T_{21}	`$00`	$\rightsquigarrow$	$R_0[21] = 0$
T_{18}	`a10`	$\rightsquigarrow$	$R_0[18] = 1$
T_{15}	`a14`	$\rightsquigarrow$	$R_0[15] = 2$
T_6	`b06`	$\rightsquigarrow$	$R_0[\ 6] = 3$
T_0	`h05`	$\rightsquigarrow$	$R_0[\ 0] = 4$
T_3	`n02`	$\rightsquigarrow$	$R_0[\ 3] = 5$
T_{12}	`n04`	$\rightsquigarrow$	$R_0[12] = 6$
T_9	`s07`	$\rightsquigarrow$	$R_0[\ 9] = 7$

DC3 / Skew algorithm – Inducing ranks example

T = `hannahbansbananasman$$$` (append 3 `$` markers)

T_0	`hannahbansbananasman$$$`
T_3	`nahbansbananasman$$$`
T_6	`bansbananasman$$$`
T_9	`sbananasman$$$`
T_{12}	`nanasman$$$`
T_{15}	`asman$$$`
T_{18}	`an$$$`
T_{21}	`$$`

`sman$$$` = T_{16}

T_1	`annahbansbananasman$$$`	$R_{1,2}[22] = 0$	T_{22}	`$`
T_2	`nnahbansbananasman$$$`	$R_{1,2}[20] = 1$	T_{20}	`$$$`
T_4	`ahbansbananasman$$$`	$R_{1,2}[\ 4] = 2$	T_4	`ahbansbananasman$$$`
T_5	`hbansbananasman$$$`	$R_{1,2}[11] = 3$	T_{11}	`ananasman$$$`
T_7	`ansbananasman$$$`	$R_{1,2}[13] = 4$	T_{13}	`anasman$$$`
T_8	`nsbananasman$$$`	$R_{1,2}[\ 1] = 5$	T_1	`annahbansbananasman$$$`
T_{10}	`bananasman$$$`	$R_{1,2}[\ 7] = 6$	T_7	`ansbananasman$$$`
T_{11}	`ananasman$$$`	$R_{1,2}[10] = 7$	T_{10}	`bananasman$$$`
T_{13}	`anasman$$$`	$R_{1,2}[\ 5] = 8$	T_5	`hbansbananasman$$$`
T_{14}	`nasman$$$`	$R_{1,2}[17] = 9$	T_{17}	`man$$$`
T_{16}	`sman$$$`	$R_{1,2}[19] = 10$	T_{19}	`n$$$`
T_{17}	`man$$$`	$R_{1,2}[14] = 11$	T_{14}	`nasman$$$`
T_{19}	`n$$$`	$R_{1,2}[\ 2] = 12$	T_2	`nnahbansbananasman$$$`
T_{20}	`$$$`	$R_{1,2}[\ 8] = 13$	T_8	`nsbananasman$$$`
T_{22}	`$`	$R_{1,2}[16] = 14$	T_{16}	`sman$$$`

$R_{1,2}$ (known)

$R_{1,2}[16] = 14$

T_0	`h05`
T_3	`n02`
T_6	`b06`
T_9	`s07`
T_{12}	`n04`
T_{15}	`a14`
T_{18}	`a10`
T_{21}	`$00`

radix sort

T_{21}	`$00`	$\rightsquigarrow$	$R_0[21] = 0$
T_{18}	`a10`	$\rightsquigarrow$	$R_0[18] = 1$
T_{15}	`a14`	$\rightsquigarrow$	$R_0[15] = 2$
T_6	`b06`	$\rightsquigarrow$	$R_0[\ 6] = 3$
T_0	`h05`	$\rightsquigarrow$	$R_0[\ 0] = 4$
T_3	`n02`	$\rightsquigarrow$	$R_0[\ 3] = 5$
T_{12}	`n04`	$\rightsquigarrow$	$R_0[12] = 6$
T_9	`s07`	$\rightsquigarrow$	$R_0[\ 9] = 7$

R_0

DC3 / Skew algorithm – Inducing ranks example

T = hannahbansbananasman$$$ (append 3 $ markers)

T_0	hannahbansbananasman$$$
T_3	nahbansbananasman$$$
T_6	bansbananasman$$$
T_9	sbananasman$$$
T_{12}	nanasman$$$
T_{15}	asman$$$
T_{18}	an$$$
T_{21}	$$

sman$$$ = T_{16}

T_1	annahbansbananasman$$$	$R_{1,2}[22] = 0$	T_{22}	$
T_2	nnahbansbananasman$$$	$R_{1,2}[20] = 1$	T_{20}	$$$
T_4	ahbansbananasman$$$	$R_{1,2}[\ 4] = 2$	T_4	ahbansbananasman$$$
T_5	hbansbananasman$$$	$R_{1,2}[11] = 3$	T_{11}	ananasman$$$
T_7	ansbananasman$$$	$R_{1,2}[13] = 4$	T_{13}	anasman$$$
T_8	nsbananasman$$$	$R_{1,2}[\ 1] = 5$	T_1	annahbansbananasman$$$
T_{10}	bananasman$$$	$R_{1,2}[\ 7] = 6$	T_7	ansbananasman$$$
T_{11}	ananasman$$$	$R_{1,2}[10] = 7$	T_{10}	bananasman$$$
T_{13}	anasman$$$	$R_{1,2}[\ 5] = 8$	T_5	hbansbananasman$$$
T_{14}	nasman$$$	$R_{1,2}[17] = 9$	T_{17}	man$$$
T_{16}	sman$$$	$R_{1,2}[19] = 10$	T_{19}	n$$$
T_{17}	man$$$	$R_{1,2}[14] = 11$	T_{14}	nasman$$$
T_{19}	n$$$	$R_{1,2}[\ 2] = 12$	T_2	nnahbansbananasman$$$
T_{20}	$$$	$R_{1,2}[\ 8] = 13$	T_8	nsbananasman$$$
T_{22}	$	$R_{1,2}[16] = 14$	T_{16}	sman$$$

$R_{1,2}$ (known)

$R_{1,2}[16] = 14$

T_0	h05
T_3	n02
T_6	b06
T_9	s07
T_{12}	n04
T_{15}	a14
T_{18}	a10
T_{21}	$00

radix sort

T_{21}	$00	⇝	$R_0[21] = 0$
T_{18}	a10	⇝	$R_0[18] = 1$
T_{15}	a14	⇝	$R_0[15] = 2$
T_6	b06	⇝	$R_0[\ 6] = 3$
T_0	h05	⇝	$R_0[\ 0] = 4$
T_3	n02	⇝	$R_0[\ 3] = 5$
T_{12}	n04	⇝	$R_0[12] = 6$
T_9	s07	⇝	$R_0[\ 9] = 7$

R_0

- sorting of pairs doable in $O(n)$ time by 2 iterations of counting sort

⇝ Obtain R_0 in $O(n)$ time

DC3 / Skew algorithm – Step 3: Merging

T_{21}	`$$`
T_{18}	`an$$$`
T_{15}	`asman$$$`
T_{6}	`bansbananasman$$$`
T_{0}	`hannahbansbananasman$$$`
T_{3}	`nahbansbananasman$$$`
T_{12}	`nanasman$$$`
T_{9}	`sbananasman$$$`

T_{22}	`$`
T_{20}	`$$$`
T_{4}	`ahbansbananasman$$$`
T_{11}	`ananasman$$$`
T_{13}	`anasman$$$`
T_{1}	`annahbansbananasman$$$`
T_{7}	`ansbananasman$$$`
T_{10}	`bananasman$$$`
T_{5}	`hbansbananasman$$$`
T_{17}	`man$$$`
T_{19}	`n$$$`
T_{14}	`nasman$$$`
T_{2}	`nnahbansbananasman$$$`
T_{8}	`nsbananasman$$$`
T_{16}	`sman$$$`

- Have:
  - sorted 1,2-list:
    $T_1, T_2, T_4, T_5, T_7, T_8, T_{10}, T_{11}, \ldots$
  - sorted 0-list:
    $T_0, T_3, T_6, T_9, \ldots$
- Task: Merge them!
  - use standard merging method from Mergesort
  - but speed up comparisons using $R_{1,2}$

DC3 / Skew algorithm – Step 3: Merging

T_{21}	`$$`
T_{18}	`an$$$`
T_{15}	`asman$$$`
T_{6}	`bansbananasman$$$`
T_{0}	`hannahbansbananasman$$$`
T_{3}	`nahbansbananasman$$$`
T_{12}	`nanasman$$$`
T_{9}	`sbananasman$$$`

T_{22}	`$`
T_{20}	`$$$`
T_{4}	`ahbansbananasman$$$`
T_{11}	`ananasman$$$`
T_{13}	`anasman$$$`
T_{1}	`annahbansbananasman$$$`
T_{7}	`ansbananasman$$$`
T_{10}	`bananasman$$$`
T_{5}	`hbansbananasman$$$`
T_{17}	`man$$$`
T_{19}	`n$$$`
T_{14}	`nasman$$$`
T_{2}	`nnahbansbananasman$$$`
T_{8}	`nsbananasman$$$`
T_{16}	`sman$$$`

