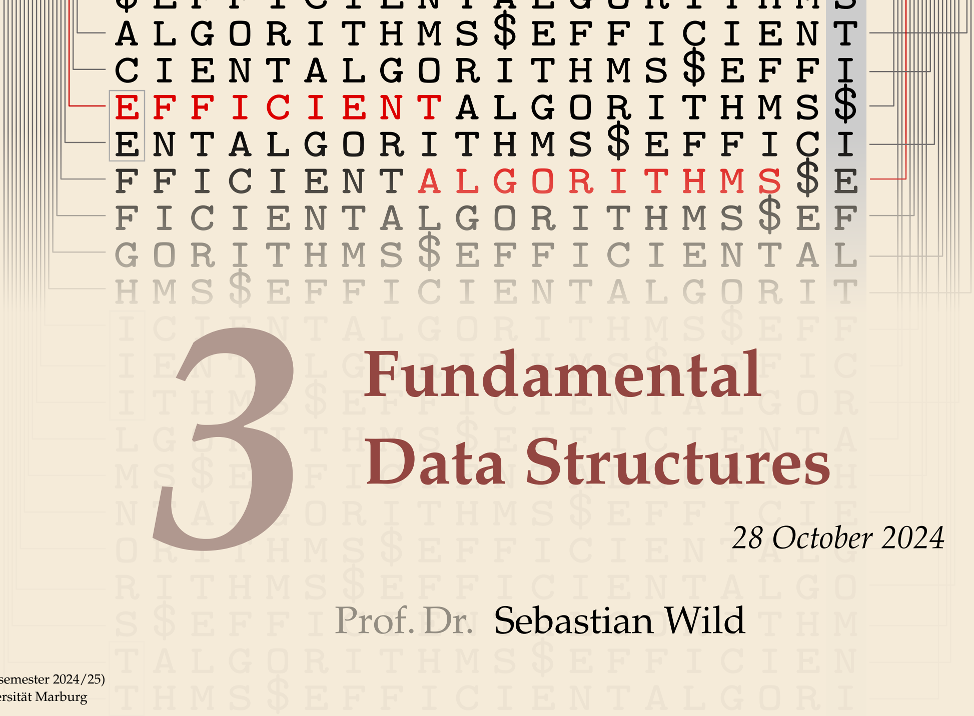

3

Fundamental Data Structures

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Learning Outcomes

Unit 3: ***Fundamental Data Structures***

1. Understand and demonstrate the difference between *abstract data type (ADT)* and its *implementation*
2. Be able to define the ADTs *stack*, *queue*, *priority queue* and *dictionary / symbol table*
3. Understand *array*-based implementations of stack and queue
4. Understand *linked lists* and the corresponding implementations of stack and queue
5. Know *binary heaps* and their performance characteristics
6. Understand *binary search trees* and their performance characteristics
7. Know high-level idea of basic *hashing strategies* and their performance characteristics

Outline

3 Fundamental Data Structures

Recap: The Random Access Machine

- Data structures make heavy use of pointers and dynamically allocated memory.

- Recall: Our RAM model supports
  - basic pseudocode (≈ simple Python/Java code)
  - creating arrays of a fixed/known size.
  - creating instances (objects) of a known class.

Python abstracts this away!

no predefined capacity!

There are *no arrays* in Python, only its built-in *lists*.

But: Python *implementations create* lists based on fixed-size arrays (stay tuned!)

⚠ Python ≠ RAM: Not every built-in Python instruction runs in $O(1)$ time!

3.1 Stacks & Queues

Abstract Data Types

abstract data type (ADT)

- list of supported operations
- **what** should happen
- **not:** how to do it
- **not:** how to store data

abstract base classes

≈ Java `interface`, Python `ABCs`
(with comments)

VS.

data structures

- specify exactly
  how data is represented
- **algorithms** for operations
- has concrete costs
  (space and running time)

≈ Java/Python `class`
(non abstract)

Why separate?

- Can swap out implementations ⇝ "drop-in replacements"

⇝ **reusable code**!

- (Often) better abstractions
- Prove generic lower bounds (⇝ Unit 3)

Stacks

Stack ADT

- `top()`
  Return the topmost item on the stack
  Does not modify the stack.

- `push(`x`)`
  Add x onto the top of the stack.

- `pop()`
  Remove the topmost item from the stack
  (and return it).

- `isEmpty()`
  Returns `true` iff stack is empty.

- `create()`
  Create and return an new empty stack.

