5 Divide & Conquer

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Learning Outcomes

Unit 5: ***Divide & Conquer***

1. Know the steps of the Divide & Conquer paradigm.
2. Be able to solve simple Divide & Conquer recurrences.
3. Be able to design and analyze new algorithms using the Divide & Conquer paradigm.
4. Know the performance characteristics of selection-by-rank algorithms.
5. Know the divide and conquer approaches for integer multiplication, matrix multiplication, finding majority elements, and the closest-pair-of-points problem.

Outline

5 Divide & Conquer

Divide and conquer

Divide and conquer *idiom* (Latin: *divide et impera*)
to make a group of people disagree and fight with one another
so that they will not join together against one (Merriam-Webster Dictionary)

⇝ in politics & algorithms, many independent, small problems are better than one big one!

Divide-and-conquer algorithms:

1. Break problem into smaller, independent subproblems. (Divide!)
2. Recursively solve all subproblems. (Conquer!)
3. Assemble solution for original problem from solutions for subproblems.

Examples:

- Mergesort
- Quicksort
- Binary search
- (arguably) Tower of Hanoi

5.1 Divide & Conquer Recurrences

Back-of-the-envelope analysis

- before working out the details of a D&C idea,
  it is often useful to get a quick indication of the resulting performance
  - don't want to waste time on something that's not competitive in the end anyways!
- since D&C is naturally recursive, running time often not obvious
  instead: given by a recursive equation
- unfortunately, rigorous analysis often tricky
  - Remember mergesort?

$$C(n) = \begin{cases} 0 & n \leq 1 \\ C(\lfloor n/2 \rfloor) + C(\lceil n/2 \rceil) + 2n & n \geq 2 \end{cases}$$

$$\rightsquigarrow \quad C(n) = 2n\lfloor \lg(n) \rfloor + 2n - 4 \cdot 2^{\lfloor \lg(n) \rfloor} \quad 😱$$
$$= \Theta(n \log n) \quad 😌$$

- the following method works for many typical cases to give the right **order of growth**

The Master Method

- Assume a stereotypical D&C algorithm
  - a recursive calls on (for some constant $a \geq 1$)
  - subproblems of size n/b (for some constant $b > 1$)
  - with non-recursive "conquer" effort $f(n)$ (for some function $f : \mathbb{R} \to \mathbb{R}$)
  - base case effort d (some constant $d > 0$)

$\rightsquigarrow$ running time $T(n)$ satisfies

$$T(n) = \begin{cases} a \cdot T\left(\frac{n}{b}\right) + f(n) & n > 1 \\ d & n \leq 1 \end{cases}$$

Theorem 5.1 (Master Theorem)

With $c := \log_b(a)$, we have for the above recurrence:

(a) $T(n) = \Theta(n^c)$ if $f(n) = O(n^{c-\varepsilon})$ for constant $\varepsilon > 0$.

(b) $T(n) = \Theta(n^c \log n)$ if $f(n) = \Theta(n^c)$.

(c) $T(n) = \Theta(f(n))$ if $f(n) = \Omega(n^{c+\varepsilon})$ for constant $\varepsilon > 0$ **and** f satisfies the regularity condition $\exists n_0, \alpha < 1 \; \forall n \geq n_0 \; : \; a \cdot f\left(\frac{n}{b}\right) \leq \alpha f(n)$.

