2 Machines & Models

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Learning Outcomes

Unit 2: ***Machines & Models***

1. Understand the difference between empirical *running time* and algorithm *analysis*.
2. Understand *worst / best / average case* models for input data.
3. Know the *RAM machine* model.
4. Know the definitions of *asymptotic notation* (Big-Oh classes and relatives).
5. Understand the reasons to make *asymptotic approximations*.
6. Be able to *analyze* simple *algorithms*.

Outline

What is an algorithm?

An algorithm is a sequence of instructions.

↖ think: recipe

More precisely:

e. g. Python script ↙

1. mechanically executable
   ⇝ no "common sense" needed
2. finite description   ≠ finite computation!
3. solves a *problem*, i. e., a class of problem instances
   ↖ $x + y$, not only $17 + 4$

- input-processing-output abstraction

Typical example: *bubblesort*

⇝ not a specific program
but the underlying idea

What is a data structure?

A data structure is

1. a rule for **encoding data** (in computer memory), plus
2. **algorithms** to work with it (queries, updates, etc.)

typical example: *binary search tree*

2.1 Algorithm analysis

Good algorithms

Our goal: Find good (best?) algorithms and data structures for a task.

Good "usually" means

can be complicated in distributed systems

- fast running *time*
- moderate memory *space* usage

Algorithm analysis is a way to

- compare different algorithms,
- predict their performance in an application

Running time experiments

Why not simply run and time it?

- results only apply to
  - single *test* machine
  - tested inputs
  - tested implementation
  - ...

  $\neq$ *universal truths*

- instead: consider and analyze algorithms on an abstract machine (survives Pentium 4)
  - ⇝ provable statements for model
  - ⇝ testable model hypotheses

⇝ Need precise model of machine (costs), input data and algorithms.

Data Models

Algorithm analysis typically uses one of the following simple data models:

- **worst-case performance:**
  consider the *worst* of all inputs as our cost metric
- **best-case performance:**
  consider the *best* of all inputs as our cost metric
- **average-case performance:**
  consider the average/expectation of a *random* input as our cost metric

Usually, we apply the above for *inputs of same size* n.

$\leadsto$ performance is only a **function of** n.

2.2 The RAM Model

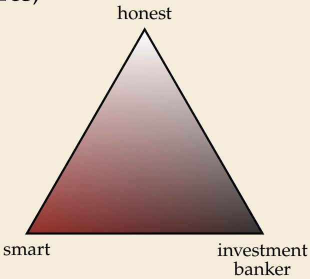
Machine models

The machine model decides

- what algorithms are possible
- how they are described (= programming language)
- what an execution *costs*

Goal: Machine models should be
detailed and powerful enough to reflect actual machines,
abstract enough to unify architectures,
simple enough to analyze.

$\rightsquigarrow$ usually some compromise is needed



Random Access Machines

Random access machine (RAM)

more detail in §2.2 of *Sequential and Parallel Algorithms and Data Structures* by Sanders, Mehlhorn, Dietzfelbinger, Dementiev

- unlimited *memory* $\texttt{MEM}[0], \texttt{MEM}[1], \texttt{MEM}[2], \ldots$
- fixed number of *registers* $R_1, \ldots, R_r$ (say $r = 100$)
- memory cells $\texttt{MEM}[i]$ and registers R_i store w-bit integers, i. e., numbers in $[0..2^w - 1]$
  w is the word width/size; typically $\boxed{w \propto \lg n}$ $\rightsquigarrow$ $2^w \approx n$
- Instructions:
  - load & store: $R_i := \texttt{MEM}[R_j]$ $\texttt{MEM}[R_j] := R_i$
  - operations on registers: $R_k := R_i + R_j$ (arithmetic is *modulo* 2^w!)
    also $R_i - R_j$, $R_i \cdot R_j$, $R_i \operatorname{div} R_j$, $R_i \bmod R_j$
    C-style operations (bitwise and/or/xor, left/right shift)
  - conditional and unconditional jumps
- cost: number of executed instructions

we will see further models later

$\rightsquigarrow$ The RAM is the standard model for sequential computation.