Linked-list implementation for Stack

Invariants:

- maintain pointer *top* to topmost element
- each element points to the element below it (or `null` if bottommost)

```
class Node
    value
    next

class Stack
    top := null
    procedure top()
        return top.value
    procedure push(x)
        top := new Node(x, top)
    procedure pop()
        t := top()
        top := top.next
        return t
```

Linked-list implementation for Stack – Discussion

Linked stacks:

👍 require $\Theta(n)$ space when n elements on stack

👍 All operations take $O(1)$ time

👎 require $\Theta(n)$ space when n elements on stack

Can we avoid extra space for pointers?

Array-based implementation for Stack

If we want no pointers $\rightsquigarrow$ array-based implementation

Invariants:

- maintain array S of elements, from bottommost to topmost
- maintain index *top* of position of topmost element in `S`.

What to do if stack is full upon `push`?

Array stacks:

- require *fixed capacity* C (decided at creation time)!
- require $\Theta(C)$ space for a capacity of C elements
- all operations take $O(1)$ time

Queues

Operations:

- `enqueue(`x`)`
  Add x at the end of the queue.
- `dequeue()`
  Remove item at the front of the queue and return it.

Implementations similar to stacks.

Bags

What do Stack and Queue have in common?

They are special cases of a ***Bag***!

Operations:

- `insert(`x`)`
  Add x to the items in the bag.
- `delAny()`
  Remove any one item from the bag and return it.
  (Not specified which; any choice is fine.)
- roughly similar to Java's `java.util.Collection`
  Python's `collections.abc.Collection`

Sometimes it is useful to state that order is irrelevant ⇝ Bag
Implementation of Bag usually just a Stack or a Queue

3.2 Resizable Arrays

Digression – Arrays as ADT

Arrays can also be seen as an ADT! . . . but are commonly seen as specific data structure

Array operations:

- `create(`n`)` *Java*: `A = new int[`n`];` *Python:* `A = [0] *` n
  Create a new array with n cells, with positions $0, 1, \ldots, n-1$;
  we write $A[0..n) = A[0..n-1]$

- `get(`i`)` *Java/Python*: `A[`i`]`
  Return the content of cell i

- `set(`i`,`x`)` *Java/Python*: `A[`i`] =` x`;`
  Set the content of cell i to x.

⇝ Arrays have *fixed* size (supplied at creation). ($\neq$ `lists` in Python)

Usually directly implemented by compiler + operating system / virtual machine.

⚠ **Difference to "real" ADTs:** *Implementation usually fixed*
to "a contiguous chunk of memory".

Doubling trick

Can we have unbounded stacks based on arrays? Yes!

Invariants:

- maintain array S of elements, from bottommost to topmost
- maintain index *top* of position of topmost element in S
- maintain capacity $C = S.length$ so that $\frac{1}{4}C \le n \le C$

⇝ can always push more elements!

How to maintain the last invariant?

- before `push`
  If $n = C$, allocate new array of size $2n$, copy all elements.
- after `pop`
  If $n < \frac{1}{4}C$, allocate new array of size $2n$, copy all elements.

⇝ ***"Resizing Arrays"***

↖ an implementation technique, not an ADT!

Amortized Analysis

- Any individual operation `push` / `pop` can be expensive!
  $\Theta(n)$ time to copy all elements to new array.

- **But:** An one expensive operation of cost T means $\Omega(T)$ next operations are cheap!

distance to boundary

since $n \le C \le 4n$

Formally: consider "credits/potential" $\Phi = \min\{n - \frac{1}{4}C, C - n\} \in [0, 0.6n]$

- amortized cost of an operation = actual cost (array accesses) $- 4 \cdot$ change in Φ
  - cheap `push`/`pop`: actual cost 1 array access, consumes ≤ 1 credits ⇝ amortized cost ≤ 5
  - copying `push`: actual cost $2n + 1$ array accesses, creates $\frac{1}{2}n + 1$ credits ⇝ amortized cost ≤ 5
  - copying `pop`: actual cost $2n + 1$ array accesses, creates $\frac{1}{2}n - 1$ credits ⇝ amortized cost 5

⇝ **sequence** of m operations: total actual cost $\le$ total amortized cost + final credits

here: $\le 5m + 4 \cdot 0.6n = \Theta(m + n)$

Deamortized Resizable Arrays

What if we need $O(1)$ ***worst case*** *time?*

- It's possible to *de-amortize* the resizing arrays solution!
- maintain **3 arrays**: S (as before) and S_2 and $S_{1/2}$
  of twice and half the size of S

- write operations go to all 3 arrays
- upon resize, "shift" arrays up/down ⇝ S_2 resp. $S_{1/2}$ become new S
  - allocate new array, but **delay filling it with elements** ← general strategy!
  - every insert or delete copies 2 slots from last resize

⇝ by time for next resize, we have caught up and S_2 resp. $S_{1/2}$ ready to use

Analysis:

assuming memory allocation in $O(1)$ ⇝ needs to be uninitialized!