T_{22}	`$`
T_{21}	`$$`
T_{20}	`$$$`
T_{4}	`ahbansbananasman$$$`
T_{18}	`an$$$`

- Have:
  - sorted 1,2-list:
    $T_1, T_2, T_4, T_5, T_7, T_8, T_{10}, T_{11}, \ldots$
  - sorted 0-list:
    $T_0, T_3, T_6, T_9, \ldots$
- Task: Merge them!
  - use standard merging method from Mergesort
  - but speed up comparisons using $R_{1,2}$

DC3 / Skew algorithm – Step 3: Merging

T_{21}	`$$`
T_{18}	`an$$$`
T_{15}	`asman$$$`
T_{6}	`bansbananasman$$$`
T_{0}	`hannahbansbananasman$$$`
T_{3}	`nahbansbananasman$$$`
T_{12}	`nanasman$$$`
T_{9}	`sbananasman$$$`

T_{22}	`$`
T_{20}	`$$$`
T_{4}	`ahbansbananasman$$$`
T_{11}	`ananasman$$$`
T_{13}	`anasman$$$`
T_{1}	`annahbansbananasman$$$`
T_{7}	`ansbananasman$$$`
T_{10}	`bananasman$$$`
T_{5}	`hbansbananasman$$$`
T_{17}	`man$$$`
T_{19}	`n$$$`
T_{14}	`nasman$$$`
T_{2}	`nnahbansbananasman$$$`
T_{8}	`nsbananasman$$$`
T_{16}	`sman$$$`

T_{22}	`$`
T_{21}	`$$`
T_{20}	`$$$`
T_{4}	`ahbansbananasman$$$`
T_{18}	`an$$$`

- Have:
  - sorted 1,2-list:
    $T_1, T_2, T_4, T_5, T_7, T_8, T_{10}, T_{11}, \ldots$
  - sorted 0-list:
    $T_0, T_3, T_6, T_9, \ldots$
- Task: Merge them!
  - use standard merging method from Mergesort
  - but speed up comparisons using $R_{1,2}$

Compare T_{15} to T_{11}

Idea: try same trick as before

$$T_{15} = \texttt{asman\$\$\$} = \texttt{a}\,\texttt{sman\$\$\$} = \texttt{a}T_{16}$$

$$T_{11} = \texttt{ananasman\$\$\$} = \texttt{a}\,\texttt{nanasman\$\$\$} = \texttt{a}T_{12}$$

DC3 / Skew algorithm – Step 3: Merging

T_{21}	`$$`
T_{18}	`an$$$`
T_{15}	`asman$$$`
T_{6}	`bansbananasman$$$`
T_{0}	`hannahbansbananasman$$$`
T_{3}	`nahbansbananasman$$$`
T_{12}	`nanasman$$$`
T_{9}	`sbananasman$$$`

T_{22}	`$`
T_{20}	`$$$`
T_{4}	`ahbansbananasman$$$`
T_{11}	`ananasman$$$`
T_{13}	`anasman$$$`
T_{1}	`annahbansbananasman$$$`
T_{7}	`ansbananasman$$$`
T_{10}	`bananasman$$$`
T_{5}	`hbansbananasman$$$`
T_{17}	`man$$$`
T_{19}	`n$$$`
T_{14}	`nasman$$$`
T_{2}	`nnahbansbananasman$$$`
T_{8}	`nsbananasman$$$`
T_{16}	`sman$$$`

T_{22}	`$`
T_{21}	`$$`
T_{20}	`$$$`
T_{4}	`ahbansbananasman$$$`
T_{18}	`an$$$`

- Have:
  - sorted 1,2-list:
    $T_1, T_2, T_4, T_5, T_7, T_8, T_{10}, T_{11}, \ldots$
  - sorted 0-list:
    $T_0, T_3, T_6, T_9, \ldots$
- Task: Merge them!
  - use standard merging method from Mergesort
  - but speed up comparisons using $R_{1,2}$

Compare T_{15} to T_{11}

Idea: try same trick as before

$T_{15} =$ `asman$$$`
$=$ `a` `sman$$$`
$= \texttt{a}T_{16}$

can't compare T_{16} and T_{12} either!

$T_{11} =$ `ananasman$$$`
$=$ `a` `nanasman$$$`
$= \texttt{a}T_{12}$

DC3 / Skew algorithm – Step 3: Merging

T_{21}	`$$`
T_{18}	`an$$$`
T_{15}	`asman$$$`
T_6	`bansbananasman$$$`
T_0	`hannahbansbananasman$$$`
T_3	`nahbansbananasman$$$`
T_{12}	`nanasman$$$`
T_9	`sbananasman$$$`

T_{22}	`$`
T_{20}	`$$$`
T_4	`ahbansbananasman$$$`
T_{11}	`ananasman$$$`
T_{13}	`anasman$$$`
T_1	`annahbansbananasman$$$`
T_7	`ansbananasman$$$`
T_{10}	`bananasman$$$`
T_5	`hbansbananasman$$$`
T_{17}	`man$$$`
T_{19}	`n$$$`
T_{14}	`nasman$$$`
T_2	`nnahbansbananasman$$$`
T_8	`nsbananasman$$$`
T_{16}	`sman$$$`

T_{22}	`$`
T_{21}	`$$`
T_{20}	`$$$`
T_4	`ahbansbananasman$$$`
T_{18}	`an$$$`

- Have:
  - sorted 1,2-list:
    $T_1, T_2, T_4, T_5, T_7, T_8, T_{10}, T_{11}, \ldots$
  - sorted 0-list:
    $T_0, T_3, T_6, T_9, \ldots$
- Task: Merge them!
  - use standard merging method from Mergesort
  - but speed up comparisons using $R_{1,2}$

Compare T_{15} to T_{11}

Idea: try same trick as before

T_{15} = `asman$$$`
= `asman$$$`
= `a`T_{16}

can't compare T_{16} and T_{12} either!

T_{11} = `ananasman$$$`
= `ananasman$$$`
= `a`T_{12}

$\rightsquigarrow$ Compare T_{16} to T_{12}

T_{16} = `sman$$$`
= `sman$$$`
= `s`T_{17}

T_{12} = `nanasman$$$`
= `nanasman$$$`
= `a`T_{13}

DC3 / Skew algorithm – Step 3: Merging

T_{21}	`$$`
T_{18}	`an$$$`
T_{15}	`asman$$$`
T_6	`bansbananasman$$$`
T_0	`hannahbansbananasman$$$`
T_3	`nahbansbananasman$$$`
T_{12}	`nanasman$$$`
T_9	`sbananasman$$$`

T_{22}	`$`
T_{20}	`$$$`
T_4	`ahbansbananasman$$$`
T_{11}	`ananasman$$$`
T_{13}	`anasman$$$`
T_1	`annahbansbananasman$$$`
T_7	`ansbananasman$$$`
T_{10}	`bananasman$$$`
T_5	`hbansbananasman$$$`
T_{17}	`man$$$`
T_{19}	`n$$$`
T_{14}	`nasman$$$`
T_2	`nnahbansbananasman$$$`
T_8	`nsbananasman$$$`
T_{16}	`sman$$$`

T_{22}	`$`
T_{21}	`$$`
T_{20}	`$$$`
T_4	`ahbansbananasman$$$`
T_{18}	`an$$$`

- Have:
  - sorted 1,2-list:
    $T_1, T_2, T_4, T_5, T_7, T_8, T_{10}, T_{11}, \ldots$
  - sorted 0-list:
    $T_0, T_3, T_6, T_9, \ldots$
- Task: Merge them!
  - use standard merging method from Mergesort
  - but speed up comparisons using $R_{1,2}$

Compare T_{15} to T_{11}

Idea: try same trick as before

T_{15} = `asman$$$`
= `asman$$$`
= `a`T_{16}

can't compare T_{16} and T_{12} either!

T_{11} = `ananasman$$$`
= `ananasman$$$`
= `a`T_{12}

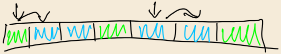
$\rightsquigarrow$ Compare T_{16} to T_{12}

T_{16} = `sman$$$`
= `sman$$$`
= `s`T_{17}

always at most 2 steps then can use $R_{1,2}$!

T_{12} = `nanasman$$$`
= `nanasman$$$`
= `a`T_{13}

DC3 / Skew algorithm – Step 3: Merging



0 mod 3 vs 2 mod ✓
" vs 1 mod ✓

T_{21}	`$$`
T_{18}	`an$$$`
$\boxed{T_{15}}$	`asman$$$`
T_{6}	`bansbananasman$$$`
T_{0}	`hannahbansbananasman$$$`
T_{3}	`nahbansbananasman$$$`
T_{12}	`nanasman$$$`
T_{9}	`sbananasman$$$`

T_{22}	`$`
T_{20}	`$$$`
T_{4}	`ahbansbananasman$$$`
$\boxed{T_{11}}$	`ananasman$$$`
T_{13}	`anasman$$$`
T_{1}	`annahbansbananasman$$$`
T_{7}	`ansbananasman$$$`
T_{10}	`bananasman$$$`
T_{5}	`hbansbananasman$$$`
T_{17}	`man$$$`
T_{19}	`n$$$`
T_{14}	`nasman$$$`
T_{2}	`nnahbansbananasman$$$`
T_{8}	`nsbananasman$$$`
T_{16}	`sman$$$`

T_{22}	`$`
T_{21}	`$$`
T_{20}	`$$$`
T_{4}	`ahbansbananasman$$$`
T_{18}	`an$$$`

- Have:
  - sorted 1,2-list:
    $T_1, T_2, T_4, T_5, T_7, T_8, T_{10}, T_{11}, \ldots$
  - sorted 0-list:
    $T_0, T_3, T_6, T_9, \ldots$
- Task: Merge them!
  - use standard merging method from Mergesort
  - but speed up comparisons using $R_{1,2}$

$\rightsquigarrow$ $O(n)$ time for merge

Compare T_{15} to T_{11}

Idea: try same trick as before

T_{15} = `asman$$$`
= `asman$$$`
= `a`T_{16}

can't compare T_{16} and T_{12} either!

