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Text Indexing – Searching entire genomes

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Learning Outcomes

Unit 13: ***Text Indexing***

1. Know and understand methods for text indexing: *inverted indices*, *suffix trees*, *(enhanced) suffix arrays*
2. Know and understand *generalized suffix trees*
3. Know properties, in particular *performance characteristics*, and limitations of the above data structures.
4. Design (simple) *algorithms based on suffix trees*.
5. Understand *construction algorithms* for suffix arrays and LCP arrays.

Outline

13 Text Indexing

13.1 Motivation

13.2 Suffix Trees

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13.5 Suffix Arrays

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13.7 Linear-Time Suffix Sorting: The DC3 Algorithm

13.8 The LCP Array

13.9 LCP Array Construction

13.1 Motivation

Text indexing

- *Text indexing* (also: *offline text search*):
  - case of string matching: find $P[0..m)$ in $T[0..n)$
  - but with *fixed* text ⇝ preprocess T (instead of P)
  - ⇝ expect many queries P, answer them without looking at all of T
  - ⇝ essentially a data structuring problem: "building an *index* of T"

Latin: "one who points out"

- application areas
  - web search engines
  - online dictionaries
  - online encyclopedia
  - DNA/RNA data bases
  - ... searching in any collection of text documents (that grows only moderately)

Inverted indices

same as “indexes”

- original indices in books: list of (key) words $\mapsto$ page numbers where they occur
- assumption: searches are only for **whole** (key) **words**

⇝ often reasonable for natural language text

Inverted index:

- collect all words in T
  - can be as simple as splitting T at whitespace
  - actual implementations typically support *stemming* of words
    `goes` $\rightarrow$ `go`, `cats` $\rightarrow$ `cat`
- store mapping from words to a list of occurrences ⇝ *how?*

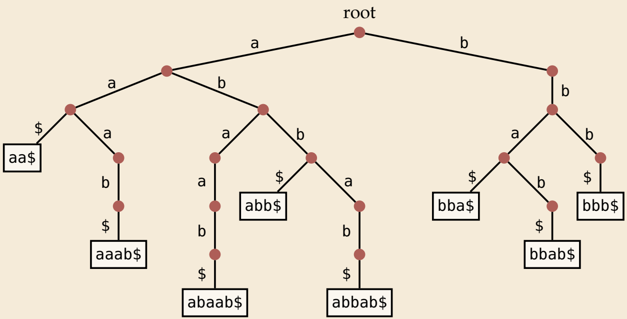
Tries

- efficient dictionary data structure for strings
- name from re**trie**val, but pronounced "try"
- tree based on symbol comparisons

- **Assumption:** stored strings are *prefix-free* (no string is a prefix of another)
  - strings of same length ✓
  - strings have "end-of-string" marker $ ✓

some character $\notin \Sigma$

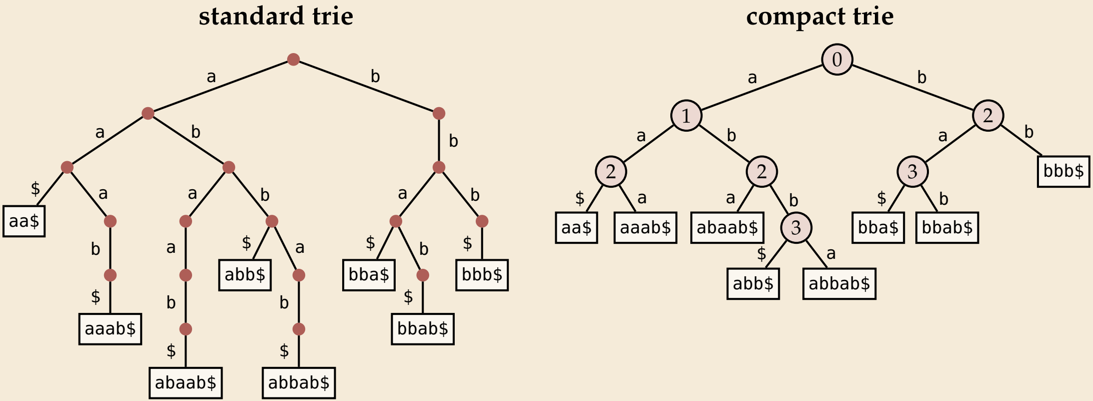
- **Example**:
  {aa$, aaab$, abaab$, abb$,
  abbab$, bba$, bbab$, bbb$}



Compact tries

=1 child

- compress paths of unary nodes into single edge
- nodes store *index* of next character to check



⇝ searching slightly trickier, but same time complexity as in trie

- all nodes ≥ 2 children ⇝ #nodes $\leq$ #leaves = #strings ⇝ linear space

Tries as inverted index

👍 simple

👍 fast lookup

👎 cannot handle more general queries:

- search part of a word
- search phrase (sequence of words)

👎 what if the ‘text’ does not even have words to begin with?!