- $O(1)$ worst case time for read/write by index, push, and pop!
- up to 7 array accesses per operation
- up to $7n$ space

other time-space trade-offs possible

Rabbit Hole: Can we do this more space-efficiently?

- It might appear as if every efficient implementation of a stack needs $\Omega(n)$ extra space on top of space for storing the n elements in the stack.
- But this is not true!
- Can get operations in $O(1)$ worst-case time with $O(\sqrt{n})$ extra space at any time (!)
  - Maintain a collection of small arrays (plus header with pointers to them)
  - Clever choice of block sizes guarantees
    $O(\sqrt{n})$ blocks of $O(\sqrt{n})$ elements throughout
    and fast calculation of address for an index.

    imaginary "superblocks" of sizes 2^k, $k = 0, 1, \ldots, \lg n$
    kth superblock consists of $2^{k/2}$ actual blocks of $2^{k/2}$ elements each.
  - $O(\sqrt{n})$ extra space is best possible

Resizable Arrays in Optimal Time and Space
Andrej Brodnik, Svante Carlsson, Erik D. Demaine, J. Ian Munro & Robert Sedgewick
WADS 1999

3.3 Priority Queues & Binary Heaps

Priority Queue ADT – min-oriented version

Now: elements in the bag have different *priorities*.

(Max-oriented) Priority Queue (MaxPQ):

- `construct(`A`)`
  Construct from from elements in array A.
- `insert(`x`,`p`)`
  Insert item x with priority p into PQ.
- `max()`
  Return item with largest priority. (Does not modify the PQ.)
- `delMax()`
  Remove the item with largest priority and return it.
- `changeKey(`x`,`p'`)`
  Update x's priority to p'.
  Sometimes restricted to *increasing* priority.
- `isEmpty()`

Fundamental building block in many applications.

PQ implementations

Elementary implementations

- unordered list $\leadsto$ $\Theta(1)$ `insert`, but $\Theta(n)$ `delMax`
- sorted list $\leadsto$ $\Theta(1)$ `delMax`, but $\Theta(n)$ `insert`

Can we get something between these extremes? Like a "slightly sorted" list?

Yes! *Binary heaps.*

Array view

Heap = array A with
$\forall i \in [n] : A[\lfloor i/2 \rfloor] \geq A[i]$

$$\equiv$$

store nodes
in level order
in $A[1..n]$

Tree view

Heap = tree that is
(i) a complete binary tree
(ii) heap ordered

all but last level full
last level flush left

parent ≥ children

Binary heap example

Why heap-shaped trees?

Why complete binary tree shape?

- only one possible tree shape $\rightsquigarrow$ keep it simple!
- complete binary trees have minimal height among all binary trees
- simple formulas for moving from a node to parent or children:

  For a node at index k in A ⚠ Recall: nodes at indices $[\mathbf{1}..n]$
  - parent at $\lfloor k/2 \rfloor$ (for $k \geq 2$)
  - left child at $2k$
  - right child at $2k + 1$

Why heap ordered?

- Maximum must be at root! $\rightsquigarrow$ `max()` is trivial!
- But: Sorted only along paths of the tree; leaves lots of leeway for fast inserts

how? . . . stay tuned

3.4 Operations on Binary Heaps

Insert

1. Add new element at only possible place: bottom-most level, next free spot.
2. Let element *swim* up to repair heap order.

Delete Max

1. Remove max (must be in root).
2. Move last element (bottom-most, rightmost) into root.
3. Let root key *sink* in heap to repair heap order.

Heap construction

- n times insert $\rightsquigarrow$ $\Theta(n \log n)$ 👎
- instead:
  1. Start with singleton heaps (one element)
  2. Repeatedly merge two heaps of height k with new element into heap of height $k + 1$

Analysis

Height of binary heaps:

- *height* of a tree: # edges on longest root-to-leaf path
- *depth/level* of a node: # edges from root ⇝ root has depth 0
- How many nodes on first k *full* levels? $\displaystyle\sum_{\ell=0}^{k} 2^{\ell} = 2^{k+1} - 1$

⇝ Height of binary heap: $h \;=\; \min k \text{ s.t. } 2^{k+1} - 1 \ge n \;=\; \lfloor \lg(n) \rfloor$

Analysis:

- `insert`: new element "swims" up ⇝ $\le h$ steps (h cmps)
- `delMax`: last element "sinks" down ⇝ $\le h$ steps ($2h$ cmps)
- `construct` from n elements:

cost = cost of letting *each node* in heap sink!