T_{11} = `ananasman$$$`
= `ananasman$$$`
= `a`T_{12}

$\rightsquigarrow$ Compare T_{16} to T_{12}

T_{16} = `sman$$$`
= `sman$$$`
= `s`T_{17}

always at most 2 steps then can use $R_{1,2}$!

T_{12} = `nanasman$$$`
= `nanasman$$$`
= `a`T_{13}

6.7 Linear-Time Suffix Sorting: The DC3 Algorithm

DC3 / Skew algorithm – Fix recursive call

- both step 2. and 3. doable in $O(n)$ time!

DC3 / Skew algorithm – Fix recursive call

- both step 2. and 3. doable in $O(n)$ time!

- But: we cheated in 1. step! *"compute rank array $R_{1,2}$ recursively"*
  - Taking a *subset* of suffixes is *not* an instance of the same problem!

DC3 / Skew algorithm – Fix recursive call

- both step 2. and 3. doable in $O(n)$ time!

- But: we cheated in 1. step! *"compute rank array $R_{1,2}$ recursively"*
  - Taking a *subset* of suffixes is *not* an instance of the same problem!

  $\rightsquigarrow$ Need a single *string* T' to recurse on, from which we can deduce $R_{1,2}$.

How can we make T' "skip" some suffixes?

DC3 / Skew algorithm – Fix recursive call

- both step 2. and 3. doable in $O(n)$ time!

- But: we cheated in 1. step! *"compute rank array $R_{1,2}$ recursively"*
  - Taking a *subset* of suffixes is *not* an instance of the same problem!

  $\rightsquigarrow$ Need a single *string* T' to recurse on, from which we can deduce $R_{1,2}$.

How can we make T' "skip" some suffixes?

redefine alphabet to be *triples of characters* $\boxed{\texttt{abc}}$

$$T = \texttt{bananaban\$\$\$}$$

$$\rightsquigarrow \quad T^{\square} = \boxed{\texttt{ban}}\boxed{\texttt{ana}}\boxed{\texttt{ban}}\boxed{\texttt{\$\$\$}}$$

$$\boxed{\texttt{ana}}\boxed{\texttt{ban}}\boxed{\texttt{\$\$\$}}$$

$$\boxed{\texttt{ban}}\boxed{\texttt{\$\$\$}}$$

$$\boxed{\texttt{\$\$\$}}$$

$\rightsquigarrow$ suffixes of $T^{\square}$ $\leftrightsquigarrow$ $T_0, T_3, T_6, T_9, \ldots$

- $T' = T[1..n)^{\square}\ \boxed{\texttt{\$\$\$}}\ T[2..n)^{\square}\ \boxed{\texttt{\$\$\$}}$ $\leftrightsquigarrow$ T_i with $i \not\equiv 0 \pmod 3$.

$\rightsquigarrow$ Can call suffix sorting recursively on T' and map result to $R_{1,2}$

DC3 / Skew algorithm – Fix alphabet explosion

- Still does not quite work!

DC3 / Skew algorithm – Fix alphabet explosion

- Still does not quite work!
  - Each recursive step *cubes* σ by using triples!
  - $\leadsto$ (Eventually) cannot use linear-time sorting anymore!

DC3 / Skew algorithm – Fix alphabet explosion

- Still does not quite work!
  - Each recursive step *cubes* σ by using triples!
  - $\rightsquigarrow$ (Eventually) cannot use linear-time sorting anymore!

- But: Have at most $\frac{2}{3}n$ different triples $\boxed{\texttt{abc}}$ in T'!

$\rightsquigarrow$ Before recursion:

1. Sort all occurring triples. (using counting sort in $O(n)$)
2. Replace them by their *rank* (in Σ).

$\rightsquigarrow$ Maintains $\sigma \le n$ without affecting order of suffixes.

DC3 / Skew algorithm – Step 3. Example

$$T' = T[1..n)^{\square}\ \boxed{\$\$\$}\ T[2..n)^{\square}\ \boxed{\$\$\$}$$

- T = `hannahbansbananasman$`

DC3 / Skew algorithm – Step 3. Example

$$T' = T[1..n)^{\square}\ \boxed{\$\$\$}\ T[2..n)^{\square}\ \boxed{\$\$\$}$$

- T = ~~h~~annahbansbananasman$ T_2 = nnahbansbananasman$

  $T' =$ $\boxed{\texttt{ann}}\boxed{\texttt{ahb}}\boxed{\texttt{ans}}\boxed{\texttt{ban}}\boxed{\texttt{ana}}\boxed{\texttt{sma}}\boxed{\texttt{n\$\$}}\ \boxed{\texttt{\$\$\$}}\ \boxed{\texttt{nna}}\boxed{\texttt{hba}}\boxed{\texttt{nsb}}\boxed{\texttt{ana}}\boxed{\texttt{nas}}\boxed{\texttt{man}}\boxed{\texttt{\$\$\$}}$

DC3 / Skew algorithm – Step 3. Example

$T' = T[1..n)^{\square}\ \boxed{\$\$\$}\ T[2..n)^{\square}\ \boxed{\$\$\$}$

- T = `hannahbansbananasman$` T_2 = `nnahbansbananasman$`

  T' = $\boxed{\text{ann}}\boxed{\text{ahb}}\boxed{\text{ans}}\boxed{\text{ban}}\boxed{\text{ana}}\boxed{\text{sma}}\boxed{\text{n\$\$}}\ \boxed{\$\$\$}\ \boxed{\text{nna}}\boxed{\text{hba}}\boxed{\text{nsb}}\boxed{\text{ana}}\boxed{\text{nas}}\boxed{\text{man}}\boxed{\$\$\$}$

- Occurring triples:

  $\boxed{\text{ann}}\boxed{\text{ahb}}\boxed{\text{ans}}\boxed{\text{ban}}\boxed{\text{ana}}\boxed{\text{sma}}\boxed{\text{n\$\$}}\ \boxed{\$\$\$}\ \boxed{\text{nna}}\boxed{\text{hba}}\boxed{\text{nsb}}\quad\boxed{\text{nas}}\boxed{\text{man}}$

DC3 / Skew algorithm – Step 3. Example

$T' = T[1..n)^{\square}$ $\boxed{\$\$\$}$ $T[2..n)^{\square}$ $\boxed{\$\$\$}$

- T = hannahbansbananasman\$ T_2 = nnahbansbananasman\$
  T' = ann ahb ans ban ana sma n\$\$ \$\$\$ nna hba nsb ana nas man \$\$\$
- Occurring triples:
  ann ahb ans ban ana sma n\$\$ \$\$\$ nna hba nsb nas man
- Sorted triples with ranks:

Rank	00	01	02	03	04	05	06	07	08	09	10	11	12
Triple	\$\$\$	ahb	ana	ann	ans	ban	hba	man	n\$\$	nas	nna	nsb	sma

DC3 / Skew algorithm – Step 3. Example

$T' = T[1..n)^{\square}$ $\boxed{\texttt{\$\$\$}}$ $T[2..n)^{\square}$ $\boxed{\texttt{\$\$\$}}$

- T = `hannahbansbananasman$` T_2 = `nnahbansbananasman$`
  T' = [ann][ahb][ans][ban][ana][sma][n$$] [$$$] [nna][hba][nsb][ana][nas][man][$$$]

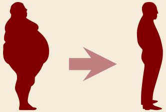
- Occurring triples:
  [ann][ahb][ans][ban][ana][sma][n$$] [$$$] [nna][hba][nsb] [nas][man]

- Sorted triples with ranks:

Rank	00	01	02	03	04	05	06	07	08	09	10	11	12
Triple	$$$	ahb	ana	ann	ans	ban	hba	man	n$$	nas	nna	nsb	sma

- T' = [ann][ahb][ans][ban][ana][sma][n$$] [$$$] [nna][hba][nsb][ana][nas][man][$$$]
  T'' = [03] [01] [04] [05] [02] [12] [08] [00] [10] [06] [11] [02] [09] [07] [00]

Suffix array – Discussion



- sleek data structure compared to suffix tree
- simple and fast $O(n \log n)$ construction (for random strings)
- more involved but optimal $O(n)$ construction
- supports efficient string matching

- string matching takes $O(m \log n)$, not optimal $O(m)$
- Cannot use more advanced suffix tree features
  e. g., for longest repeated substrings

6.8 The LCP Array

Clicker Question

Which feature of suffix **trees** did we use to find the *length* of a longest repeated substring?