Master Theorem – Intuition & Proof Idea

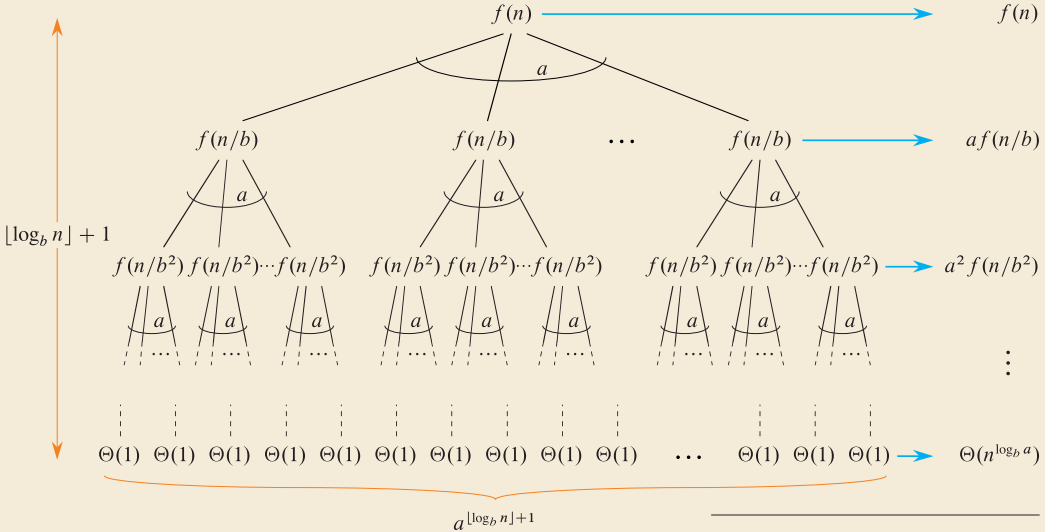


Figure 4.3 of Cormen et al. *Introduction to Algorithms* 4th ed.

When it's fine to ignore floors and ceilings

The ***polynomial-growth condition***

- $f : \mathbb{R}_{>0} \to \mathbb{R}$ satisfies the *polynomial-growth condition* if

$$\exists n_0\ \forall C \geq 1\ \exists D > 1 \quad \forall n \geq n_0\ \forall c \in [1, C] \ : \ \tfrac{1}{D} f(n) \leq f(cn) \leq D f(n)$$

- intuitively: increasing n by up to a factor C (and anywhere in between!) changes the function value by at most a factor $D = D(C)$
  (for sufficiently large n)
- examples: $f(n) = \Theta(n^\alpha \log^\beta(n) \log\log^\gamma(n))$ for constants α, β, γ (zero allowed)
  $\rightsquigarrow$ f satisfies the polynomial-growth condition

Lemma 5.2 (Polynomial-growth master method)

If the toll function $f(n)$ satisfies the polynomial-growth condition, then the Θ-class of the solution of a D&C recurrence remains the same when ignoring floors and ceilings on subproblem sizes. ◀

A Rigorous and Stronger Meta Theorem

Theorem 5.3 (Roura's Discrete Master Theorem)

Let $T(n)$ be recursively defined as

$$T(n) = \begin{cases} b_n & 0 \le n < n_0, \\ f(n) + \sum_{d=1}^{D} a_d \cdot T\left(\frac{n}{b_d} + r_{n,d}\right) & n \ge n_0, \end{cases}$$

where $D \in \mathbb{N}$, $a_d > 0$, $b_d > 1$, for $d = 1, \ldots, D$ are constants, functions $r_{n,d}$ satisfy $|r_{n,d}| = O(1)$ as $n \to \infty$, and function $f(n)$ satisfies $f(n) \sim B \cdot n^{\alpha}(\ln n)^{\gamma}$ for constants $B > 0$, α, γ.
Set $H = 1 - \sum_{d=1}^{D} a_d \, (1/b_d)^{\alpha}$; then we have:

- **(a)** If $H < 0$, then $T(n) = O(n^{\tilde{\alpha}})$, for $\tilde{\alpha}$ the unique value of α that would make $H = 0$.
- **(b)** If $H = 0$ and $\gamma > -1$, then $T(n) \sim f(n)\ln(n)/\tilde{H}$ with constant $\tilde{H} = (\gamma + 1) \sum_{d=1}^{D} a_d \, b_d^{-\alpha} \ln(b_d)$.
- **(c)** If $H = 0$ and $\gamma = -1$, then $T(n) \sim f(n)\ln(n)\ln(\ln(n))/\hat{H}$ with constant $\hat{H} = \sum_{d=1}^{D} a_d \, b_d^{-\alpha} \ln(b_d)$.
- **(d)** If $H = 0$ and $\gamma < -1$, then $T(n) = O(n^{\alpha})$.
- **(e)** If $H > 0$, then $T(n) \sim f(n)/H$.