RAM-Program Example

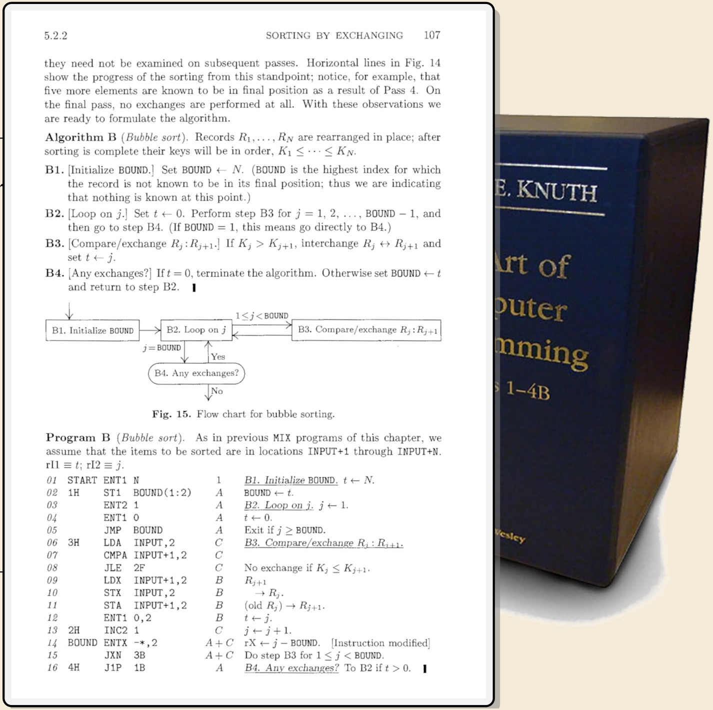
Example RAM program

1 *// Assume:* R_1 *stores number* N
2 *// Assume:* $\texttt{MEM}[0..N)$ *contains list of* N *numbers*
3 $R_2 := R_1$;
4 $R_3 := R_1 - 2$;
5 $R_4 := \texttt{MEM}[R_3]$;
6 $R_5 := R_3 + 1$;
7 $R_6 := \texttt{MEM}[R_5]$;
8 **if** $(R_4 \leq R_6)$ goto line 11;
9 $\texttt{MEM}[R_3] := R_6$;
10 $\texttt{MEM}[R_5] := R_4$;
11 $R_3 := R_3 - 1$;
12 **if** $(R_3 \geq 0)$ goto line 5;
13 $R_2 := R_2 - 1$;
14 **if** $(R_2 > 0)$ goto line 4;
15 *// Done:*

RAM-Program Example

Example RAM program

1 // *Assume:* R_1 *stores number* N
2 // *Assume:* `MEM`$[0..N)$ *contains list of* N *number*
3 $R_2 := R_1$;
4 $R_3 := R_1 - 2$;
5 $R_4 :=$ `MEM`$[R_3]$;
6 $R_5 := R_3 + 1$;
7 $R_6 :=$ `MEM`$[R_5]$;
8 **if** $(R_4 \le R_6)$ goto line 11;
9 `MEM`$[R_3] := R_6$;
10 `MEM`$[R_5] := R_4$;
11 $R_3 := R_3 - 1$;
12 **if** $(R_3 \ge 0)$ goto line 5;
13 $R_2 := R_2 - 1$;
14 **if** $(R_2 > 0)$ goto line 4;
15 // *Done:* `MEM`$[0..N)$ *sorted*


5.2.2 SORTING BY EXCHANGING 107

they need not be examined on subsequent passes. Horizontal lines in Fig. 14 show the progress of the sorting from this standpoint; notice, for example, that five more elements are known to be in final position as a result of Pass 4. On the final pass, no exchanges are performed at all. With these observations we are ready to formulate the algorithm.