- biological sequences

```
ACAAGATGCCATTGTCCCCCGGCCTCCTGCTGCTGCTGCTCTCCGGGGCCACGGCCACCGCTGCCCTGCCCCTGGAGGGTGGCCCCACCGGC
CGAGACAGCGAGCATATGCAGGAAGCGGCAGGAATAAGGAAAAGCAGCCTCCTGACTTTCCTCGCTTGGTGGTTTGAGTGGACCTCCCAGGC
CAGTGCCGGGCCCCTCATAGGAGAGGAAGCTCGGGAGGTGGCCAGGCGGCAGGAAGGCGCACCCCCCCAGCAATCCGCGCGCCGGGACAGAA
TGCCCTGCAGGAACTTCTTCTGGAAGACCTTCTCCTCCTGCAAATAAAACCTCACCCATGAATGCTCACGCAAGTTTAATTACAGACCTGAA
```

- binary streams

```
0000001010100111101011100000111110001111101111100110110100001110001001101111000001000110101O
0110110000110101101000000001000000001110101100000100001111010111011001000110010110111011111111
1100010100010110010100000011101010100110000000011011000011001111100001O1 01010111011110000011
1010111001001010101010000001111101001100000011110011010100000000100100100000101100011000110111
```

⇝ need new ideas

13.2 Suffix Trees

Suffix trees – A 'magic' data structure

Appetizer: Longest common substring problem

- Given: strings $S_1, \ldots, S_k$
- Goal: find the longest substring that occurs in all k strings

Example: S_1 = `superiorcalifornialives`, S_2 = `sealiver`

$\rightsquigarrow$ `alive`

Can we do this in time $O(|S_1| + \cdots + |S_k|)$? How??

Enter: ***suffix trees***

- versatile data structure for index with full-text search
- linear time (for construction) and linear space
- allows efficient solutions for many advanced string problems

"Although the longest common substring problem looks trivial now, given our knowledge of suffix trees, it is very interesting to note that in 1970 Don Knuth conjectured that a linear-time algorithm for this problem would be impossible." *[Gusfield: Algorithms on Strings, Trees, and Sequences (1997)]*

Suffix trees – Definition

- suffix tree $\mathcal{T}$ for text $T = T[0..n)$ = compact trie of all suffixes of $T\$$ (set $T[n] := \$$)
- except: in leaves, store *start index* (instead of copy of actual string)

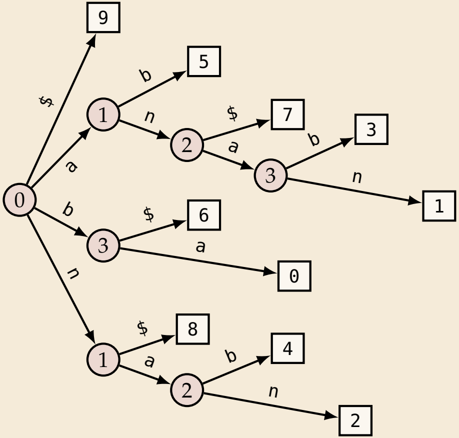
Example:

T = bananaban$

suffixes: {bananaban$, ananaban$, nanaban$, anaban$, naban$, aban$, ban$, an$, n$, $}

	0	1	2	3	4	5	6	7	8	9
T =	b	a	n	a	n	a	b	a	n	$

- also: edge labels like in compact trie
- (more readable form on slides to explain algorithms)



Suffix trees – Construction

- $T[0..n]$ has $n+1$ suffixes (starting at character $i \in [0..n]$)
- We can build the suffix tree by inserting each suffix of T into a compressed trie. But that takes time $\Theta(n^2)$. $\rightsquigarrow$ not interesting!

same order of growth as reading the text!

Amazing result: Can construct the suffix tree of T in $\Theta(n)$ time!

- algorithms are a bit tricky to understand
- but were a theoretical breakthrough
- and they are efficient in practice (and heavily used)!

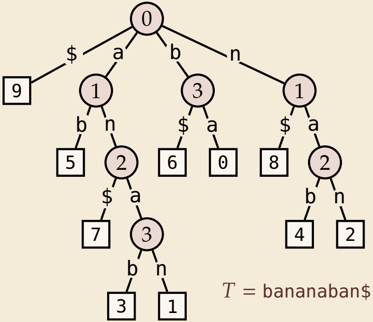
$\rightsquigarrow$ for now, take linear-time construction for granted. What can we do with them?

13.3 Applications

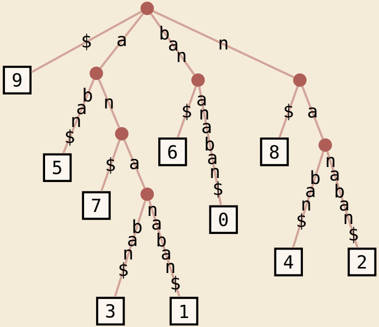
Applications of suffix trees

- In this section, always assume suffix tree $\mathcal{T}$ for T given.

Recall: $\mathcal{T}$ stored like this: but think about this:



T = bananaban$



- Moreover: assume internal nodes store pointer to *leftmost leaf in subtree*.
- Notation: $T_i = T[i..n]$ (including $)

Application 1: Text Indexing / String Matching

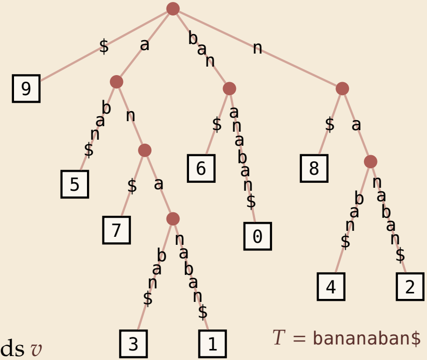
- $\boxed{P \text{ occurs in } T \iff P \text{ is a prefix of a suffix of } T}$
- we have all suffixes in $\mathcal{T}$!