5.2 Order Statistics

Selection by Rank

- Standard data summary of numerical data: (Data scientists, listen up!)
  - mean, standard deviation
  - min/max (range)
  - histograms

  } easy to compute in $\Theta(n)$ time
  - median, quartiles, other quantiles (a.k.a. order statistics)

computable in $\Theta(n)$ time?

General form of problem: **Selection by Rank**

- **Given:** array $A[0..n)$ of numbers and number $k \in [0..n)$.
- **Goal:** find element that would be in position k if A was sorted (kth smallest element).

but 0-based & counting dups

- $k = \lfloor n/2 \rfloor \rightsquigarrow$ median; $k = \lfloor n/4 \rfloor \rightsquigarrow$ lower quartile
  $k = 0 \rightsquigarrow$ minimum; $k = n - \ell \rightsquigarrow \ell$th largest

Quickselect

- Key observation: Finding the element of rank k seems hard.
  But computing the rank of a given element is easy!
  (count smaller elements)

⇝ Pick any element $A[b]$ and find its rank j.

- $j = k$? ⇝ Lucky Duck! Return chosen element and stop
- $j < k$? ⇝ . . . not done yet. But: The $j + 1$ elements smaller than $\leq A[b]$ can be excluded!
- $j > k$? ⇝ similarly exclude the $n - j$ elements $\geq A[b]$

- partition function from Quicksort:
  - returns the rank of pivot
  - separates elements into smaller/larger

⇝ can use same building blocks

```
1  procedure quickselect(A[l..r), k)
2      if r − l ≤ 1 then return A[l]
3      b := choosePivot(A[l..r))
4      j := partition(A[l..r), b)
5      if j == k
6          return A[j]
7      else if j < k
8          quickselect(A[j + 1..r), k)
9      else // j > k
10         quickselect(A[l..j), k)
```

Quickselect – Iterative Code

Recursion can be replaced by loop (*tail-recursion elimination*)

```
1   procedure quickselect(A[l..r), k)
2       if r − l ≤ 1 then return A[l]
3       b := choosePivot(A[l..r))
4       j := partition(A[l..r), b)
5       if j == k
6           return A[j]
7       else if j < k
8           quickselect(A[j + 1..r), k)
9       else // j > k
10          quickselect(A[l..j), k)
```

```
1  procedure quickselectIterative(A[0..n), k)
2      l := 0;  r := n
3      while r − l > 1
4          b := choosePivot(A[l..r))
5          j := partition(A[l..r), b)
6          if j ≥ k then r := j − 1
7          if j ≤ k then l := j + 1
8      return A[k]
```

- implementations should usually prefer iterative version
- analysis more intuitive with recursive version

Quickselect – Analysis

```
1  procedure quickselect(A[l..r], k)
2      if r − l ≤ 1 then return A[l]
3      b := choosePivot(A[l..r])
4      j := partition(A[l..r], b)
5      if j == k
6          return A[j]
7      else if j < k
8          quickselect(A[j + 1..r), k)
9      else // j > k
10         quickselect(A[l..j), k)
```

- cost = #cmps
- costs depend on n **and** k
- **worst case:** $k = 0$, but always $j = n - 2$
  - $\rightsquigarrow$ each recursive call makes n one smaller at cost $\Theta(n)$
  - $\rightsquigarrow$ $T(n,k) = \Theta(n^2)$ worst case cost

average case:

- let $T(n,k)$ expected cost when we choose a pivot uniformly from $A[0..n)$

$\rightsquigarrow$ formulate recurrence for $T(n,k)$ similar to BST/Quicksort recurrence

$$T(n,k) \;=\; n \;+\; \frac{1}{n}\sum_{r=0}^{n-1}[r=k]\cdot 0 \;+\; [k<r]\cdot T(r,k) \;+\; [k>r]\cdot T(n-r-1,k-r-1)$$

Quickselect – Average Case Analysis

- $T(n,k) \;=\; n \;+\; \frac{1}{n}\sum_{r=0}^{n-1}[r=k]\cdot 0 \;+\; [k<r]\cdot T(r,k) \;+\; [k>r]\cdot T(n-r-1,k-r-1)$
- Set $\hat{T}(n) \;=\; \max_{k\in[0..n)} T(n,k)$

$\rightsquigarrow\; \hat{T}(n) \;\leq\; n \;+\; \frac{1}{n}\sum_{r=0}^{n-1}\max\{\hat{T}(r),\hat{T}(n-r-1)\}$

- analyze hypothetical, worse algorithm:
  if $r \notin [\frac{1}{4}n, \frac{3}{4}n)$, discard pivot and repeat with new one!