$$\le 1 \cdot h + 2 \cdot (h-1) + 4 \cdot (h-2) + \cdots + 2^{\ell} \cdot (h-\ell) + \cdots + 2^{h-1} \cdot 1 + 2^{h} \cdot 0$$

$$= \sum_{\ell=0}^{h} 2^{\ell}(h-\ell) \;=\; \sum_{i=0}^{h} \frac{2^h}{2^i} i \;=\; 2^h \sum_{i=0}^{h} \frac{i}{2^i} \;\le\; 2 \cdot 2^h \;\le\; 4n$$

Binary heap summary

Operation	**Running Time**
`construct(`$A[1..n]$`)`	$O(n)$
`max()`	$O(1)$
`insert(`x`,`p`)`	$O(\log n)$
`delMax()`	$O(\log n)$
`changeKey(`x`,`p'`)`	$O(\log n)$
`isEmpty()`	$O(1)$
`size()`	$O(1)$

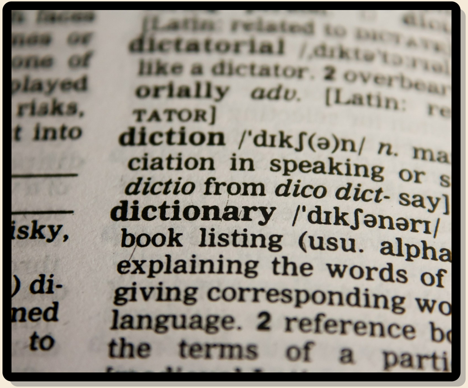
3.5 Symbol Tables

Symbol table ADT

Java: `java.util.Map<K,V>`

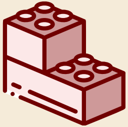
Symbol table / Dictionary / Map / Associative array / key-value store:

Python dict `{k:v}`



[Latin: related to DICTATE]
dictatorial /dɪktəˈtɔːrɪəl/
like a dictator. 2 overbear
orially adv. [Latin: re
TATOR]
diction /ˈdɪkʃ(ə)n/ n. ma
ciation in speaking or s
dictio from dico dict- say]
dictionary /ˈdɪkʃənərɪ/
book listing (usu. alpha
explaining the words of
giving corresponding wo
language. 2 reference bo
the terms of a parti

- `put(`k`,`v`)` *Python* dict: `d[`k`] = `v
  Put key-value pair (k, v) into table
- `get(`k`)` *Python* dict: `d[`k`]`
  Return value associated with key k
- `delete(`k`)` *Python* dict: `del d[`k`]`
  Remove key k (any associated value) form table
- `contains(`k`)` *Python* dict: k `in d`
  Returns whether the table has a value for key k
- `isEmpty(), size()`
- `create()`



Most fundamental building block in computer science.

(Every programming library has a symbol table implementation.)

Symbol tables vs. mathematical functions

- similar interface
- but: mathematical functions are *static/immutable* (never change their mapping)

  (Different mapping is a *different* function)
- symbol table $=$ *dynamic* mapping

  Function may change over time

Elementary implementations

Unordered (linked) list:

- 👍 Fast `put`
- 👎 $\Theta(n)$ time for `get`
- ⇝ Too slow to be useful

Sorted *linked* list:

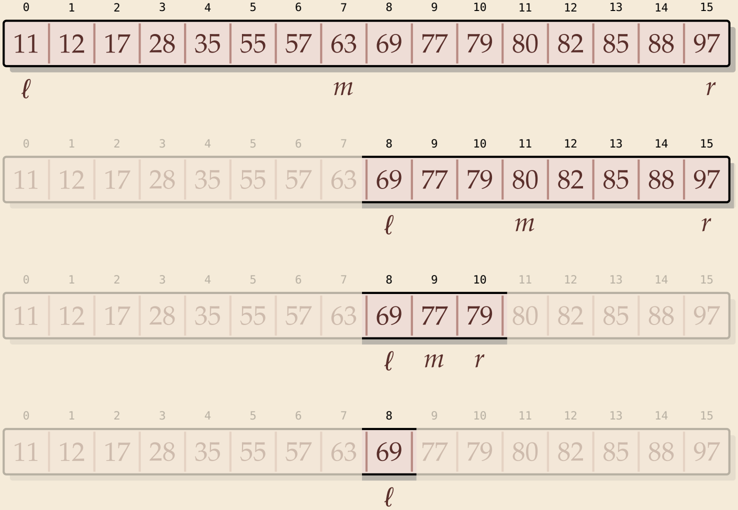
- 👎 $\Theta(n)$ time for `put`
- 👎 $\Theta(n)$ time for `get`
- ⇝ Too slow to be useful

⇝ *Sorted order does not help us at all?!*

Binary search

It does help . . . if we have a sorted ***array****!*

Example: search for 69



Binary search:

± 1

- ▶ halve remaining list in each step

$\rightsquigarrow$ $\le \lfloor \lg n \rfloor + 1$ cmps in the worst case

⚠ needs random access!