- **A** order of leaves
- **B** path label of internal nodes
- **C** string depth of internal nodes
- **D** constant-time traversal to child nodes
- **E** constant-time traversal to parent nodes
- **F** constant-time traversal to leftmost leaf in subtree

`sli.do/comp526`

Clicker Question

Which feature of suffix **trees** did we use to find the *length* of a longest repeated substring?

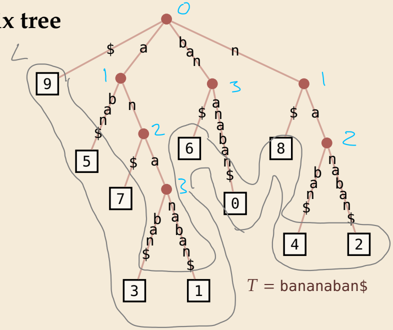
- **A** ~~order of leaves~~
- **B** ~~path label of internal nodes~~
- **C** string depth of internal nodes ✓
- **D** ~~constant-time traversal to child nodes~~
- **E** ~~constant-time traversal to parent nodes~~
- **F** ~~constant-time traversal to leftmost leaf in subtree~~

`sli.do/comp526`

String depths of internal nodes

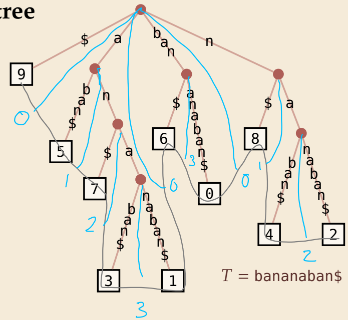
- Recall algorithm for longest repeated substring in **suffix tree**
  1. Compute *string depth* of nodes
  2. Find *path label to node* with maximal string depth

- Can we do this using **suffix *arrays***?



String depths of internal nodes

- Recall algorithm for longest repeated substring in **suffix tree**
  1. Compute *string depth* of nodes
  2. Find *path label to node* with maximal string depth



T = bananaban$

- Can we do this using **suffix *arrays***?

- Yes, by **enhancing** the suffix array with the ***LCP array***!
  $\mathrm{LCP}[1..n]$
  $\mathrm{LCP}[r] \;=\; \mathrm{LCP}(T_{L[r]}, T_{L[r-1]})$

length of longest common prefix of suffixes of rank r and $r-1$

⇝ longest repeated substring $=$ find maximum in $\mathrm{LCP}[1..n]$

LCP array and internal nodes

$L[0..n]$

9
5
7
3
1
6
0
8
4
2

LCP array and internal nodes

	$L[0..n]$
`$`	9
`aban$`	5
`an$`	7
`anaban$`	3
`ananaban$`	1
`ban$`	6
`bananaban$`	0
`n$`	8
`naban$`	4
`nanaban$`	2

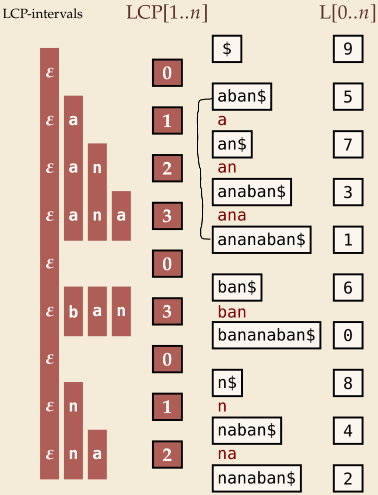
LCP array and internal nodes

	L[0..n]
$	9
aban$	5
a	
an$	7
an	
anaban$	3
ana	
ananaban$	1
ban$	6
ban	
bananaban$	0
n$	8
n	
naban$	4
na	
nanaban$	2

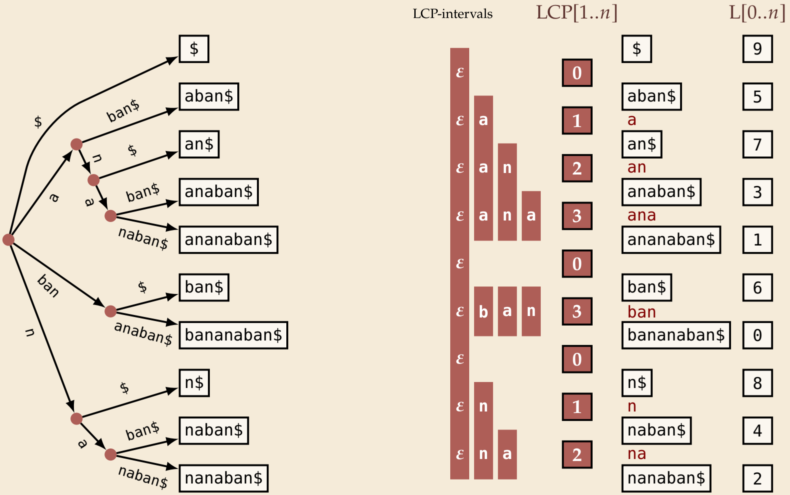
LCP array and internal nodes

LCP[1..n]		L[0..n]
	$	9
0		
	aban$	5
1	a	
	an$	7
2	an	
	anaban$	3
3	ana	
	ananaban$	1
0		
	ban$	6
3	ban	
	bananaban$	0
0		
	n$	8
1	n	
	naban$	4
2	na	
	nanaban$	2

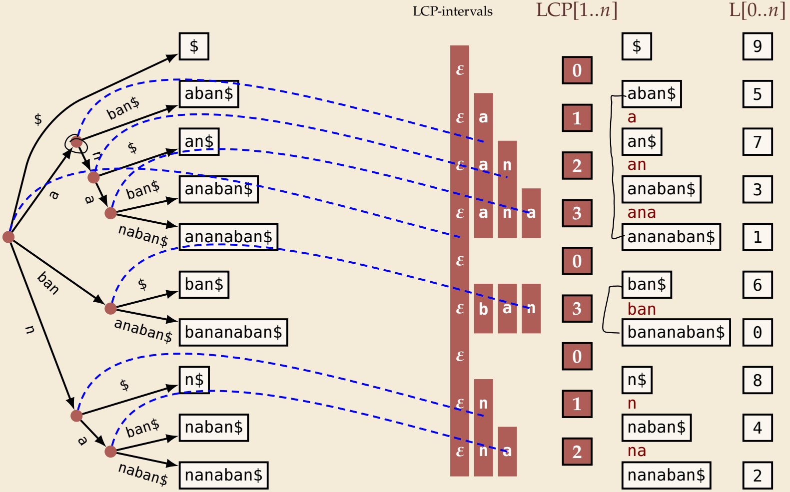
LCP array and internal nodes



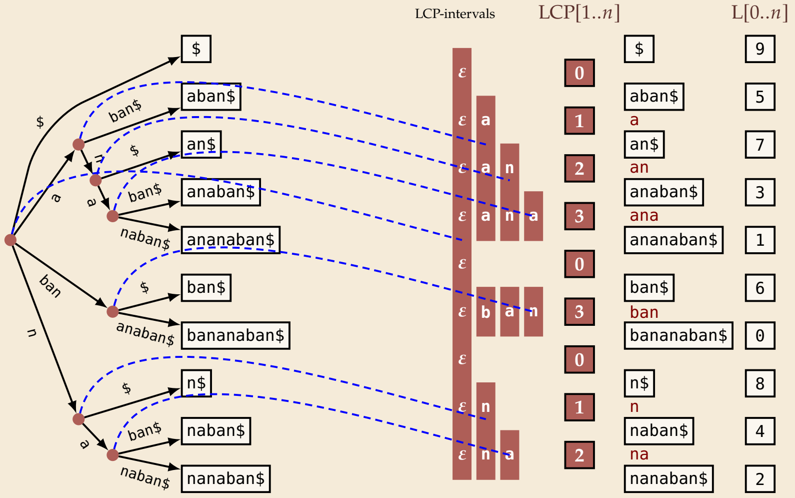
LCP array and internal nodes



LCP array and internal nodes



LCP array and internal nodes



⇝ Leaf array $L[0..n]$ plus LCP array $\mathrm{LCP}[1..n]$ encode full tree!

6.9 LCP Array Construction

LCP array construction

- computing $\mathrm{LCP}[1..n]$ naively too expensive
  - each value could take $\Theta(n)$ time
  - $\Theta(n^2)$ in total

LCP array construction

- computing $\mathrm{LCP}[1..n]$ naively too expensive
  - each value could take $\Theta(n)$ time

  $\Theta(n^2)$ in total

- but: seeing one large (= costly) LCP value $\rightsquigarrow$ can find another large one!