Algorithm B (*Bubble sort*). Records $R_1, \ldots, R_N$ are rearranged in place; after sorting is complete their keys will be in order, $K_1 \le \cdots \le K_N$.

B1. [Initialize BOUND.] Set BOUND $\leftarrow N$. (BOUND is the highest index for which the record is not known to be in its final position; thus we are indicating that nothing is known at this point.)

B2. [Loop on j.] Set $t \leftarrow 0$. Perform step B3 for $j = 1, 2, \ldots,$ BOUND $- 1$, and then go to step B4. (If BOUND $= 1$, this means go directly to B4.)

B3. [Compare/exchange $R_j : R_{j+1}$.] If $K_j > K_{j+1}$, interchange $R_j \leftrightarrow R_{j+1}$ and set $t \leftarrow j$.

B4. [Any exchanges?] If $t = 0$, terminate the algorithm. Otherwise set BOUND $\leftarrow t$ and return to step B2. ▌

Fig. 15. Flow chart for bubble sorting.

Program B (*Bubble sort*). As in previous MIX programs of this chapter, we assume that the items to be sorted are in locations INPUT+1 through INPUT+N. rI1 $\equiv t$; rI2 $\equiv j$.

```
01 START ENT1 N            1     B1. Initialize BOUND. t ← N.
02 1H    ST1  BOUND(1:2)   A     BOUND ← t.
03       ENT2 1            A     B2. Loop on j. j ← 1.
04       ENT1 0            A     t ← 0.
05       JMP  BOUND        A     Exit if j ≥ BOUND.
06 3H    LDA  INPUT,2      C     B3. Compare/exchange R_j : R_{j+1}.
07       CMPA INPUT+1,2    C
08       JLE  2F           C     No exchange if K_j ≤ K_{j+1}.
09       LDX  INPUT+1,2    B     R_{j+1}
10       STX  INPUT,2      B       → R_j.
11       STA  INPUT+1,2    B     (old R_j) → R_{j+1}.
12       ENT1 0,2          B     t ← j.
13 2H    INC2 1            C     j ← j + 1.
14 BOUND ENTX -*,2         A+C   rX ← j − BOUND.  [Instruction modified]
15       JXN  3B           A+C   Do step B3 for 1 ≤ j < BOUND.
16 4H    J1P  1B           A     B4. Any exchanges? To B2 if t > 0. ▌
```

Pseudocode

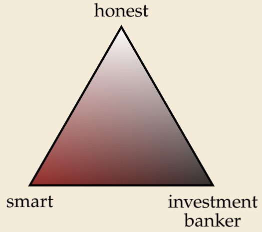
- Programs for the random-access machine are very low level and detailed

≈ assembly/machine language

Typical simplifications when describing and analyzing algorithms:

- more abstract *pseudocode* ← code that humans understand (easily)
  - control flow using **if**, **for**, **while**, etc.
  - variable names instead of fixed registers and memory cells
  - memory management (more below)
- count dominant *elementary operations* (e. g. memory accesses) instead of all RAM instructions

In both cases: We *can* go to full detail where needed/desired.



Pseudocode – Example

RAM-Program

```
1  // Bubblesort
2  // Assume: R_1 stores number N
3  // Assume: MEM[0..N) contains list of N numbers
4  R_2 := R_1;
5  R_3 := R_1 − 2;
6  R_4 := MEM[R_3];
7  R_5 := R_3 + 1;
8  R_6 := MEM[R_5];
9  if (R_4 ≤ R_6) goto line 12;
10 MEM[R_3] := R_6;
11 MEM[R_5] := R_4;
12 R_3 := R_3 − 1;
13 if (R_3 ≥ 0) goto line 6;
14 R_2 := R_2 − 1;
15 if (R_2 > 0) goto line 5;
16 // Done: MEM[0..N) sorted
```

Pseudocode Algorithm

```
procedure bubblesort(A[0..N)):
    for i := N, N − 1, ..., 1
        for j := N − 2, N − 3, ..., 0
            if A[j] > A[j + 1]:
                Swap A[j] and A[j + 1]
            end if
        end for
    end for
```