$\rightsquigarrow\; \hat{T}(n) \leq \tilde{T}(n)$ defined by $\qquad \tilde{T}(n) \;\leq\; n \;+\; \frac{1}{2}\tilde{T}(n) \;+\; \frac{1}{2}\tilde{T}(\frac{3}{4}n)$

$\rightsquigarrow\; \tilde{T}(n) \;\leq\; 2n \;+\; \tilde{T}(\frac{3}{4}n)$

- Master Theorem Case 3: $\tilde{T}(n) = \Theta(n)$

Quickselect Discussion

- 👎 $\Theta(n^2)$ worst case (like Quicksort)
- 👍 expected cost $\Theta(n)$ (best possible)
- 👍 no extra space needed
- 👍 adaptations possible to find several order statistics at once

- 👍 expected cost can be further improved by choosing pivot from a small sorted sample
  - $\rightsquigarrow$ asymptotically optimal randomized cost: $n + \min\{k, n-k\}$ comparisons in expectation achieved asymptotically by the *Floyd-Rivest algorithm*

5.3 Linear-Time Selection

Interlude – A recurring conversation

Cast of Characters:

Hi! I'm a *computer science practitioner*.
I love algorithms for the sometimes miraculous applications they enable.
I care for **things** I can implement and **that actually work in practice**.

Hi! I'm a *theoretical computer science researcher*.
I find beauty in elegant and **definitive** answers to questions about complexity.
I care for **eternal truths** and mathematically proven facts;
asymptotically optimal is what counts! (Constant factors are secondary.)

Quickselect Disagreements

For practical purposes, (randomized) Quickselect is perfect.

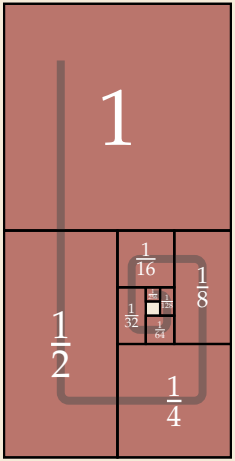
e. g. used in C++ STL `std::nth_element`

Yeah . . . maybe. But can we select by rank in $O(n)$ deterministic **worst case** time?

Better Pivots

It turns out, we can!

- All we need is better pivots!
  - If pivot was the exact median,
    we would at least halve #elements in each step
  - Then the total cost of all partitioning steps is $\leq 2n = \Theta(n)$.



But: finding medians is (basically) our original problem!

It totally suffices to find an element of rank αn for $\alpha \in (\varepsilon, 1-\varepsilon)$
to get overall costs $\Theta(n)$!

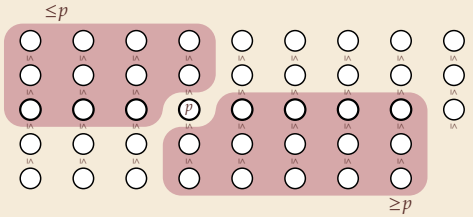
The Median-of-Medians Algorithm

```
1  procedure choosePivotMoM(A[l..r])
2      m := ⌊n/5⌋
3      for i := 0, ..., m − 1
4          sort(A[5i..5i + 4])
5          // collect median of 5
6          Swap A[i] and A[5i + 2]
7      return quickselectMoM(A[0..m), ⌊(m−1)/2⌋)
8
9  procedure quickselectMoM(A[l..r), k)
10     if r − l ≤ 1 then return A[l]
11     b := choosePivotMoM(A[l..r))
12     j := partition(A[l..r), b)
13     if j == k
14         return A[j]
15     else if j < k
16         quickselectMoM(A[j + 1..r), k)
17     else // j > k
18         quickselectMoM(A[l..j), k)
```

Analysis:

- Note: 2 mutually recursive procedures
  - $\rightsquigarrow$ effectively 2 recursive calls!