$\rightsquigarrow$ (try to) follow path with label P, until

1. **we get stuck**
   at internal node (no node with next character of P)
   or *inside edge* (mismatch of next characters)
   $\rightsquigarrow$ P does not occur in T
2. **we run out of pattern**
   reach end of P at internal node v or inside edge towards v
   $\rightsquigarrow$ P occurs at all leaves in subtree of v
3. **we run out of tree**
   reach a leaf ℓ with part of P left $\rightsquigarrow$ compare P to ℓ.

   This cannot happen when testing edge labels since $\$ \notin \Sigma$,
   but needs check(s) in compact trie implementation!

- Finding first match (or `NO_MATCH`) takes $O(|P|)$ time!



T = bananaban$

Examples:

- P = ann
- P = baa
- P = ana
- P = ba
- P = briar

Application 2: Longest repeated substring

- **Goal:** Find longest substring $T[i..i+\ell)$ that occurs also at $j \neq i$: $T[j..j+\ell) = T[i..i+\ell)$.

e. g. for compression ⇝ Unit 7

How can we efficiently check *all possible substrings?*

Repeated substrings = shared paths in *suffix tree*

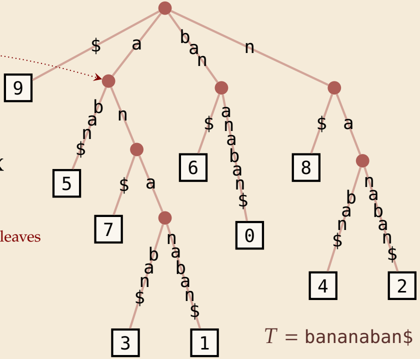
- T_5 = aban\$ and T_7 = an\$ have *longest common prefix* 'a'

⇝ ∃ internal node with path label 'a'

here single edge, can be longer path

⇝ longest repeated substring = longest common prefix (LCP) of two suffixes

actually: adjacent leaves



T = bananaban\$

- Algorithm:
  1. Compute *string depth* (=length of path label) of nodes
  2. Find internal nodes with maximal string depth

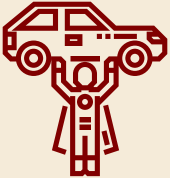
- Both can be done in depth-first traversal ⇝ $\Theta(n)$ time

Generalized suffix trees

- longest *repeated* substring (of one string) feels very similar to longest *common* substring of several strings $T^{(1)}, \ldots, T^{(k)}$ with $T^{(j)} \in \Sigma^{n_j}$
- can we solve that in the same way?
- could build the suffix tree for each $T^{(j)}$ ... but doesn't seem to help

$\rightsquigarrow$ need a *single/joint* suffix tree for *several* texts

Enter: ***generalized suffix tree***



- Define $T := T^{(1)}\$_1 T^{(2)}\$_2 \cdots T^{(k)}\$_k$ for k new end-of-word symbols
- Construct suffix tree $\mathcal{T}$ for T

$\rightsquigarrow$ $\$_j$-edges always leads to leaves $\rightsquigarrow$ $\exists$ leaf (j, i) for each suffix $T_i^{(j)} = T^{(j)}[i..n_j]$

Application 3: Longest common substring

- With that new idea, we can find longest common substrings:

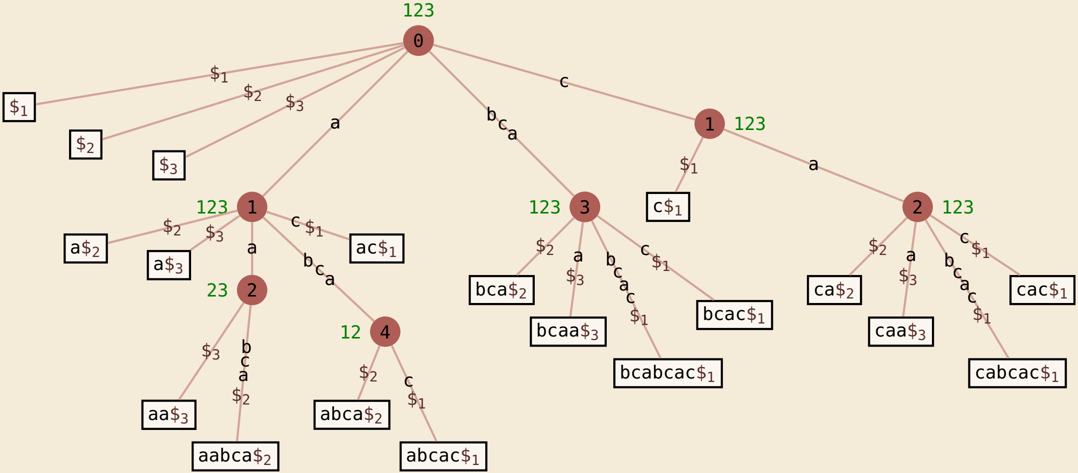
  1. Compute generalized suffix tree $\mathcal{T}$.
  2. Store with each node the *subset of strings* that contain its path label:
     - 2.1. Traverse $\mathcal{T}$ bottom-up.
     - 2.2. For a leaf (j, i), the subset is $\{j\}$.
     - 2.3. For an internal node, the subset is the union of its children.
  3. In top-down traversal, compute *string depths* of nodes. (as above)
  4. Report deepest node (by string depth) whose subset is $\{1, \ldots, k\}$.

- Each step takes time $\Theta(n)$ for $n = n_1 + \cdots + n_k$ the total length of all texts.