⇝ much more **readable**

- closer to modern high-level programming languages
- **but:** only allow primitive operations that correspond to $O(1)$ RAM instructions
  ⇝ analysis

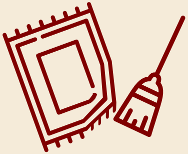
Memory management & Pointers

- A random-access machine is a bit like a bare CPU . . . without any operating system
  ⇝ cumbersome to use

- All high-level programming languages / operating systems add *memory management*:
  - Instruction to *allocate* a contiguous piece of memory of a given size (like `malloc`).
    - used to allocate a `new` array (of a fixed size) or
    - a `new` object/record (with a known list of instance variables)
    - There's a similar instruction to `free` allocated memory again or an automated garbage collector.

  ⇝ A *pointer* is a memory address (i. e., the i of $\texttt{MEM}[i]$).

  - Support for procedures (a.k.a. functions, methods) calls including recursive calls
    - (this internally requires maintaining call stack)



We will mostly ignore *how* all this works here.

2.3 Asymptotics & Big-Oh

Why asymptotics?

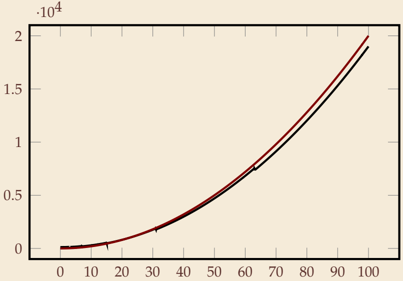
Algorithm analysis focuses on (the limiting behavior for infinitely) **large** inputs.

- abstracts from unnecessary detail
- simplifies analysis
- often necessary for sensible comparison

Asymptotics = approximation around ∞

Example: Consider a function $f(n)$ given by

$$2n^2 - 3n\lfloor \log_2(n+1)\rfloor + 7n - 3\lfloor \log_2(n+1)\rfloor + 120 \quad \sim \quad \mathbf{2n^2}$$



Asymptotic tools – Formal & definitive definition

- **"Tilde Notation":** $f(n) \sim g(n)$ iff $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$ (iff: if, and only if)

  „f and g are *asymptotically equivalent*"

- **"Big-Oh Notation":** $f(n) \in O(g(n))$ iff $\left|\frac{f(n)}{g(n)}\right|$ is bounded for $n \ge n_0$ (also write '=' instead)

  iff $\limsup_{n\to\infty} \left|\frac{f(n)}{g(n)}\right| < \infty$ (need supremum since limit might not exist!)

  Variants:

  - $f(n) \in \Omega(g(n))$ iff $g(n) \in O(f(n))$ ("Big-Omega")
  - $f(n) \in \Theta(g(n))$ iff $f(n) \in O(g(n))$ **and** $f(n) \in \Omega(g(n))$ ("Big-Theta")

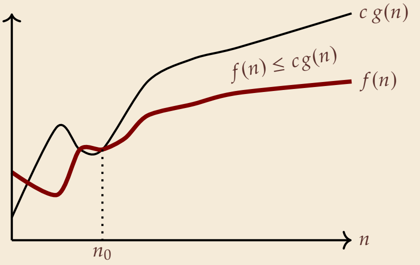
- **"Little-Oh Notation":** $f(n) \in o(g(n))$ iff $\lim_{n\to\infty} \left|\frac{f(n)}{g(n)}\right| = 0$

  similarly: $f(n) \in \omega(g(n))$ if $\lim = \infty$

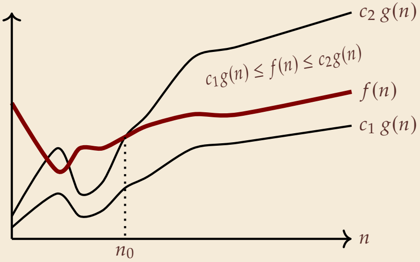
(Benefit of this definition: Works for any $f, g : \mathbb{R} \to \mathbb{R}$ and is easy to generalize to limits other than $n \to \infty$)

Asymptotic tools – Intuition

- $f(n) = O(g(n))$: $f(n)$ is **at most** $g(n)$ up to constant factors and for sufficiently large n



- $f(n) = \Theta(g(n))$: $f(n)$ is **equal to** $g(n)$ up to constant factors and for sufficiently large n



⚠ Plots can be misleading!