3.6 Binary Search Trees

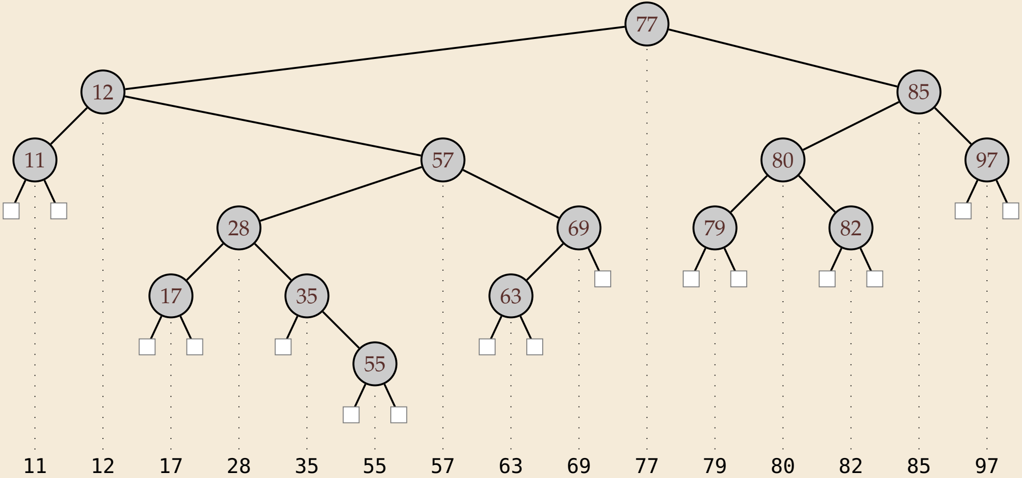
Binary search trees

Binary search trees (BSTs) $\approx$ dynamic sorted array

- binary tree
    - Each node has left and right child
    - Either can be empty (`null`)
- Keys satisfy ***search-tree property***

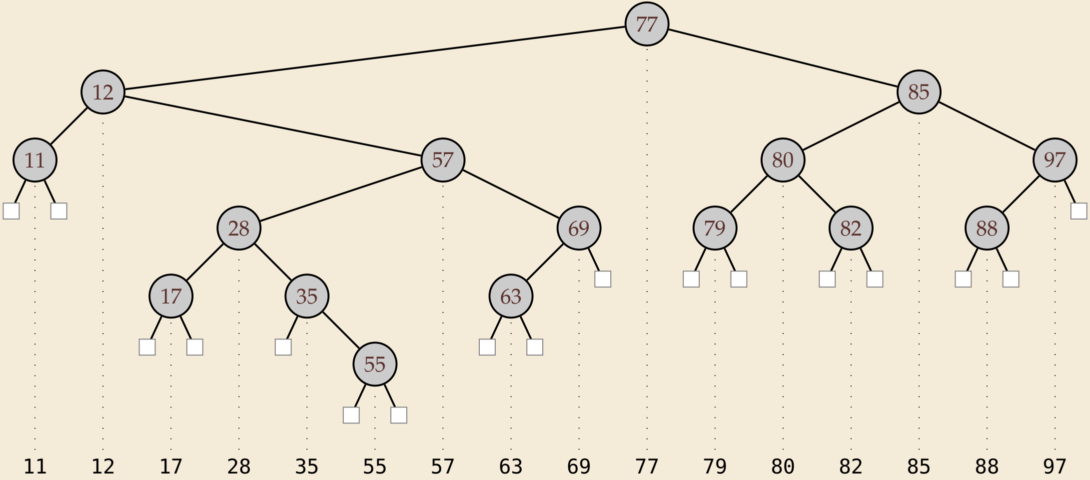
all keys in left subtree $\le$ root key $\le$ all keys in right subtree