1. recursive call inside choosePivotMoM on $m \leq \frac{n}{5}$ elements
2. recursive call inside quickselectMoM



$\rightsquigarrow$ partition excludes $\sim 3 \cdot \frac{m}{2} \sim \frac{3}{10}n$ elem.

$$\rightsquigarrow\ C(n) \leq \Theta(n) + C(\tfrac{1}{5}n) + C(\tfrac{7}{10}n)$$
$$\leq \Theta(n) + C(\tfrac{1}{5}n + \tfrac{7}{10}n)$$
$$= \Theta(n) + C(\tfrac{9}{10}n) \quad \rightsquigarrow \quad C(n) = \Theta(n)$$

ansatz: overall cost linear

5.4 Fast Multiplication

Integer Multiplication

- What's the cost of computing $x \cdot y$ for two integers x and y?

$\rightsquigarrow$ depends on how big the numbers are!

  - If x and y have $O(w)$ bits, multiplication takes $O(1)$ time on word-RAM
  - otherwise, need a dedicated algorithm!

Long multiplication (»Schulmethode«)

- Given $x = \sum_{i=0}^{n-1} x_i 2^i$ and $y = \sum_{i=0}^{n-1} y_i 2^i$, want $z = \sum_{i=0}^{2n-1} z_i 2^i$

```
1 for i := 0,...,n-1
2     c := 0
3     for j := 0,...,n-1
4         z_{i+j} := z_{i+j} + c + x_i · y_j
5         c := ⌊z_{i+j}/2⌋
6         z_{i+j} := z_{i+j} mod 2
7     end for
8     z_{i+n} := c
9 end for
```

- $\Theta(n^2)$ bit operations
- could work with base 2^w instead of 2

  $\rightsquigarrow$ $\Theta((n/w)^2)$ time

- here: count bit operations for simplicity can be generalized

Example:
easier in binary!
("shift and add")

```
1001010101 * 101101
-------------------
    1001010101
   0000000000
  1001010101
 1001010101
0000000000
 1001010101
-------------------
110100011110001
```

Divide & Conquer Multiplication

- assume n is power of 2 (fill up with 0-bits otherwise)
- We can write
  - $x = a_1 2^{n/2} + a_2$ and
  - $y = b_1 2^{n/2} + b_2$
  - for a_1, a_2, b_1, b_2 integers with $n/2$ bits

$\rightsquigarrow\ x \cdot y = (a_1 2^{n/2} + a_2) \cdot (b_1 2^{n/2} + b_2) = \mathbf{a_1 b_1} 2^n + (\mathbf{a_1 b_2} + \mathbf{a_2 b_1}) 2^{n/2} + \mathbf{a_2 b_2}$

  - recursively compute **4** smaller products
  - combine with shifts and additions ($O(n)$ bit operations)

- . . . but is this any good?
  - $T(n) = \mathbf{4} \cdot T(n/2) + \Theta(n)$
  - $\rightsquigarrow$ Master Theorem Case 1: $T(n) = \Theta(n^2)$ *. . . just like the primary school method!?*
    - but Master Theorem gives us a hint: cost is dominated by the leaves
    - $\rightsquigarrow$ try to do more work in conquer step!

Karatsuba Multiplication

- how can we do "less divide and more conquer"?