- Example: $T = \texttt{Buffalo␣buffalo␣buffalo␣buffalo\$}$
  - first few suffixes in sorted order:

$T_{L[0]} = \texttt{\$}$
$T_{L[1]} = \texttt{alo␣buffalo\$}$
$T_{L[2]} = \texttt{alo␣buffalo␣buffalo\$}$
$\quad\quad\quad\;\;\texttt{alo␣buffalo␣buffalo}$ $\rightsquigarrow$ $\mathrm{LCP}[3] = \mathbf{19}$
$T_{L[3]} = \texttt{alo␣buffalo␣buffalo␣buffalo\$}$

LCP array construction

- computing LCP$[1..n]$ naively too expensive
  - each value could take $\Theta(n)$ time
  - $\Theta(n^2)$ in total
- but: seeing one large (= costly) LCP value $\leadsto$ can find another large one!
- Example: T = `Buffalo␣buffalo␣buffalo␣buffalo$`
  - first few suffixes in sorted order:

$T_{L[0]}$ = `$`
$T_{L[1]}$ = `alo␣buffalo$`
$T_{L[2]}$ = ~~a~~`lo␣buffalo␣buffalo$`
`alo␣buffalo␣buffalo` $\leadsto$ LCP$[3]$ = **19**
$T_{L[3]}$ = ~~a~~`lo␣buffalo␣buffalo␣buffalo$`

$\leadsto$ **Removing first character** from $T_{L[2]}$ and $T_{L[3]}$ gives two new suffixes:

$T_{L[?]}$ = `lo␣buffalo␣buffalo$`
`lo␣buffalo␣buffalo` $\leadsto$ LCP$[?]$ = **18**
$T_{L[?]}$ = `lo␣buffalo␣buffalo␣buffalo$`

unclear where...

LCP array construction

- computing $\mathrm{LCP}[1..n]$ naively too expensive
  - each value could take $\Theta(n)$ time
  - $\Theta(n^2)$ in total
- but: seeing one large (= costly) LCP value $\rightsquigarrow$ can find another large one!
- Example: T = `Buffalo␣buffalo␣buffalo␣buffalo$`
  - first few suffixes in sorted order:

$T_{L[0]}$ = `$`
$T_{L[1]}$ = `alo␣buffalo$`
$T_{L[2]}$ = `alo␣buffalo␣buffalo$`
`alo␣buffalo␣buffalo` $\rightsquigarrow$ $\mathrm{LCP}[3] = \mathbf{19}$
$T_{L[3]}$ = `alo␣buffalo␣buffalo␣buffalo$`

$\rightsquigarrow$ **Removing first character** from $T_{L[2]}$ and $T_{L[3]}$ gives two new suffixes:

$T_{L[?]}$ = `lo␣buffalo␣buffalo$`
`lo␣buffalo␣buffalo` $\rightsquigarrow$ $\mathrm{LCP}[?] = \mathbf{18}$
$T_{L[?]}$ = `lo␣buffalo␣buffalo␣buffalo$`

unclear where...

Shortened suffixes might *not* be *adjacent* in sorted order!
$\rightsquigarrow$ no LCP entry for them!

Kasai's algorithm – Example

- Kasai et al. used above observation systematically
- Key idea: *compute* LCP values in ***text order***
- Dropping first character of adjacent suffixes might not lead to *adjacent* shorter suffixes, but LCP entry can only be *longer*.

i	$R[i]$	T_i
0	6^{th}	bananaban$
1	4^{th}	ananaban$
2	9^{th}	nanaban$
3	3^{th}	anaban$
4	8^{th}	naban$
5	1^{th}	aban$
6	5^{th}	ban$
7	2^{th}	an$
8	7^{th}	n$
9	0^{th}	$

r	$L[r]$	$T_{L[r]}$	$\mathrm{LCP}[r]$
0	9	$	–
1	5	aban$	
2	7	an$	
3	3	anaban$	
4	1	ananaban$	
5	6	ban$	
6	0	bananaban$	
7	8	n$	
8	4	naban$	
9	2	nanaban$	

Kasai's algorithm – Example

- Kasai et al. used above observation systematically
- Key idea: *compute* LCP values in ***text order***
- Dropping first character of adjacent suffixes might not lead to *adjacent* shorter suffixes, but LCP entry can only be *longer*.

i	$R[i]$	T_i
0	6^{th}	bananaban$
1	4^{th}	ananaban$
2	9^{th}	nanaban$
3	3^{th}	anaban$
4	8^{th}	naban$
5	1^{th}	aban$
6	5^{th}	ban$
7	2^{th}	an$
8	7^{th}	n$
9	0^{th}	$

r	$L[r]$	$T_{L[r]}$	LCP$[r]$
0	9	$	–
1	5	aban$	
2	7	an$	
3	3	anaban$	
4	1	ananaban$	
5	6	ban$	
6	0	bananaban$	
7	8	n$	
8	4	naban$	
9	2	nanaban$	

Kasai's algorithm – Example

- Kasai et al. used above observation systematically
- Key idea: *compute* LCP values in ***text order***
- Dropping first character of adjacent suffixes might not lead to *adjacent* shorter suffixes, but LCP entry can only be *longer*.

i	$R[i]$	T_i
0	6^{th}	`bananaban$`
1	4^{th}	`ananaban$`
2	9^{th}	`nanaban$`
3	3^{th}	`anaban$`
4	8^{th}	`naban$`
5	1^{th}	`aban$`
6	5^{th}	`ban$`
7	2^{th}	`an$`
8	7^{th}	`n$`
9	0^{th}	`$`

r	$L[r]$	$T_{L[r]}$	$\mathrm{LCP}[r]$
0	9	`$`	–
1	5	`aban$`	
2	7	`an$`	
3	3	`anaban$`	
4	1	`ananaban$`	
5	6	`ban$`	
6	0	`bananaban$`	
7	8	`n$`	
8	4	`naban$`	
9	2	`nanaban$`	

Kasai's algorithm – Example

- Kasai et al. used above observation systematically
- Key idea: *compute* LCP values in ***text order***
- Dropping first character of adjacent suffixes might not lead to *adjacent* shorter suffixes, but LCP entry can only be *longer*.

i	$R[i]$	T_i
0	6th	bananaban$
1	4th	ananaban$
2	9th	nanaban$
3	3th	anaban$
4	8th	naban$
5	1th	aban$
6	5th	ban$
7	2th	an$
8	7th	n$
9	0th	$

r	$L[r]$	$T_{L[r]}$	$\mathrm{LCP}[r]$
0	9	$	–
1	5	aban$	
2	7	an$	
3	3	anaban$	
4	1	ananaban$	
5	6	ban$	
6	0	bananaban$	
7	8	n$	
8	4	naban$	
9	2	nanaban$	

Kasai's algorithm – Example

- Kasai et al. used above observation systematically
- Key idea: *compute* LCP values in ***text order***
- Dropping first character of adjacent suffixes might not lead to *adjacent* shorter suffixes, but LCP entry can only be *longer*.

i	$R[i]$	T_i
0	6th	bananaban$
1	4th	ananaban$
2	9th	nanaban$
3	3th	anaban$
4	8th	naban$
5	1th	aban$
6	5th	ban$
7	2th	an$
8	7th	n$
9	0th	$

r	$L[r]$	$T_{L[r]}$	$\mathrm{LCP}[r]$
0	9	$	–
1	5	aban$	
2	7	an$	
3	3	anaban$	
4	1	ananaban$	
5	6	**ba**n$	
6	0	**ba**nanaban$	
7	8	n$	
8	4	naban$	
9	2	nanaban$	

Kasai's algorithm – Example

- Kasai et al. used above observation systematically
- Key idea: *compute* LCP values in ***text order***
- Dropping first character of adjacent suffixes might not lead to *adjacent* shorter suffixes, but LCP entry can only be *longer*.

	i	$R[i]$	T_i
→	0	6^{th}	bananaban$
	1	4^{th}	ananaban$
	2	9^{th}	nanaban$
	3	3^{th}	anaban$
	4	8^{th}	naban$
	5	1^{th}	aban$
	6	5^{th}	ban$
	7	2^{th}	an$
	8	7^{th}	n$
	9	0^{th}	$

	r	$L[r]$	$T_{L[r]}$	$\mathrm{LCP}[r]$
	0	9	$	–
	1	5	aban$	
	2	7	an$	
	3	3	anaban$	
	4	1	ananaban$	
	5	6	**ban**$	
→	6	0	**ban**anaban$	
	7	8	n$	
	8	4	naban$	
	9	2	nanaban$	

Kasai's algorithm – Example

- Kasai et al. used above observation systematically
- Key idea: *compute* LCP values in ***text order***
- Dropping first character of adjacent suffixes might not lead to *adjacent* shorter suffixes, but LCP entry can only be *longer*.

i	$R[i]$	T_i
→ 0	6^{th}	bananaban$
1	4^{th}	ananaban$
2	9^{th}	nanaban$
3	3^{th}	anaban$
4	8^{th}	naban$
5	1^{th}	aban$
6	5^{th}	ban$
7	2^{th}	an$
8	7^{th}	n$
9	0^{th}	$

r	$L[r]$	$T_{L[r]}$	$\mathrm{LCP}[r]$
0	9	$	–
1	5	aban$	
2	7	an$	
3	3	anaban$	
4	1	ananaban$	
5	6	**ban**$	
→ 6	0	**ban**anaban$	**3**
7	8	n$	
8	4	naban$	
9	2	nanaban$	