BST example & find



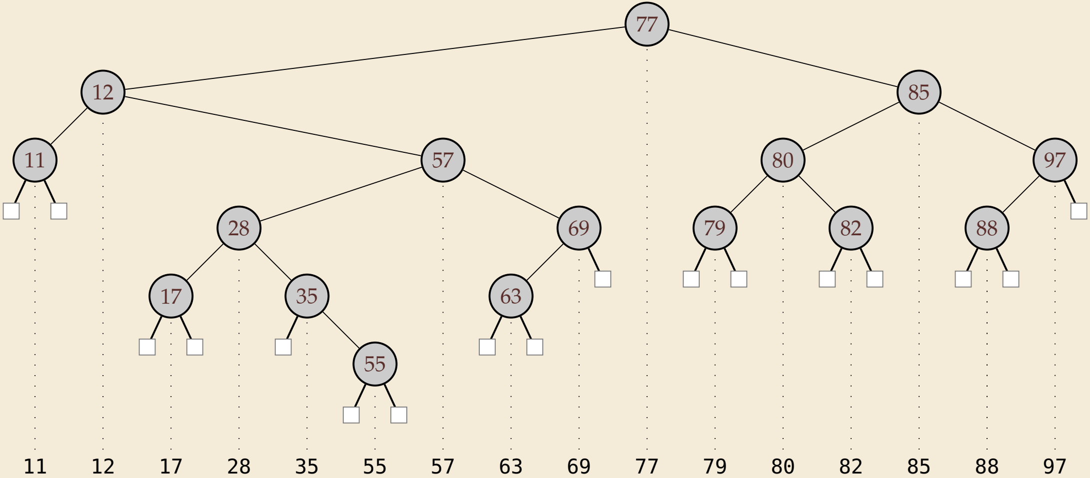
BST insert

Example: Insert 88



BST delete

- Easy case: remove leaf, e. g., 11 ⇝ replace by `null`
- Medium case: remove unary, e. g., 69 ⇝ replace by unique child
- Hard case: remove binary, e. g., 85 ⇝ swap with predecessor, recurse



Analysis

- **Search:**
- **Insert:**
- **Delete:**

BST summary

Operation	**Running Time**
`construct(`$A[1..n]$`)`	$O(nh)$
`put(`k`,`v`)`	$O(h)$
`get(`k`)`	$O(h)$
`delete(`k`)`	$O(h)$
`contains(`k`)`	$O(h)$
`isEmpty()`	$O(1)$
`size()`	$O(1)$

What is the height of a BST?

Worst Case:

- $h = n - 1 = \Theta(n)$

Average Case:

- Assumption: insertions come in random order
  no deletions

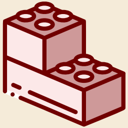
$\rightsquigarrow$ $h = \Theta(\log n)$ in expectation

↑
even "with high probability":
$\forall d\, \exists c : \Pr[h \geq c \lg(n)] \leq n^{-d}$

3.7 Ordered Symbol Tables

Ordered symbol tables

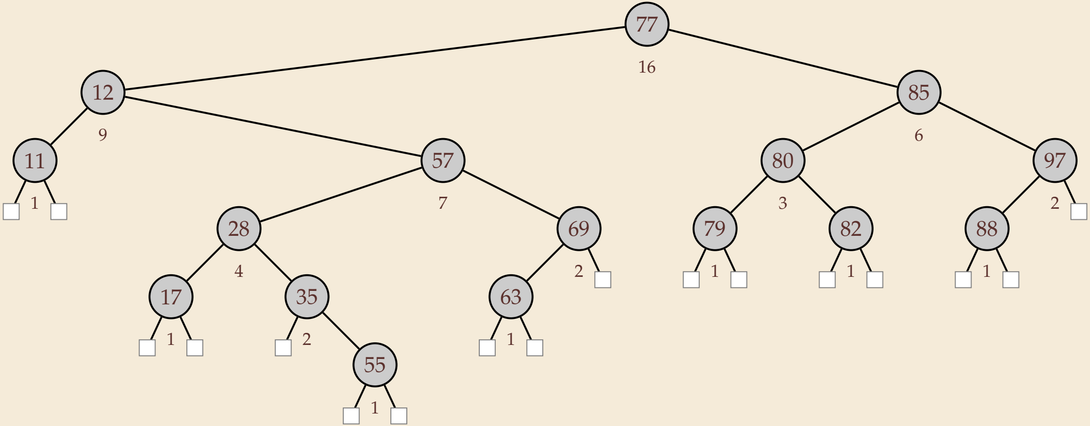
- `min()`, `max()`
  Return the smallest resp. largest key in the ST
- `floor(`x`)`, $\lfloor x \rfloor = \mathbb{Z}$`.floor(`$x$`)`
  Return largest key k in ST with $k \le x$.
- `ceiling(`x`)`
  Return smallest key k in ST with $k \ge x$.
- `rank(`x`)`
  Return the number of keys k in ST $k < x$.
- `select(`i`)`
  Return the ith smallest key in ST (zero-based, i. e., $i \in [0..n)$)