Example

Asymptotics – Example 1

Basic examples:

- $20n^3 + 10n\ln(n) + 5 \;\sim\; 20n^3 \;=\; \Theta(n^3)$
- $3\lg(n^2) + \lg(\lg(n)) \;=\; \Theta(\log n)$
- $10^{100} \;=\; O(1)$

Use *wolfram alpha* to compute/check limits, but also practice it with pen and paper!

Asymptotics – Basic facts

Rules to work with Big-Oh classes:

- $f = \Theta(f)$ (reflexivity)
- $f = \Theta(g) \wedge g = \Theta(h) \implies f = \Theta(h)$
- $c \cdot f(n) = \Theta(f(n))$ for constant $c \neq 0$
- $f \sim g \iff f = g \cdot (1 \pm o(1))$
- $\Theta(f) \cdot \Theta(g) = \Theta(f \cdot g)$
- $\Theta(f) + \Theta(g) = \Theta(f + g) = \Theta(\max\{f, g\})$ largest summand determines Θ-class

Asymptotics – Frequently encountered classes

Frequently used orders of growth:

- constant $\Theta(1)$
- logarithmic $\Theta(\log n)$ Note: $a, b > 0$ constants $\rightsquigarrow$ $\Theta(\log_a(n)) = \Theta(\log_b(n))$
- linear $\Theta(n)$
- linearithmic $\Theta(n \log n)$
- quadratic $\Theta(n^2)$
- cubic $\Theta(n^3)$
- polynomial $O(n^c)$ for some constant c
- exponential $O(c^n)$ for some constant $c > 1$ Note: $a > b > 0$ constants $\rightsquigarrow$ $b^n = o(a^n)$

Asymptotics – Example 2

Square-and-multiply algorithm

for computing x^m with $m \in \mathbb{N}$

Inputs:

- m as binary number (array of bits)
- n = #bits in m
- x a floating-point number

```
1 def pow(x, m):
2     # compute binary representation of exponent
3     exponent_bits = bin(m)[2:]
4     result = 1
5     for bit in exponent_bits:
6         result *= result
7         if bit == '1':
8             result *= x
9     return result
```

- Cost: C = # multiplications
- $C = n$ (line 6) $+$ #one-bits in binary representation of m (line 8)

$\rightsquigarrow n \le C \le 2n$

Asymptotics with several variables

- **Example:** Algorithms on graphs with n vertices and m edges.
  - want to say: Algorithm A takes time $\Theta(n+m)$.
  - But what does that even mean formally?!

⚠ Inconsistent and incompatible definitions used in the literature!

- **Here:**
  - (implicitly) always have a single *"main" variable* n: with $n \to \infty$
  - all other variables are *functions* of n: $m = m(n)$
  - must make *conditions* on functions explicit: $m(n) \in \Omega(n)$ and $m(n) \in O(n^2)$.

$\rightsquigarrow$ Can make statements like

$$O(n+m) \subseteq O(nm) \qquad (n \to \infty,\ m \in \Omega(1))$$

2.4 Teaser: Maximum subarray problem

Bring on the puzzles!

Time for a concrete example of algorithm design!

- we will illustrate the algorithm design process on a "toy problem"
- clean abstract problem, but nontrivial to solve!