  Recall: $x \cdot y = \mathbf{a_1 b_1} 2^n + (\mathbf{a_1 b_2} + \mathbf{a_2 b_1}) 2^{n/2} + \mathbf{a_2 b_2}$

Let's do some algebra.

$$\begin{aligned} c &:= (a_1 + a_2) \cdot (b_1 + b_2) \\ &= a_1 b_1 + (a_1 b_2 + a_2 b_1) + a_2 b_2 \end{aligned}$$

$$\rightsquigarrow \quad (\mathbf{a_1 b_2} + \mathbf{a_2 b_1}) = c - a_1 b_1 - a_2 b_2$$

this can be computed with **3** recursive multiplications

$a_1 + a_2$ and $b_1 + b_2$ still have roughly $n/2$ bits

```
1 procedure karatsuba(x, y):
2     // Assume x and y are n = 2^k bit integers
3     a1 := ⌊x/2^{n/2}⌋; a2 := x mod 2^{n/2} // implemented by shifts
4     b1 := ⌊y/2^{n/2}⌋; b2 := y mod 2^{n/2}
5     c1 := karatsuba(a1, b1)
6     c2 := karatsuba(a2, b2)
7     c  := karatsuba(a1 + a2, b1 + b2) − c1 − c2
8     return c1 2^n + c 2^{n/2} + c2 // shifts and additions
```

Analysis:

- nonrecursive cost: only additions and shifts
- all numbers $O(n)$ bits

$\rightsquigarrow$ conquer cost $f(n) = \Theta(n)$

Recurrence:

- $T(n) = 3T(n/2) + \Theta(n)$
- Master Theorem Case 1

$\rightsquigarrow$ $T(n) = \Theta(n^{\lg 3}) = O(n^{1.585})$

much cheaper (for large n)!

Integer Multiplication

- until 1960, integer multiplication was conjectured to take $\Omega(n^2)$ bit operations

$\rightsquigarrow$ Karatsuba's algorithm was a big breakthrough
  - which he discovered as a student!

- idea can be generalized to breaking numbers into $k \geq 2$ parts (*Toom-Cook algorithm*)

- asymptotically *much* better algorithms are now known!
  - e. g., the *Schönhage-Strassen algorithm* with $O(n \log n \log\log n)$ bit operations (!)
  - these are based on the *Fast Fourier Transform* (FFT) algorithm
    - numbers = polynomials evaluated at base (e. g., $z = 2$)
    - $\rightsquigarrow$ multiplication of numbers = convolution of polynomials
    - FFT makes computation of this convolution cheap by computing the polynomial via interpolation
    - Schönhage-Strassen adds careful finite-field algebra to make computations efficient

Matrix Multiplication

- The same trick can also be used for faster matrix multiplication
- Recall: For $A, B \in \mathbb{R}^{n \times n}$ we define $C = A \cdot B$ via $c_{i,j} = \sum_{k=1}^{n} a_{i,k}\, b_{k,j}$ (entry of A in row i and column k)

$\rightsquigarrow$ Naive cost: n^2 sums with n terms each $\rightsquigarrow$ $\Theta(n^3)$ arithmetic operations

- Can use D&C as follows (assuming n is a power of 2 again)
  - Decompose (cut in half hor. & vert.) $A = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix}$, $B = \begin{pmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{pmatrix}$, $C = \begin{pmatrix} C_{1,1} & C_{1,2} \\ C_{2,1} & C_{2,2} \end{pmatrix}$
  - $\rightsquigarrow$ We get C as

$$\begin{aligned} C_{1,1} &= A_{1,1} \cdot B_{1,1} + A_{1,2} \cdot B_{2,1} \\ C_{1,2} &= A_{1,1} \cdot B_{1,2} + A_{1,2} \cdot B_{2,2} \\ C_{2,1} &= A_{2,1} \cdot B_{1,1} + A_{2,2} \cdot B_{2,1} \\ C_{2,2} &= A_{2,1} \cdot B_{1,2} + A_{2,2} \cdot B_{2,2} \end{aligned}$$

(note "$\cdot$" and "$+$" operate on matrices here)

4 matrix sums with $(\frac{n}{2})^2$ entries each

  - 8 recursive matrix multiplications on two $\frac{n}{2} \times \frac{n}{2}$ matrices + $\Theta(n^2)$ summations
  - # operations $T(n) = 8T(n/2) + \Theta(n^2)$
  - $\rightsquigarrow$ Master Theorem Case 1: $T(n) = \Theta(n^3)$ 😒 (but: still useful for better memory locality!)