"Although the longest common substring problem looks trivial now, given our knowledge of suffix trees,
it is very interesting to note that in 1970 Don Knuth conjectured that
a linear-time algorithm for this problem would be impossible." *[Gusfield: Algorithms on Strings, Trees, and Sequences (1997)]*

Longest common substring – Example

$T^{(1)} = \texttt{bcabcac}, \quad T^{(2)} = \texttt{aabca}, \quad T^{(3)} = \texttt{bcaa}$



13.4 Longest Common Extensions

Application 4: Longest Common Extensions

- We implicitly used a special case of a more general, versatile idea:

  The ***longest common extension (LCE)*** data structure:

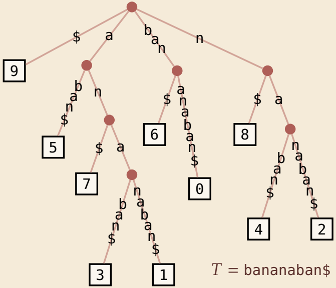
  - **Given:** String $T[0..n)$
  - **Goal:** Answer LCE queries, i. e.,
    given positions i, j in T,
    how far can we read the same text from there?
    formally: $\mathrm{LCE}(i,j) = \max\{\ell : T[i..i+\ell) = T[j..j+\ell)\}$

$\rightsquigarrow$ use suffix tree of T!

(length of) longest common prefix
of ith and jth suffix

- In $\mathcal{T}$: $\mathrm{LCE}(i,j) = \mathrm{LCP}(T_i, T_j)$ $\rightsquigarrow$ same thing, different name!
  = string depth of
  lowest common ancester (LCA) of
  leaves $\boxed{i}$ and $\boxed{j}$

- in short: $\boxed{\mathrm{LCE}(i,j) = \mathrm{LCP}(T_i,T_j) = \mathrm{stringDepth}\big(\mathrm{LCA}(\boxed{i},\boxed{j})\big)}$



T = bananaban$

Efficient LCA

How to find lowest common ancestors?

- Could walk up the tree to find LCA $\rightsquigarrow$ $\Theta(n)$ worst case 👎
- Could store all LCAs in big table $\rightsquigarrow$ $\Theta(n^2)$ space and preprocessing 👎

Amazing result: Can compute data structure in $\Theta(n)$ time and space that finds any LCA is **constant(!) time**.

- a bit tricky to understand
- but a theoretical breakthrough
- and useful in practice

and suffix tree construction inside . . .

$\rightsquigarrow$ for now, use $O(1)$ LCA as black box.

$\rightsquigarrow$ After linear preprocessing (time & space), we can find LCEs in $O(1)$ time.

Application 5: Approximate matching

***k*-mismatch matching**:

- **Input:** text $T[0..n)$, pattern $P[0..m)$, $k \in [0..m)$
- **Output:**

  “Hamming distance $\le k$”

  - smallest i so that $T[i..i+m)$ are P differ in at most k characters
  - or `NO_MATCH` if there is no such i

⇝ searching with typos

- Adapted brute-force algorithm ⇝ $O(n \cdot m)$

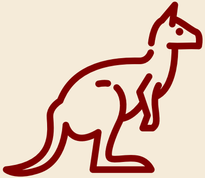
- Assume longest common extensions in $T\$_1P\$_2$ can be found in $O(1)$

  ⇝ generalized suffix tree $\mathcal{T}$ has been built

  ⇝ string depths of all internal nodes have been computed

  ⇝ constant-time LCA data structure for $\mathcal{T}$ has been built

Kangaroo Algorithm for approximate matching



```
**procedure** kMismatch($T[0..n-1]$, $P[0..m-1]$):
    // *build LCE data structure*
    **for** $i := 0, \ldots, n-m-1$ **do**
        $\mathrm{mismatches} := 0;\ \ t := i;\ \ p := 0$
        **while** $\mathrm{mismatches} \le k \land p < m$ **do**
            $\ell := \mathrm{LCE}(t, p)$ // *jump over matching part*
            $t := t + \ell + 1;\ \ p := p + \ell + 1$
            $\mathrm{mismatches} := \mathrm{mismatches} + 1$
        **if** $p == m$ **then**
            **return** $i$
```

- **Analysis:** $\Theta(n+m)$ preprocessing $+$ $O(n \cdot k)$ matching

⇝ very efficient for small k

- State of the art
  - $O\left(n\frac{k^2 \log k}{m}\right)$ possible with complicated algorithms
  - extensions for edit distance $\le k$ possible

Application 6: Matching with wildcards

- Allow a wildcard character in pattern

  stands for arbitrary (single) character

```
    unit*         P
in␣unit5␣we␣will  T
```

- similar algorithm as for k-mismatch ⇝ $O(n \cdot k + m)$ when P has k wildcards

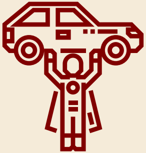
* * *

Many more applications, in particular for problems on biological sequences

20+ described in Gusfield, *Algorithms on strings, trees, and sequences* (1999)

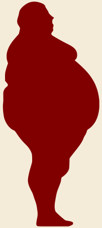
Suffix trees – Discussion

- Suffix trees were a threshold invention



👍 linear time and space

👍 suddenly many questions efficiently solvable in theory



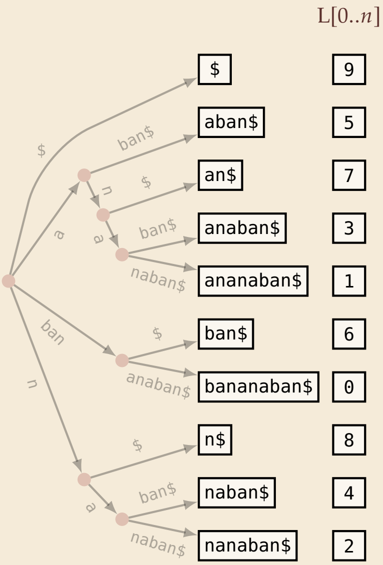
👎 construction of suffix trees:
linear time, but significant overhead