Kasai's algorithm – Example

- Kasai et al. used above observation systematically
- Key idea: *compute* LCP values in ***text order***
- Dropping first character of adjacent suffixes might not lead to *adjacent* shorter suffixes, but LCP entry can only be *longer*.

i	$R[i]$	T_i
0	6^{th}	`bananaban$`
1	4^{th}	`ananaban$`
2	9^{th}	`nanaban$`
3	3^{th}	`anaban$`
4	8^{th}	`naban$`
5	1^{th}	`aban$`
6	5^{th}	`ban$`
7	2^{th}	`an$`
8	7^{th}	`n$`
9	0^{th}	`$`

r	$L[r]$	$T_{L[r]}$	$\mathrm{LCP}[r]$
0	9	`$`	–
1	5	`aban$`	
2	7	`an$`	
3	3	`anaban$`	
4	1	`ananaban$`	
5	6	`ban$`	
6	0	`bananaban$`	3
7	8	`n$`	
8	4	`naban$`	
9	2	`nanaban$`	

Kasai's algorithm – Example

- Kasai et al. used above observation systematically
- Key idea: *compute* LCP values in ***text order***
- Dropping first character of adjacent suffixes might not lead to *adjacent* shorter suffixes, but LCP entry can only be *longer*.

i	$R[i]$	T_i
0	6th	bananaban$
1	4th	ananaban$
2	9th	nanaban$
3	3th	anaban$
4	8th	naban$
5	1th	aban$
6	5th	ban$
7	2th	an$
8	7th	n$
9	0th	$

r	$L[r]$	$T_{L[r]}$	$\mathrm{LCP}[r]$
0	9	$	–
1	5	aban$	
2	7	an$	
3	3	anaban$	
4	1	ananaban$	
5	6	ban$	
6	0	bananaban$	3
7	8	n$	
8	4	naban$	
9	2	nanaban$	

Kasai's algorithm – Example

- Kasai et al. used above observation systematically
- Key idea: *compute* LCP values in ***text order***
- Dropping first character of adjacent suffixes might not lead to *adjacent* shorter suffixes, but LCP entry can only be *longer*.

i	$R[i]$	T_i
0	6th	bananaban$
1	4th	ananaban$
2	9th	nanaban$
3	3th	anaban$
4	8th	naban$
5	1th	aban$
6	5th	ban$
7	2th	an$
8	7th	n$
9	0th	$

r	$L[r]$	$T_{L[r]}$	LCP$[r]$
0	9	$	–
1	5	aban$	
2	7	an$	
3	3	anaban$	
4	1	ananaban$	
5	6	ban$	
6	0	bananaban$	3
7	8	n$	
8	4	naban$	
9	2	nanaban$	

Kasai's algorithm – Example

- Kasai et al. used above observation systematically
- Key idea: *compute* LCP values in ***text order***
- Dropping first character of adjacent suffixes might not lead to *adjacent* shorter suffixes, but LCP entry can only be *longer*.

i	$R[i]$	T_i
0	6th	bananaban$
1	4th	ananaban$
2	9th	nanaban$
3	3th	anaban$
4	8th	naban$
5	1th	aban$
6	5th	ban$
7	2th	an$
8	7th	n$
9	0th	$

r	$L[r]$	$T_{L[r]}$	$\mathrm{LCP}[r]$
0	9	$	–
1	5	aban$	
2	7	an$	
3	3	anaban$	
4	1	ananaban$	
5	6	ban$	
6	0	bananaban$	3
7	8	n$	
8	4	naban$	
9	2	nanaban$	

Kasai's algorithm – Example

- Kasai et al. used above observation systematically
- Key idea: *compute* LCP values in ***text order***
- Dropping first character of adjacent suffixes might not lead to *adjacent* shorter suffixes, but LCP entry can only be *longer*.

i	$R[i]$	T_i
0	6^{th}	bananaban$
1	4^{th}	ananaban$
2	9^{th}	nanaban$
3	3^{th}	anaban$
4	8^{th}	naban$
5	1^{th}	aban$
6	5^{th}	ban$
7	2^{th}	an$
8	7^{th}	n$
9	0^{th}	$

r	$L[r]$	$T_{L[r]}$	LCP$[r]$
0	9	$	–
1	5	aban$	
2	7	an$	
3	3	anaban$	
4	1	ananaban$	
5	6	ban$	
6	0	bananaban$	3
7	8	n$	
8	4	naban$	
9	2	nanaban$	

Kasai's algorithm – Example

- Kasai et al. used above observation systematically
- Key idea: *compute* LCP values in ***text order***
- Dropping first character of adjacent suffixes might not lead to *adjacent* shorter suffixes, but LCP entry can only be *longer*.

i	$R[i]$	T_i
0	6th	bananaban$
1	4th	ananaban$
2	9th	nanaban$
3	3th	anaban$
4	8th	naban$
5	1th	aban$
6	5th	ban$
7	2th	an$
8	7th	n$
9	0th	$

r	$L[r]$	$T_{L[r]}$	LCP$[r]$
0	9	$	–
1	5	aban$	
2	7	an$	
3	3	anaban$	
4	1	ananaban$	3
5	6	ban$	
6	0	bananaban$	3
7	8	n$	
8	4	naban$	
9	2	nanaban$	

Kasai's algorithm – Example

- Kasai et al. used above observation systematically
- Key idea: *compute* LCP values in ***text order***
- Dropping first character of adjacent suffixes might not lead to *adjacent* shorter suffixes, but LCP entry can only be *longer*.

i	$R[i]$	T_i
0	6th	bananaban$
1	4th	ananaban$
2	9th	nanaban$
3	3th	anaban$
4	8th	naban$
5	1th	aban$
6	5th	ban$
7	2th	an$
8	7th	n$
9	0th	$

r	$L[r]$	$T_{L[r]}$	$\mathrm{LCP}[r]$
0	9	$	–
1	5	aban$	
2	7	an$	
3	3	anaban$	
4	1	ananaban$	3
5	6	ban$	
6	0	bananaban$	3
7	8	n$	
8	4	naban$	
9	2	nanaban$	

Kasai's algorithm – Example

- Kasai et al. used above observation systematically
- Key idea: *compute* LCP values in ***text order***
- Dropping first character of adjacent suffixes might not lead to *adjacent* shorter suffixes, but LCP entry can only be *longer*.

i	$R[i]$	T_i
0	6^{th}	bananaban$
1	4^{th}	ananaban$
2	9^{th}	nanaban$
3	3^{th}	anaban$
4	8^{th}	naban$
5	1^{th}	aban$
6	5^{th}	ban$
7	2^{th}	an$
8	7^{th}	n$
9	0^{th}	$

r	$L[r]$	$T_{L[r]}$	$\mathrm{LCP}[r]$
0	9	$	–
1	5	aban$	
2	7	an$	
3	3	anaban$	
4	1	ananaban$	3
5	6	ban$	
6	0	bananaban$	3
7	8	n$	
8	4	naban$	
9	2	nanaban$	

Kasai's algorithm – Example

- Kasai et al. used above observation systematically
- Key idea: *compute* LCP values in ***text order***
- Dropping first character of adjacent suffixes might not lead to *adjacent* shorter suffixes, but LCP entry can only be *longer*.

i	$R[i]$	T_i
0	6^{th}	`bananaban$`
1	4^{th}	`ananaban$`
2	9^{th}	`nanaban$`
3	3^{th}	`anaban$`
4	8^{th}	`naban$`
5	1^{th}	`aban$`
6	5^{th}	`ban$`
7	2^{th}	`an$`
8	7^{th}	`n$`
9	0^{th}	`$`

r	$L[r]$	$T_{L[r]}$	LCP$[r]$
0	9	`$`	–
1	5	`aban$`	
2	7	`an$`	
3	3	`anaban$`	
4	1	`ananaban$`	3
5	6	`ban$`	
6	0	`bananaban$`	3
7	8	`n$`	
8	4	`naban$`	
9	2	`nanaban$`	

Kasai's algorithm – Example

- Kasai et al. used above observation systematically
- Key idea: *compute* LCP values in ***text order***
- Dropping first character of adjacent suffixes might not lead to *adjacent* shorter suffixes, but LCP entry can only be *longer*.

i	$R[i]$	T_i
0	6th	bananaban$
1	4th	ananaban$
2	9th	nanaban$
3	3th	anaban$
4	8th	naban$
5	1th	aban$
6	5th	ban$
7	2th	an$
8	7th	n$
9	0th	$

r	$L[r]$	$T_{L[r]}$	$\mathrm{LCP}[r]$
0	9	$	–
1	5	aban$	
2	7	an$	
3	3	anaban$	
4	1	ananaban$	3
5	6	ban$	
6	0	bananaban$	3
7	8	n$	
8	4	naban$	
9	2	nanaban$	

Kasai's algorithm – Example

- Kasai et al. used above observation systematically
- Key idea: *compute* LCP values in ***text order***
- Dropping first character of adjacent suffixes might not lead to *adjacent* shorter suffixes, but LCP entry can only be *longer*.