Strassen Algorithm for Matrix Multiplication

- Observation (again): Can do more conquer for less divide!
- We recursively compute the following 7 products:

$$
\begin{aligned}
M_1 &:= (A_{1,2} - A_{2,2}) \cdot (B_{2,1} + B_{2,2}) \\
M_2 &:= (A_{1,1} + A_{2,2}) \cdot (B_{1,1} + B_{2,2}) \\
M_3 &:= (A_{1,1} - A_{2,1}) \cdot (B_{1,1} + B_{1,2}) \\
M_4 &:= (A_{1,1} + A_{1,2}) \cdot B_{2,2} \\
M_5 &:= A_{1,1} \cdot (B_{1,2} - B_{2,2}) \\
M_6 &:= A_{2,2} \cdot (B_{2,1} - B_{1,1}) \\
M_7 &:= (A_{2,1} + A_{2,2}) \cdot B_{1,1}
\end{aligned}
$$

$\rightsquigarrow$ We then obtain the 4 parts of C as

$$
\begin{aligned}
C_{1,1} &= M_1 + M_2 - M_4 + M_6 \\
C_{1,2} &= M_4 + M_5 \\
C_{2,1} &= M_6 + M_7 \\
C_{2,2} &= M_2 - M_3 + M_5 - M_7
\end{aligned}
$$

(Proof: left as exercise 😉)

Analysis:

- **conquer step:** larger but still $O(1)$ # matrix add/subtract

$\rightsquigarrow$ $\Theta(n^2)$ operations for conquer

$\rightsquigarrow$ total # arithmetic operations
$T(n) = 7\,T(n/2) + \Theta(n^2)$

$\rightsquigarrow$ Master Theorem Case 1:
$T(n) = \Theta(n^{\lg 7}) = O(n^{2.808})$

Open Problems

*Multiplication is extremely fundamental, but its **computational complexity** is an **open problem** and subject of active research!*

Integer multiplication:

- **conjectured** to require $\Omega(n \log n)$ bit operations (no proof known!)
- Harvey & van der Hoeven **2021**: $O(n \log n)$ algorithm possible!

Matrix multiplication (MM):

- more relevant than it might seem since complexity identical to
  - computing inverse matrices, determinants
  - Gaussian elimination ($\rightsquigarrow$ solving systems of linear equations)
  - recognition of context free languages

$\rightsquigarrow$ best exponent even has standard notation:
smallest $\omega \in [2, 3)$ so that MM takes $O(n^\omega)$ operations

- Big open question: Is $\omega > 2$?
- best known bound: $\omega \leq 2.371339$ (from 2024!)

Timeline of matrix multiplication exponent

Year	Bound on omega	Authors
1969	2.8074	Strassen[1]
1978	2.796	Pan[10]
1979	2.780	Bini, Capovani [it], Romani[11]
1981	2.522	Schönhage[12]
1981	2.517	Romani[13]
1981	2.496	Coppersmith, Winograd[14]
1986	2.479	Strassen[15]
1990	2.3755	Coppersmith, Winograd[16]
2010	2.3737	Stothers[17]
2012	2.3729	Williams[18][19]
2014	2.3728639	Le Gall[20]
2020	2.3728596	Alman, Williams[21][22]
2022	2.371866	Duan, Wu, Zhou[23]
2024	2.371552	Williams, Xu, Xu, and Zhou[2]
2024	2.371339	Alman, Duan, Williams, Xu, Xu, and Zhou[24]

5.5 Majority

Majority

- **Given:** Array $A[0..n)$ of objects
- **Goal:** Check of there is an object x that occurs at $> \frac{n}{2}$ positions in A
  if so, return x
- Naive solution: check each $A[i]$ whether it is a majority $\rightsquigarrow$ $\Theta(n^2)$ time

Majority – Divide & Conquer

Can be solved faster using a simple Divide & Conquer approach:

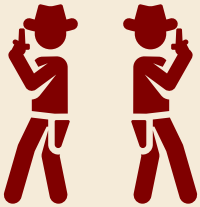
- If A has a majority, that element must also be a majority of at least one half of A.
- $\rightsquigarrow$ Can find majority (if it exists) of left half and right half recursively
- $\rightsquigarrow$ Check these ≤ 2 candidates.
- Costs similar to mergesort $\Theta(n \log n)$

```
1  procedure majority(A[0..n))
2      if n == 1 then return A[0] end if
3      k := ⌊n/2⌋
4      M_ℓ := majority(A[0..k))
5      M_r := majority(A[k..n))
6      if M_ℓ == M_r then return M_ℓ end if
7      m_ℓ := 0;  m_r := 0
8      for i := 0, ..., n − 1
9          if A[i] == M_ℓ then m_ℓ = m_ℓ + 1 end if
10         if A[i] == M_r then m_r = m_r + 1 end if
11     end for
12     if m_ℓ ≥ k + 1
13         return M_ℓ
14     else if m_r ≥ k + 1
15         return M_r
16     else
17         return NO_MAJORITY_ELEMENT
```

Majority – Linear Time

We can actually do much better!

```
1 def MJRTY(A[0..n))
2     c := 0
3     for i := 1, . . . , n − 1
4         if c == 0
5             x := A[i]; c := 1
6         else
7             if A[i] == x then c := c + 1 else c := c − 1
8     return x
```



- MJRTY($A[0..n)$) returns *candidate* majority element
- either that candidate is the majority element or none exists(!)

Clearly $\Theta(n)$ time

5.6 Closest Pair of Points in the Plane

Closest Pair of Points in the Plane

- **Given:** Array $P[0..n)$ of points in the plane ($\mathbb{R}^2$)
  each has x and y coordinates: $P[i].x$ and $P[i].y$
- **Goal:** Find pair $P[i]$, $P[j]$ that is closest in (Euclidean) distance
  i. e., i and j that minimize $\sqrt{(P[i].x - P[j].x)^2 + (P[i].y - P[j].y)^2}$
- Naive solution: compute distance of each pair $\rightsquigarrow$ $\Theta(n^2)$ time
  - cost here = # arithmetic operations
  - ignore numerical accuracy

  $\rightsquigarrow$ formally work on the ***real RAM***
  - like word-RAM, but words contain **exact** real numbers
  - support arithmetic operations and comparisons,
    but **not** bitwise operations or $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$

Closest Pair – Divide & Conquer

Closest Pair – Refined Conquer

Closest Pair – Code

```
1  procedure closestDist(P[0..N), byX[0..n), byY[0..n))
2      // P contains all N ≥ n points
3      // P[byX[0]].x ≤ P[byX[1]].x ≤ ··· ≤ P[byX[n]].x
4      // P[byY[0]].y ≤ P[byY[1]].y ≤ ··· ≤ P[byY[n]].y
5      if n == 2 return d_2(P[byX[0]], P[byX[1]])
6      if n == 3 return min{d_2(P[byX[0]], P[byX[1]]),
7                           d_2(P[byX[1]], P[byX[2]]),
8                           d_2(P[byX[0]], P[byX[2]])}
9      // 1. Split by median x and recurse
10     k := ⌊n/2⌋;
11     m := P[byX[k]]
12     byX_L := byX[0..k);  byX_R := byX[k..n)
13     byY_L, byY_R := new empty array list
14     for i := 0, ..., n − 1
15         if P[byY[i]] ≤ m // breaking ties as in byX
16             byY_L.append(byY[i])
17         else
18             byY_R.append(byY[i])
19         end if
20     end for
21     // ...
22     // ... closestDist continued
23     δ_L := closestDist(P, byX_L, byY_L)
24     δ_R := closestDist(P, byX_R, byY_R)
25     δ := min{δ_L, δ_R}
26     // 2. Check straddling pairs
27     // Find points close to dividing line
28     for i := 0, ..., n − 1
29         if |P[byY[i]].x − m.x| ≤ δ
30             C.append(byY[i])
31         end if
32     end for
33     // Distance ≤ δ implies within 8 positions in C
34     for i := 0, ..., C.size()
35         for j := i + 1, ..., i + 7
36             δ := min{δ, d_2(P[C[i]], P[C[j]])}
37         end for
38     end for
39     return δ
40
41 procedure d_2(P, Q)
42     return √((P.x − Q.x)^2 + (P.y − Q.y)^2)
```