👎 construction methods fairly complicated

👎 many pointers in tree incur large space overhead

13.5 Suffix Arrays

Putting suffix trees on a diet



	L[0..n]
$	9
aban$	5
an$	7
anaban$	3
ananaban$	1
ban$	6
bananaban$	0
n$	8
naban$	4
nanaban$	2

- **Observation:** order of leaves in suffix tree
  = suffixes lexicographically *sorted*

- Idea: only store list of leaves $L[0..n]$

- Enough to do efficient string matching!
  1. Use binary search for pattern P
  2. check if P is prefix of suffix after position found

- **Example:** P = ana

⇝ $L[0..n]$ is called ***suffix array***:

$L[r]$ = (start index of) rth suffix in sorted order

- using L, can do string matching with
  $\le (\lg n + 2) \cdot m$ character comparisons

Suffix arrays – Construction

How to compute $L[0..n]$?

- from suffix tree
  - possible with traversal . . .
  - 👎 but we are trying to avoid constructing suffix trees!

- sorting the suffixes of T using general purpose sorting method
  - 👍 trivial to code!
  - but: comparing two suffixes can take $\Theta(n)$ character comparisons
  - 👎 $\Theta(n^2 \log n)$ time in worst case

- We can do better!

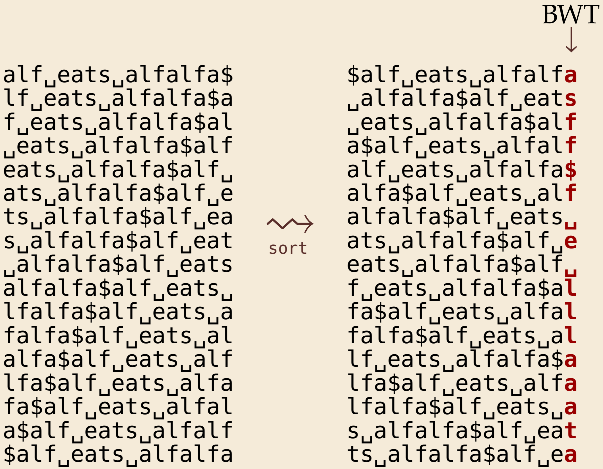
Digression: Recall BWT

Burrows-Wheeler Transform

1. Take all cyclic shifts of S
2. Sort cyclic shifts
3. Extract last column

S = `alf␣eats␣alfalfa$`

B = `asff$f␣e␣lllaaata`



```
                                          BWT
                                           ↓
alf␣eats␣alfalfa$          $alf␣eats␣alfalfa
lf␣eats␣alfalfa$a          ␣alfalfa$alf␣eats
f␣eats␣alfalfa$al          ␣eats␣alfalfa$alf
␣eats␣alfalfa$alf          a$alf␣eats␣alfalf
eats␣alfalfa$alf␣          alf␣eats␣alfalfa$
ats␣alfalfa$alf␣e          alfa$alf␣eats␣alf
ts␣alfalfa$alf␣ea   ⇝      alfalfa$alf␣eats␣
s␣alfalfa$alf␣eat  sort    ats␣alfalfa$alf␣e
␣alfalfa$alf␣eats          eats␣alfalfa$alf␣
alfalfa$alf␣eats␣          f␣eats␣alfalfa$al
lfalfa$alf␣eats␣a          fa$alf␣eats␣alfal
falfa$alf␣eats␣al          falfa$alf␣eats␣al
alfa$alf␣eats␣alf          lf␣eats␣alfalfa$a
lfa$alf␣eats␣alfa          lfa$alf␣eats␣alfa
fa$alf␣eats␣alfal          lfalfa$alf␣eats␣a
a$alf␣eats␣alfalf          s␣alfalfa$alf␣eat
$alf␣eats␣alfalfa          ts␣alfalfa$alf␣ea
```

Digression: Computing the BWT

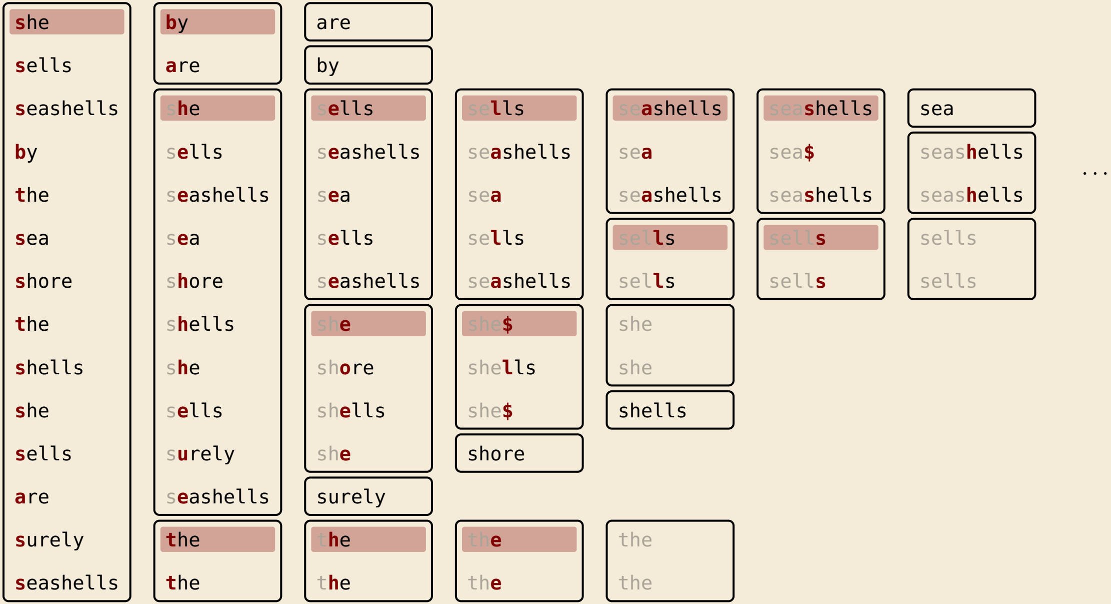
How can we compute the BWT of a text efficiently?