i	$R[i]$	T_i
0	6th	bananaban$
1	4th	ananaban$
2	9th	nanaban$
3	3th	anaban$
4	8th	naban$
5	1th	aban$
6	5th	ban$
7	2th	an$
8	7th	n$
9	0th	$

r	$L[r]$	$T_{L[r]}$	LCP$[r]$
0	9	$	–
1	5	aban$	
2	7	an$	
3	3	anaban$	
4	1	ananaban$	3
5	6	ban$	
6	0	bananaban$	3
7	8	n$	
8	4	naban$	
9	2	nanaban$	2

Kasai's algorithm – Example

- Kasai et al. used above observation systematically
- Key idea: *compute* LCP values in ***text order***
- Dropping first character of adjacent suffixes might not lead to *adjacent* shorter suffixes, but LCP entry can only be *longer*.

i	$R[i]$	T_i
0	6^{th}	bananaban$
1	4^{th}	ananaban$
2	9^{th}	nanaban$
3	3^{th}	anaban$
4	8^{th}	naban$
5	1^{th}	aban$
6	5^{th}	ban$
7	2^{th}	an$
8	7^{th}	n$
9	0^{th}	$

r	$L[r]$	$T_{L[r]}$	LCP$[r]$
0	9	$	–
1	5	aban$	
2	7	an$	
3	3	anaban$	
4	1	ananaban$	3
5	6	ban$	
6	0	bananaban$	3
7	8	n$	
8	4	naban$	
9	2	nanaban$	2

Kasai's algorithm – Example

- Kasai et al. used above observation systematically
- Key idea: *compute* LCP values in ***text order***
- Dropping first character of adjacent suffixes might not lead to *adjacent* shorter suffixes, but LCP entry can only be *longer*.

i	$R[i]$	T_i
0	6th	bananaban$
1	4th	ananaban$
2	9th	nanaban$
3	3th	anaban$
4	8th	naban$
5	1th	aban$
6	5th	ban$
7	2th	an$
8	7th	n$
9	0th	$

r	$L[r]$	$T_{L[r]}$	$\mathrm{LCP}[r]$
0	9	$	–
1	5	aban$	
2	7	an$	
3	3	anaban$	
4	1	ananaban$	3
5	6	ban$	
6	0	bananaban$	3
7	8	n$	
8	4	naban$	
9	2	nanaban$	2

Kasai's algorithm – Example

- Kasai et al. used above observation systematically
- Key idea: *compute* LCP values in ***text order***
- Dropping first character of adjacent suffixes might not lead to *adjacent* shorter suffixes, but LCP entry can only be *longer*.

i	$R[i]$	T_i
0	6th	bananaban$
1	4th	ananaban$
2	9th	nanaban$
3	3th	anaban$
4	8th	naban$
5	1th	aban$
6	5th	ban$
7	2th	an$
8	7th	n$
9	0th	$

r	$L[r]$	$T_{L[r]}$	$\mathrm{LCP}[r]$
0	9	$	–
1	5	aban$	
2	7	an$	
3	3	anaban$	
4	1	ananaban$	3
5	6	ban$	
6	0	bananaban$	3
7	8	n$	
8	4	naban$	
9	2	nanaban$	2

Kasai's algorithm – Example

- Kasai et al. used above observation systematically
- Key idea: *compute* LCP values in ***text order***
- Dropping first character of adjacent suffixes might not lead to *adjacent* shorter suffixes, but LCP entry can only be *longer*.

	i	$R[i]$	T_i
	0	6^{th}	bananaban$
	1	4^{th}	ananaban$
	2	9^{th}	nanaban$
→	3	3^{th}	anaban$
	4	8^{th}	naban$
	5	1^{th}	aban$
	6	5^{th}	ban$
	7	2^{th}	an$
	8	7^{th}	n$
	9	0^{th}	$

	r	$L[r]$	$T_{L[r]}$	LCP$[r]$
	0	9	$	–
	1	5	aban$	
	2	7	an$	
→	3	3	anaban$	
	4	1	ananaban$	3
	5	6	ban$	
	6	0	bananaban$	3
	7	8	n$	
	8	4	naban$	
	9	2	nanaban$	2

Kasai's algorithm – Example

- Kasai et al. used above observation systematically
- Key idea: *compute* LCP values in ***text order***
- Dropping first character of adjacent suffixes might not lead to *adjacent* shorter suffixes, but LCP entry can only be *longer*.

	i	$R[i]$	T_i
	0	6^{th}	bananaban$
	1	4^{th}	ananaban$
	2	9^{th}	nanaban$
→	3	3^{th}	anaban$
	4	8^{th}	naban$
	5	1^{th}	aban$
	6	5^{th}	ban$
	7	2^{th}	an$
	8	7^{th}	n$
	9	0^{th}	$

	r	$L[r]$	$T_{L[r]}$	$\mathrm{LCP}[r]$
	0	9	$	–
	1	5	aban$	
	2	7	an$	
→	3	3	anaban$	
	4	1	ananaban$	3
	5	6	ban$	
	6	0	bananaban$	3
	7	8	n$	
	8	4	naban$	
	9	2	nanaban$	2

Kasai's algorithm – Example

- Kasai et al. used above observation systematically
- Key idea: *compute* LCP values in ***text order***
- Dropping first character of adjacent suffixes might not lead to *adjacent* shorter suffixes, but LCP entry can only be *longer*.

	i	$R[i]$	T_i
	0	6^{th}	bananaban$
	1	4^{th}	ananaban$
	2	9^{th}	nanaban$
→	3	3^{th}	anaban$
	4	8^{th}	naban$
	5	1^{th}	aban$
	6	5^{th}	ban$
	7	2^{th}	an$
	8	7^{th}	n$
	9	0^{th}	$

	r	$L[r]$	$T_{L[r]}$	LCP$[r]$
	0	9	$	–
	1	5	aban$	
	2	7	an$	
→	3	3	anaban$	2
	4	1	ananaban$	3
	5	6	ban$	
	6	0	bananaban$	3
	7	8	n$	
	8	4	naban$	
	9	2	nanaban$	2

Kasai's algorithm – Example

- Kasai et al. used above observation systematically
- Key idea: *compute* LCP values in ***text order***
- Dropping first character of adjacent suffixes might not lead to *adjacent* shorter suffixes, but LCP entry can only be *longer*.

i	$R[i]$	T_i	r	$L[r]$	$T_{L[r]}$	$\mathrm{LCP}[r]$
0	6th	bananaban$	0	9	$	–
1	4th	ananaban$	1	5	aban$	
2	9th	nanaban$	2	7	an$	
3	3th	anaban$	3	3	anaban$	2
4	8th	naban$	4	1	ananaban$	3
5	1th	aban$	5	6	ban$	
6	5th	ban$	6	0	bananaban$	3
7	2th	an$	7	8	n$	
8	7th	n$	8	4	naban$	
9	0th	$	9	2	nanaban$	2

Kasai's algorithm – Example

- Kasai et al. used above observation systematically
- Key idea: *compute* LCP values in ***text order***
- Dropping first character of adjacent suffixes might not lead to *adjacent* shorter suffixes, but LCP entry can only be *longer*.

i	$R[i]$	T_i
0	6^{th}	bananaban$
1	4^{th}	ananaban$
2	9^{th}	nanaban$
3	3^{th}	anaban$
→ 4	8^{th}	naban$
5	1^{th}	aban$
6	5^{th}	ban$
7	2^{th}	an$
8	7^{th}	n$
9	0^{th}	$

r	$L[r]$	$T_{L[r]}$	$\mathrm{LCP}[r]$
0	9	$	–
1	5	aban$	
2	7	an$	
3	3	anaban$	2
4	1	ananaban$	3
5	6	ban$	
6	0	bananaban$	3
7	8	n$	
→ 8	4	naban$	
9	2	nanaban$	2

Kasai's algorithm – Example

- Kasai et al. used above observation systematically
- Key idea: *compute* LCP values in ***text order***
- Dropping first character of adjacent suffixes might not lead to *adjacent* shorter suffixes, but LCP entry can only be *longer*.

i	$R[i]$	T_i
0	6^{th}	bananaban$
1	4^{th}	ananaban$
2	9^{th}	nanaban$
3	3^{th}	anaban$
4	8^{th}	naban$
5	1^{th}	aban$
6	5^{th}	ban$
7	2^{th}	an$
8	7^{th}	n$
9	0^{th}	$

r	$L[r]$	$T_{L[r]}$	LCP$[r]$
0	9	$	–
1	5	aban$	
2	7	an$	
3	3	anaban$	2
4	1	ananaban$	3
5	6	ban$	
6	0	bananaban$	3
7	8	n$	
8	4	naban$	1
9	2	nanaban$	2

Kasai's algorithm – Example

- Kasai et al. used above observation systematically
- Key idea: *compute* LCP values in ***text order***
- Dropping first character of adjacent suffixes might not lead to *adjacent* shorter suffixes, but LCP entry can only be *longer*.

i	$R[i]$	T_i
0	6^{th}	bananaban$
1	4^{th}	ananaban$
2	9^{th}	nanaban$
3	3^{th}	anaban$
4	8^{th}	naban$
5	1^{th}	aban$
6	5^{th}	ban$
7	2^{th}	an$
8	7^{th}	n$
9	0^{th}	$

r	$L[r]$	$T_{L[r]}$	$\mathrm{LCP}[r]$
0	9	$	–
1	5	aban$	
2	7	an$	
3	3	anaban$	2
4	1	ananaban$	3
5	6	ban$	
6	0	bananaban$	3
7	8	n$	
8	4	naban$	1
9	2	nanaban$	2