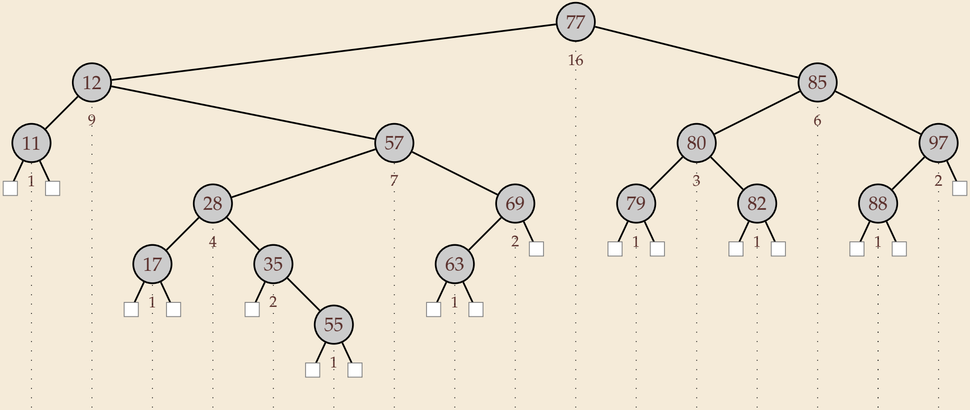
With select, we can simulate access as in a truly dynamic array!.

(Might not need any keys at all then!)

Augmented BSTs

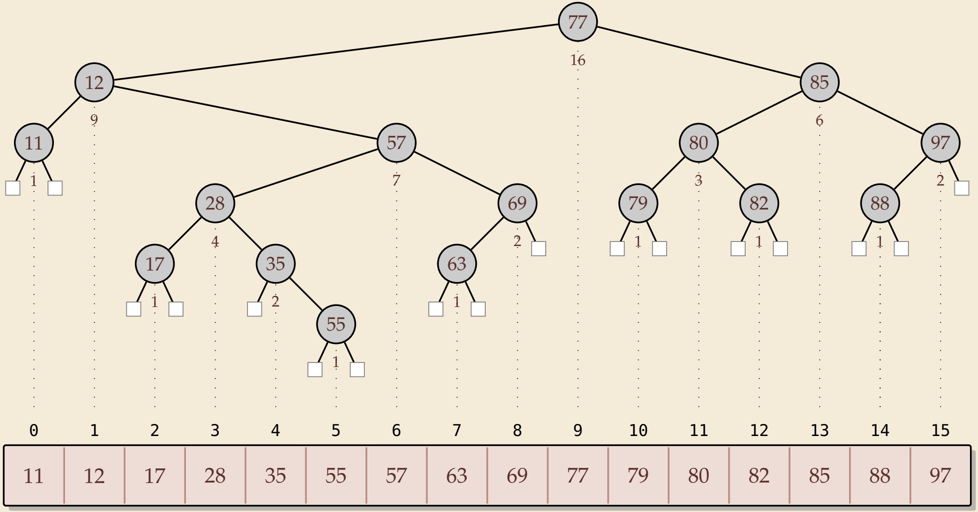


Rank



0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
11	12	17	28	35	55	57	63	69	77	79	80	82	85	88	97

Select



0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
11	12	17	28	35	55	57	63	69	77	79	80	82	85	88	97

Why store subtree sizes?

- Note that in an augmented BST, each node stores the **size of its subtree**.
- . . . why not directly store the **rank**? Would make rank/select much simpler!

- Problem: Single insertion/deletion can change *all* node ranks!

⇝ Cannot efficiently maintain node ranks.

👍 Subtree sizes only change along search path ⇝ $O(h)$ nodes affected

3.8 Balanced BSTs

Balanced BSTs

Balanced binary search trees:

- imposes shape invariant that guarantees $O(\log n)$ height
- adds rules to restore invariant after updates
- many examples known
  - *AVL trees* (height-balanced trees)
  - *red-black trees*
  - *weight-balanced trees* ($\mathrm{BB}[\alpha]$ trees)
  - . . .

Other options:

I'd love to talk more about all of these . . .
(Maybe another time)

- **amortization:** *splay trees, scapegoat trees*
  COLA (cache oblivious lookahead array)
- **randomization:** *randomized BSTs, treaps, skip lists*

BSTs vs. Heaps

Balanced binary search tree

Operation	Running Time
`construct`($A[1..n]$)	$O(n \log n)$
`put`(k,v)	$O(\log n)$
`get`(k)	$O(\log n)$
`delete`(k)	$O(\log n)$
`contains`(k)	$O(\log n)$
`isEmpty()`	$O(1)$
`size()`	$O(1)$
`min()` / `max()`	$O(\log n)$ ⇝ $O(1)$
`floor`(x)	$O(\log n)$
`ceiling`(x)	$O(\log n)$
`rank`(x)	$O(\log n)$
`select`(i)	$O(\log n)$

~~Binary heaps~~ *Strict Fibonacci heaps*

Operation	Running Time
`construct`($A[1..n]$)	$O(n)$
`insert`(x,p)	~~$O(\log n)$~~ $O(1)$
`delMax()`	$O(\log n)$
`changeKey`(x,p')	~~$O(\log n)$~~ $O(1)$
`max()`	$O(1)$
`isEmpty()`	$O(1)$
`size()`	$O(1)$

- apart from faster `construct`,
  BSTs always as good as binary heaps
- MaxPQ abstraction still helpful
- and faster heaps exist!