- cyclic shifts S $\mathrel{\widehat{=}}$ suffixes of S
  - comparing cyclic shifts stops at first `$`
  - for comparisons, anything after `$` irrelevant!
- BWT is essentially suffix sorting!
  - $B[i] = S[L[i] - 1]$
  - where $L[i] = 0$, $B[i] = \$$

⇝ Can compute B in $O(n)$ time from L

	r	↓	$L[r]$
alf␣eats␣alfalfa$	0	$alf␣eats␣alfalf**a**	16
lf␣eats␣alfalfa$a	1	␣alfalfa$alf␣eat**s**	8
f␣eats␣alfalfa$al	2	␣eats␣alfalfa$al**f**	3
␣eats␣alfalfa$alf	3	a$alf␣eats␣alfal**f**	15
eats␣alfalfa$alf␣	4	alf␣eats␣alfalfa**$**	0
ats␣alfalfa$alf␣e	5	alfa$alf␣eats␣al**f**	12
ts␣alfalfa$alf␣ea	6	alfalfa$alf␣eats**␣**	9
s␣alfalfa$alf␣eat	7	ats␣alfalfa$alf␣**e**	5
␣alfalfa$alf␣eats	8	eats␣alfalfa$alf**␣**	4
alfalfa$alf␣eats␣	9	f␣eats␣alfalfa$a**l**	2
lfalfa$alf␣eats␣a	10	fa$alf␣eats␣alfa**l**	14
falfa$alf␣eats␣al	11	falfa$alf␣eats␣a**l**	11
alfa$alf␣eats␣alf	12	lf␣eats␣alfalfa$**a**	1
lfa$alf␣eats␣alfa	13	lfa$alf␣eats␣alf**a**	13
fa$alf␣eats␣alfal	14	lfalfa$alf␣eats␣**a**	10
a$alf␣eats␣alfalf	15	s␣alfalfa$alf␣ea**t**	7
$alf␣eats␣alfalfa	16	ts␣alfalfa$alf␣e**a**	6

Fat-pivot radix quicksort – Example



she
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Fat-pivot radix quicksort

details in §5.1 of Sedgewick, Wayne *Algorithms 4th ed.* (2011), Pearson

- **partition** based on ***d*th character** only (initially $d = 0$)

⇝ 3 segments: smaller, equal, or larger than dth symbol of pivot

- recurse on smaller and large with same d, on equal with $d + 1$
  - ⇝ never compare equal prefixes twice

for random strings

⇝ can show: $\sim 2\ln(2) \cdot n \lg n \approx 1.39 n \lg n$ character comparisons on average

👍 simple to code

👍 efficient for sorting many lists of strings

random string

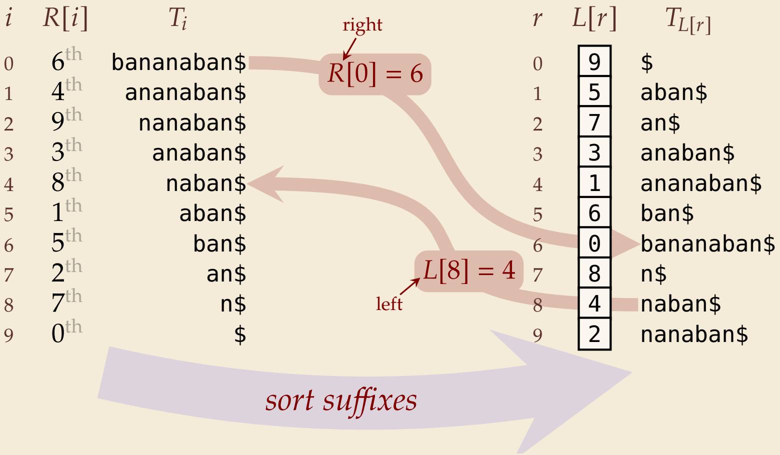
- fat-pivot radix quicksort finds suffix array in $O(n \log n)$ expected time

*but we can do $O(n)$ time **worst case**!*

13.6 Linear-Time Suffix Sorting: Overview

Inverse suffix array: going left & right

- to understand the fastest algorithm, it is helpful to define the *inverse suffix array*:
  - $R[i] = r \iff L[r] = i$ L = *leaf array*
    $\iff$ there are r suffixes that come before T_i in sorted order
    $\iff$ T_i has (0-based) *rank* r ⇝ call $R[0..n]$ the ***rank array***



i	$R[i]$	T_i
0	6th	`bananaban$`
1	4th	`ananaban$`
2	9th	`nanaban$`
3	3th	`anaban$`
4	8th	`naban$`
5	1th	`aban$`
6	5th	`ban$`
7	2th	`an$`
8	7th	`n$`
9	0th	`$`

r	$L[r]$	$T_{L[r]}$
0	9	`$`
1	5	`aban$`
2	7	`an$`
3	3	`anaban$`
4	1	`ananaban$`
5	6	`ban$`
6	0	`bananaban$`
7	8	`n$`
8	4	`naban$`
9	2	`nanaban$`