Kasai's algorithm – Example

- Kasai et al. used above observation systematically
- Key idea: *compute* LCP values in ***text order***
- Dropping first character of adjacent suffixes might not lead to *adjacent* shorter suffixes, but LCP entry can only be *longer*.

i	$R[i]$	T_i
0	6th	bananaban$
1	4th	ananaban$
2	9th	nanaban$
3	3th	anaban$
4	8th	naban$
5	1th	aban$
6	5th	ban$
7	2th	an$
8	7th	n$
9	0th	$

r	$L[r]$	$T_{L[r]}$	LCP$[r]$
0	9	$	–
1	5	aban$	0
2	7	an$	
3	3	anaban$	2
4	1	ananaban$	3
5	6	ban$	
6	0	bananaban$	3
7	8	n$	
8	4	naban$	1
9	2	nanaban$	2

Kasai's algorithm – Example

- Kasai et al. used above observation systematically
- Key idea: *compute* LCP values in ***text order***
- Dropping first character of adjacent suffixes might not lead to *adjacent* shorter suffixes, but LCP entry can only be *longer*.

i	$R[i]$	T_i
0	6th	bananaban$
1	4th	ananaban$
2	9th	nanaban$
3	3th	anaban$
4	8th	naban$
5	1th	aban$
6	5th	ban$
7	2th	an$
8	7th	n$
9	0th	$

r	$L[r]$	$T_{L[r]}$	$\mathrm{LCP}[r]$
0	9	$	–
1	5	aban$	0
2	7	an$	
3	3	anaban$	2
4	1	ananaban$	3
5	6	ban$	
6	0	bananaban$	3
7	8	n$	
8	4	naban$	1
9	2	nanaban$	2

Kasai's algorithm – Example

- Kasai et al. used above observation systematically
- Key idea: *compute* LCP values in ***text order***
- Dropping first character of adjacent suffixes might not lead to *adjacent* shorter suffixes, but LCP entry can only be *longer*.

i	$R[i]$	T_i
0	6th	bananaban$
1	4th	ananaban$
2	9th	nanaban$
3	3th	anaban$
4	8th	naban$
5	1th	aban$
→ 6	5th	ban$
7	2th	an$
8	7th	n$
9	0th	$

r	$L[r]$	$T_{L[r]}$	$\mathrm{LCP}[r]$
0	9	$	–
1	5	aban$	0
2	7	an$	
3	3	anaban$	2
4	1	ananaban$	3
→ 5	6	ban$	**0**
6	0	bananaban$	3
7	8	n$	
8	4	naban$	1
9	2	nanaban$	2

Kasai's algorithm – Example

- Kasai et al. used above observation systematically
- Key idea: *compute* LCP values in ***text order***
- Dropping first character of adjacent suffixes might not lead to *adjacent* shorter suffixes, but LCP entry can only be *longer*.

i	$R[i]$	T_i
0	6^{th}	bananaban$
1	4^{th}	ananaban$
2	9^{th}	nanaban$
3	3^{th}	anaban$
4	8^{th}	naban$
5	1^{th}	aban$
6	5^{th}	ban$
7	2^{th}	an$
8	7^{th}	n$
9	0^{th}	$

r	$L[r]$	$T_{L[r]}$	$\mathrm{LCP}[r]$
0	9	$	–
1	5	aban$	0
2	7	an$	
3	3	anaban$	2
4	1	ananaban$	3
5	6	ban$	0
6	0	bananaban$	3
7	8	n$	
8	4	naban$	1
9	2	nanaban$	2

Kasai's algorithm – Example

- Kasai et al. used above observation systematically
- Key idea: *compute* LCP values in ***text order***
- Dropping first character of adjacent suffixes might not lead to *adjacent* shorter suffixes, but LCP entry can only be *longer*.

i	$R[i]$	T_i
0	6^{th}	bananaban$
1	4^{th}	ananaban$
2	9^{th}	nanaban$
3	3^{th}	anaban$
4	8^{th}	naban$
5	1^{th}	aban$
6	5^{th}	ban$
7	2^{th}	an$
8	7^{th}	n$
9	0^{th}	$

r	$L[r]$	$T_{L[r]}$	$\mathrm{LCP}[r]$
0	9	$	–
1	5	aban$	0
2	7	an$	1
3	3	anaban$	2
4	1	ananaban$	3
5	6	ban$	0
6	0	bananaban$	3
7	8	n$	
8	4	naban$	1
9	2	nanaban$	2

Kasai's algorithm – Example

- Kasai et al. used above observation systematically
- Key idea: *compute* LCP values in ***text order***
- Dropping first character of adjacent suffixes might not lead to *adjacent* shorter suffixes, but LCP entry can only be *longer*.

i	$R[i]$	T_i
0	6th	bananaban$
1	4th	ananaban$
2	9th	nanaban$
3	3th	anaban$
4	8th	naban$
5	1th	aban$
6	5th	ban$
7	2th	an$
8	7th	n$
9	0th	$

r	$L[r]$	$T_{L[r]}$	$\mathrm{LCP}[r]$
0	9	$	–
1	5	aban$	0
2	7	an$	1
3	3	anaban$	2
4	1	ananaban$	3
5	6	ban$	0
6	0	bananaban$	3
7	8	n$	
8	4	naban$	1
9	2	nanaban$	2

Kasai's algorithm – Example

- Kasai et al. used above observation systematically
- Key idea: *compute* LCP values in ***text order***
- Dropping first character of adjacent suffixes might not lead to *adjacent* shorter suffixes, but LCP entry can only be *longer*.

i	$R[i]$	T_i
0	6^{th}	bananaban$
1	4^{th}	ananaban$
2	9^{th}	nanaban$
3	3^{th}	anaban$
4	8^{th}	naban$
5	1^{th}	aban$
6	5^{th}	ban$
7	2^{th}	an$
8	7^{th}	n$
9	0^{th}	$

r	$L[r]$	$T_{L[r]}$	LCP$[r]$
0	9	$	–
1	5	aban$	0
2	7	an$	1
3	3	anaban$	2
4	1	ananaban$	3
5	6	ban$	0
6	0	bananaban$	3
7	8	n$	0
8	4	naban$	1
9	2	nanaban$	2

Kasai's algorithm – Code

```
procedure computeLCP(T[0..n], L[0..n], R[0..n])
    // Assume T[n] = $, L and R are suffix array and inverse
    ℓ := 0
    for i := 0, ..., n − 1 // Consider T_i now
        r := R[i]
        // compute LCP[r]; note that r > 0 since R[n] = 0
        i_{−1} := L[r − 1]
        while T[i + ℓ] == T[i_{−1} + ℓ] do
            ℓ := ℓ + 1
        LCP[r] := ℓ
        ℓ := max{ℓ − 1, 0}
    return LCP[1..n]
```

- remember length ℓ of induced common prefix
- use L to get start index of suffixes

Kasai's algorithm – Code

```
procedure computeLCP(T[0..n], L[0..n], R[0..n])
    // Assume T[n] = $, L and R are suffix array and inverse
    ℓ := 0
    for i := 0, ..., n − 1 // Consider T_i now
        r := R[i]
        // compute LCP[r]; note that r > 0 since R[n] = 0
        i_{−1} := L[r − 1]
        while T[i + ℓ] == T[i_{−1} + ℓ] do
            ℓ := ℓ + 1
        LCP[r] := ℓ
        ℓ := max{ℓ − 1, 0}
    return LCP[1..n]
```

- remember length ℓ of induced common prefix
- use L to get start index of suffixes

Analysis:

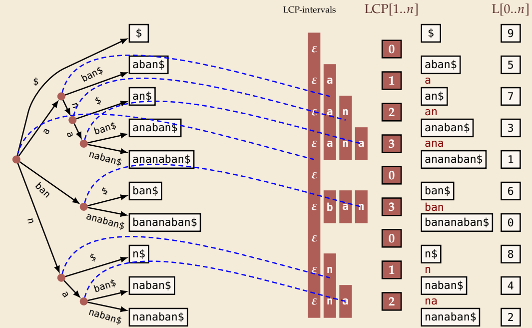
- dominant operation: character comparisons
- separately count those with outcomes "=" resp. "≠"
- each ≠ ends iteration of for-loop
  $\rightsquigarrow$ $\le n$ cmps
- each = implies increment of ℓ, but $\ell \le n$ and decremented $\le n$ times
  $\rightsquigarrow$ $\le 2n$ cmps

$\rightsquigarrow$ $\Theta(n)$ overall time

Back to suffix trees

We can finally look into the black box of linear-time suffix-array construction!

1. Compute suffix array for T.
2. Compute LCP array for T.
3. Construct $\mathcal{T}$ from suffix array and LCP array.



Conclusion

- *(Enhanced) Suffix Arrays* are the modern version of suffix trees

👎 can be harder to reason about

👍 can support same algorithms as suffix trees

👍 but use much less space

👍 simpler linear-time construction