Maximum (sum) subarray problem

- **Given:** $A[0..n)$ with $A[i] \in \mathbb{Z}$ for $0 \le i < n$.
- Abbreviate $s(i,j) := \sum_{k=i}^{j-1} A[k]$
- **Goal:** Compute $s := \max\{s(i,j) : 0 \le i \le j \le n\}$
  and a pair (i,j) with $s = s(i,j)$.

  will ignore that here; easy to modify algorithms

Applications:

- largest gain of a stock
  $A[i]$ price change on day i
- signal detection in biological sequence analysis
- 2D generalization used in image analysis

Modeling decisions:

- input size: #numbers n
- assume all integers (and sums) fit in $O(1)$ words

$\rightsquigarrow$ count #additions as elementary operation

Template for Describing an Algorithm

1. **Algorithmic Idea**
   Abstract idea that makes the algorithm work (prose)
   (an expert could fill in the rest from here)

2. </> **Pseudocode**
   structured description of procedure including edge cases
   should be unambiguous and close to real code

3. **Correctness proof**
   argument why the correct result is computed
   often uses induction and invariants

4. **Algorithm analysis**
   analysis of the efficiency of the algorithm
   usually want Θ-class of worst-case running time
   where interesting, also space usage

Brute force approach

- Let's start with the simplest thinkable solution

Maximal subarray problem

- **Given:** $A[0..n)$ with $A[i] \in \mathbb{Z}$ for $0 \le i < n$.
- Abbreviate $s(i,j) := \sum_{k=i}^{j-1} A[k]$
- **Goal:** Compute $s := \max\{s(i,j) : 0 \le i \le j \le n\}$ and a pair (i,j) with $s = s(i,j)$.

1. Algorithmic Idea

try all contiguous subarrays $A[i..j)$

2. </> Pseudocode

```
s = 0
for i = 0, ..., n − 1
    for j = i, ..., n
        t = 0
        for k = i, ..., j − 1
            t = t + A[k]
        end for
        if t > s then s := t
    end for
end for
```

3. Correctness proof

direct by definition of s

4. Algorithm analysis

\# additions

$$= \sum_{i=0}^{n-1} \sum_{j=i}^{n} \sum_{k=i}^{j-1} 1 = \sum_{i=0}^{n-1} \sum_{j=i}^{n} (j-i)$$

$$= \sum_{i=0}^{n-1} \sum_{j=0}^{n-i} j = \sum_{i=0}^{n-1} \frac{(n-i)(n-i+1)}{2}$$

$$= \frac{1}{2} \sum_{i=1}^{n} i(i+1) = \frac{1}{2} \sum_{i=1}^{n} i^2 + \frac{1}{2} \sum_{i=1}^{n} i$$

$$= \frac{n(n+1)(2n+1)}{12} + \frac{n(n+1)}{4}$$

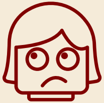
$$= \frac{n(n+1)(n+2)}{6} \sim \frac{1}{6} n^3 = \Theta(n^3)$$

Reusing sums

1. **Algorithmic Idea**
   - brute force algorithm is unnecessarily wasteful!
   - can use $s(i, j) = s(i, j-1) + A[j-1]$

2. **Pseudocode**

```
s = 0
for i = 0, ..., n − 1
    t = 0
    for j = i + 1, ..., n
        t = t + A[j − 1]
        if t > s then s := t
    end for
end for
```



Can we possibly do better?

- There are $\binom{n}{2} \sim \frac{1}{2}n^2$ different $s(i, j)$ . . .

$\rightsquigarrow$ Can't look at all of them

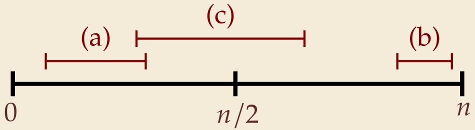
3. **Correctness proof:** as above

4. **Algorithm analysis:** $\displaystyle\sum_{i=0}^{n-1} \sum_{j=i+1}^{n} 1 = \frac{n(n+1)}{2} \sim \frac{1}{2}n^2 = \Theta(n^2)$ additions

A subquadratic solution

- **Algorithmic idea:**
  Consider $n/2$-mark.
  Only 3 options for optimal solution $s(i,j)$:

(a) $0 \le i \le j < \lceil \frac{n}{2} \rceil$ (left)
(b) $\lceil \frac{n}{2} \rceil \le i \le j \le n$ (right)
(c) $i < \lceil \frac{n}{2} \rceil \le j$ (straddle)



- optimal straddle easy to compute!
  - **independently** find best left endpoint i for $s(i, \lceil \frac{n}{2} \rceil)$ and best right endpoint j for $s(\lceil \frac{n}{2} \rceil, j)$

- for (a) and (b), recurse on instance of half the size!