Linear-time suffix sorting

DC3 / Skew algorithm

not a multiple of 3

1. Compute rank array $R_{1,2}$ for suffixes T_i starting at $i \not\equiv \overset{\swarrow}{0} \pmod 3$ *recursively.*
2. Induce rank array R_3 for suffixes $T_0, T_3, T_6, T_9, \ldots$ from $R_{1,2}$.
3. Merge $R_{1,2}$ and R_0 using $R_{1,2}$.
   $\rightsquigarrow$ rank array R for entire input

- We will show that steps 2. and 3. take $\Theta(n)$ time

$\rightsquigarrow$ Total complexity is $n + \frac{2}{3}n + \left(\frac{2}{3}\right)^2 n + \left(\frac{2}{3}\right)^3 n + \cdots \;\le\; n \cdot \sum_{i \ge 0} \left(\frac{2}{3}\right)^i \;=\; 3n \;=\; \Theta(n)$

- **Note:** L can easily be computed from R in one pass, and vice versa.
  $\rightsquigarrow$ Can use whichever is more convenient.

DC3 / Skew algorithm – Step 2: Inducing ranks

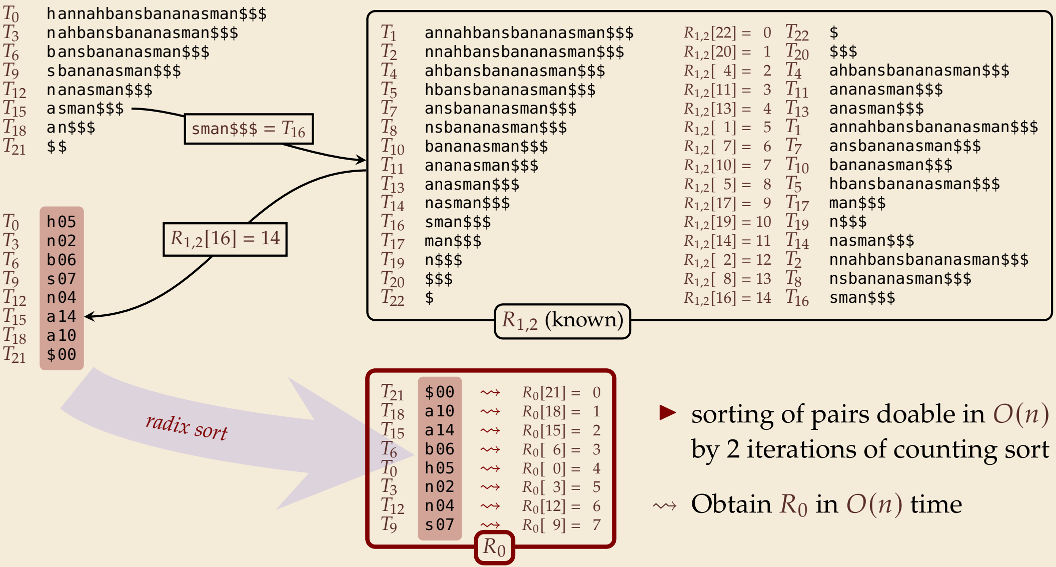
- **Assume:** rank array $R_{1,2}$ known:
  - $$R_{1,2}[i] = \begin{cases} \text{rank of } T_i \text{ among } T_1, T_2, T_4, T_5, T_7, T_8, \ldots & \text{for } i = 1,2,4,5,7,8,\ldots \\ \text{undefined} & \text{for } i = 0,3,6,9,\ldots \end{cases}$$
- **Task:** sort the suffixes $T_0, T_3, T_6, T_9, \ldots$ in linear time (!)

- Suppose we want to compare T_0 and T_3.
  - Characterwise comparisons too expensive
  - but: after removing first character, we obtain T_1 and T_4
  - these two can be compared in *constant time* by comparing $R_{1,2}[1]$ and $R_{1,2}[4]$!

$\rightsquigarrow$ T_0 comes before T_3 in lexicographic order
iff pair $(T[0], R_{1,2}[1])$ comes before pair $(T[3], R_{1,2}[4])$ in lexicographic order

DC3 / Skew algorithm – Inducing ranks example

T = `hannahbansbananasman$$$` (append 3 $ markers)



T_0	`hannahbansbananasman$$$`
T_3	`nahbansbananasman$$$`
T_6	`bansbananasman$$$`
T_9	`sbananasman$$$`
T_{12}	`nanasman$$$`
T_{15}	`asman$$$`
T_{18}	`an$$$`
T_{21}	`$$`