3.9 Hashing

Lower bound for search

The fastest implementations of the ordered symbol table ADT require $\Theta(\log n)$ time to search among n items. Is this the best possible?

Theorem: In the comparison model (on the keys),
$\Omega(\log n)$ comparisons are required to search a size-n dictionary.

Proof: Similar to lower bound for sorting (see Unit 4).

Any algorithm defines a binary decision tree with comparisons at the nodes and actions at the leaves.

There are at least $n + 1$ different actions (return an item, or "not found").

So there are $\Omega(n)$ leaves, and therefore the height is $\Omega(\log n)$. □

What if we don't need the ***ordered*** *symbol table operations?*

⇝ Focus on symbol table operations: `get`, `put`, `contains`, `delete`

Symbol Table without Sorting

- key idea in hashing: everything is ultimately an integer, or can be turned into one!

⇝ hash function $h : U \to [0..m)$

  - maps elements from universe U to integers
  - $h(x)$ used as index in a hash table $T[0..m)$

⇝ if h is quick to compute and **all stored elements hash to different indices**
`get`, `put`, `contains`, `delete` become simple array operations!

⇝ symbol table with $O(1)$ time per operation

(can make it so (“*perfect hashing*”), but usually too expensive)

ϟ Generally hash function h is not injective, so many keys can map to the same integer.

- We get *collisions*: we want to insert (k, v) into the table, but $T[h(k)]$ is already occupied.
  - *Birthday Paradox:* quite likely! Some collision with prob. $\geq \frac{1}{e}$ when $n \geq 2\sqrt{m}$

  ⇝ need to deal with them

Handling Collision

- Two basic strategies to deal with collisions:
  - *Buckets/Chaining*: Allow multiple items at each table location
    each table location points to linked list
  - *Open addressing*: Allow each item to go into multiple locations
    need strategy to define and search these locations
    - linear probing
    - quadratic probing
    - Robin Hood hashing
    - Cuckoo hashing

  (for full details of these strategies, see *Algorithms and Data Structures*)

- We evaluate strategies by the average cost of `get`, `put`, `delete` in terms of n, m, and/or the *load factor* $\alpha = n/m$.

⇝ Might have to rebuild the whole hash table and change the value of m when the load factor gets too large or too small.
  - This is called *rehashing*, and costs $\Theta(m+n)$.
  - alternative: *dynamic hashing* (not here; examples in *Algorithms and Data Structures*)

Comparison of Classic Hashing Schemes

Hash table design	**Search hit**	**Search miss**	**Insert**	**Space**	**good α**
Separate Chaining	$\sim \frac{1}{2}\alpha$	$\sim \alpha$	= miss	$n + m$	≈ 2
Linear Probing	$\sim \frac{1}{2}\left(1 + \frac{1}{1-\alpha}\right)$	$\sim \frac{1}{2}\left(1 + \frac{1}{(1-\alpha)^2}\right)$	= miss	m	≤ 0.5
Quadratic Probing	$\sim 1 + \ln\left(\frac{1}{1-\alpha}\right) - \frac{1}{2}\alpha$	$\sim \frac{1}{1-\alpha} - \alpha + \ln\left(\frac{1}{1-\alpha}\right)$	= miss	m	≤ 0.7
Robin Hood Hashing	$O(1)$	$O(1)$	= miss	m	≤ 1 (=any!)
d-way Cuckoo Hashing	$\le d$ worst case	$\le d$ worst case	amort.	m	$< c_d$

- Assumption: uniform hashing (all m^n hash sequences equally likely)
- Cost: expected # (equality) comparisons
- Space usage in words on top of space for items (without space for optional optimizations)

More improvements possible with word-RAM bitwise tricks ⇝ *Advanced Data Structures*

Hashing vs. Balanced Search Trees

Advantages of Balanced Search Trees

- $O(\log n)$ worst-case operation cost
- Does not require any assumptions, special functions, or known properties of input distribution
- Predictable (and often smaller) space usage (exactly n nodes)
- Never need to rebuild the entire structure
- supports ordered dictionary operations (rank, select etc.)

Advantages of Hash Tables

- $O(1)$ operations (if hashes well-spread and load factor small)
- We can choose space-time tradeoff via load factor
- Cuckoo hashing achieves $O(1)$ worst-case for search & delete