A subquadratic solution – Pseudocode & Correctness

```
procedure findMaxSubarraySum(A[ℓ..r)):
    if r − ℓ ≤ 0
        return 0
    if r − ℓ == 1
        return max{0, A[ℓ]}
    m := ⌈(ℓ + r)/2⌉
    s_(a) := findMaxSubarraySum(A[ℓ, m))
    s_(b) := findMaxSubarraySum(A[m, r))
    // Find left endpoint of straddle:
    s_ℓ := 0;  t := 0
    for i = m − 1, m − 2, . . . , ℓ
        t := A[i] + t
        s_ℓ := max{s_ℓ, t}
    end for
    // Find right endpoint of straddle:
    s_r := 0;  t := 0
    for j = m + 1, . . . , r
        t := t + A[j − 1]
        s_r := max{s_r, t}
    end for
    s_(c) := s_ℓ + s_r
    return max{s_(a), s_(b), s_(c)}
```

Correctness proof:

- Induction over $n = r - \ell$
  - **basis:** for $n \le 1$ ✓
  - **hypothesis:** Assume findMaxSubarraySum returns correct result for all arrays of up to $n - 1$ elements
  - **step:** For array of $n \ge 2$ elements, distinguish cases (a), (b), (c)

    (a) and (b) $\rightsquigarrow$ IH ✓

    (c) “from inspection of the code”

A subquadratic solution – Analysis

```
procedure findMaxSubarraySum(A[ℓ..r)):
    if r − ℓ ≤ 0
        return 0
    if r − ℓ == 1
        return max{0, A[ℓ]}
    m := ⌈(ℓ + r)/2⌉
    s_(a) := findMaxSubarraySum(A[ℓ, m))
    s_(b) := findMaxSubarraySum(A[m, r))
    // Find left endpoint of straddle:
    s_ℓ := 0; t := 0
    for i = m − 1, m − 2, . . . , ℓ
        t := A[i] + t
        s_ℓ := max{s_ℓ, t}
    end for
    // Find right endpoint of straddle:
    s_r := 0; t := 0
    for j = m + 1, . . . , r
        t := t + A[j − 1]
        s_r := max{s_r, t}
    end for
    s_(c) := s_ℓ + s_r
    return max{s_(a), s_(b), s_(c)}
```

Algorithm analysis:

- Write $n = r - \ell$
- # additions in non-recursive part:
  $(m - \ell) + (r - m) + 1 \quad = \quad n + 1$
- Write $C(n)$ for total # additions for n elements

$\rightsquigarrow$ $C(n) = C(\lceil \frac{n}{2} \rceil) + C(\lfloor \frac{n}{2} \rfloor) + n + 1$

- for $n = 2^k$ for $k \in \mathbb{N}_0$, this simplifies to
  $C(2^k) = 2C(2^{k-1}) + 2^k + 1$

$\rightsquigarrow$ $C(n) \sim n \log_2(n)$

A lower bound

- **Theorem:** Every correct algorithm has a running time of $\Omega(n)$.

An optimal algorithm

Algorithmic idea:
In a clever sweep, we can compute best $s(i, r)$ and best $s(i, j)$ with $i \le j \le r$ for all r.

Pseudocode

```
procedure findMaxSubarraySum(A[0..n))
    suffixMax := 0;  globalMax := 0
    for r = 1, ..., n
        suffixMax := max{suffixMax + A[r − 1], 0}
        globalMax := max{globalMax, suffixMax}
    return globalMax
```