`sman$$$` = T_{16}

T_1	`annahbansbananasman$$$`	$R_{1,2}[22] = 0$	T_{22}	`$`
T_2	`nnahbansbananasman$$$`	$R_{1,2}[20] = 1$	T_{20}	`$$$`
T_4	`ahbansbananasman$$$`	$R_{1,2}[4] = 2$	T_4	`ahbansbananasman$$$`
T_5	`hbansbananasman$$$`	$R_{1,2}[11] = 3$	T_{11}	`ananasman$$$`
T_7	`ansbananasman$$$`	$R_{1,2}[13] = 4$	T_{13}	`anasman$$$`
T_8	`nsbananasman$$$`	$R_{1,2}[1] = 5$	T_1	`annahbansbananasman$$$`
T_{10}	`bananasman$$$`	$R_{1,2}[7] = 6$	T_7	`ansbananasman$$$`
T_{11}	`ananasman$$$`	$R_{1,2}[10] = 7$	T_{10}	`bananasman$$$`
T_{13}	`anasman$$$`	$R_{1,2}[5] = 8$	T_5	`hbansbananasman$$$`
T_{14}	`nasman$$$`	$R_{1,2}[17] = 9$	T_{17}	`man$$$`
T_{16}	`sman$$$`	$R_{1,2}[19] = 10$	T_{19}	`n$$$`
T_{17}	`man$$$`	$R_{1,2}[14] = 11$	T_{14}	`nasman$$$`
T_{19}	`n$$$`	$R_{1,2}[2] = 12$	T_2	`nnahbansbananasman$$$`
T_{20}	`$$$`	$R_{1,2}[8] = 13$	T_8	`nsbananasman$$$`
T_{22}	`$`	$R_{1,2}[16] = 14$	T_{16}	`sman$$$`

$R_{1,2}$ (known)

$R_{1,2}[16] = 14$

T_0	`h05`
T_3	`n02`
T_6	`b06`
T_9	`s07`
T_{12}	`n04`
T_{15}	`a14`
T_{18}	`a10`
T_{21}	`$00`

radix sort

T_{21}	`$00`	⇝	$R_0[21] = 0$
T_{18}	`a10`	⇝	$R_0[18] = 1$
T_{15}	`a14`	⇝	$R_0[15] = 2$
T_6	`b06`	⇝	$R_0[6] = 3$
T_0	`h05`	⇝	$R_0[0] = 4$
T_3	`n02`	⇝	$R_0[3] = 5$
T_{12}	`n04`	⇝	$R_0[12] = 6$
T_9	`s07`	⇝	$R_0[9] = 7$

R_0

- sorting of pairs doable in $O(n)$ time by 2 iterations of counting sort

⇝ Obtain R_0 in $O(n)$ time

DC3 / Skew algorithm – Step 3: Merging

T_{21}	`$$`
T_{18}	`an$$$`
T_{15}	`asman$$$`
T_{6}	`bansbananasman$$$`
T_{0}	`hannahbansbananasman$$$`
T_{3}	`nahbansbananasman$$$`
T_{12}	`nanasman$$$`
T_{9}	`sbananasman$$$`

T_{22}	`$`
T_{20}	`$$$`
T_{4}	`ahbansbananasman$$$`
T_{11}	`ananasman$$$`
T_{13}	`anasman$$$`
T_{1}	`annahbansbananasman$$$`
T_{7}	`ansbananasman$$$`
T_{10}	`bananasman$$$`
T_{5}	`hbansbananasman$$$`
T_{17}	`man$$$`
T_{19}	`n$$$`
T_{14}	`nasman$$$`
T_{2}	`nnahbansbananasman$$$`
T_{8}	`nsbananasman$$$`
T_{16}	`sman$$$`

T_{22}	`$`
T_{21}	`$$`
T_{20}	`$$$`
T_{4}	`ahbansbananasman$$$`
T_{18}	`an$$$`

- Have:
  - sorted 1,2-list:
    $T_1, T_2, T_4, T_5, T_7, T_8, T_{10}, T_{11}, \ldots$
  - sorted 0-list:
    $T_0, T_3, T_6, T_9, \ldots$
- Task: Merge them!
  - use standard merging method from Mergesort
  - but speed up comparisons using $R_{1,2}$
  - ⇝ $O(n)$ time for merge

Compare T_{15} to T_{11}

Idea: try same trick as before

$$\begin{aligned} T_{15} &= \texttt{asman\$\$\$} \\ &= \texttt{asman\$\$\$} \\ &= \texttt{a}T_{16} \end{aligned}$$

can't compare T_{16} and T_{12} either!

$$\begin{aligned} T_{11} &= \texttt{ananasman\$\$\$} \\ &= \texttt{ananasman\$\$\$} \\ &= \texttt{a}T_{12} \end{aligned}$$

⇝ Compare T_{16} to T_{12}

$$\begin{aligned} T_{16} &= \texttt{sman\$\$\$} \\ &= \texttt{sman\$\$\$} \\ &= \texttt{s}T_{17} \end{aligned}$$

always at most 2 steps then can use $R_{1,2}$!

$$\begin{aligned} T_{12} &= \texttt{nanasman\$\$\$} \\ &= \texttt{aanasman\$\$\$} \\ &= \texttt{a}T_{13} \end{aligned}$$

13.7 Linear-Time Suffix Sorting: The DC3 Algorithm

DC3 / Skew algorithm – Fix recursive call

- both step 2. and 3. doable in $O(n)$ time!

- But: we cheated in 1. step! *"compute rank array $R_{1,2}$ recursively"*
  - Taking a *subset* of suffixes is *not* an instance of the same problem!

$\rightsquigarrow$ Need a single *string* T' to recurse on, from which we can deduce $R_{1,2}$.

How can we make T' "skip" some suffixes?

redefine alphabet to be *triples of characters* $\boxed{\texttt{abc}}$

$T = \texttt{bananaban\$\$\$}$

$\rightsquigarrow \quad T^{\square} = \boxed{\texttt{ban}}\boxed{\texttt{ana}}\boxed{\texttt{ban}}\boxed{\texttt{\$\$\$}}$

$\boxed{\texttt{ana}}\boxed{\texttt{ban}}\boxed{\texttt{\$\$\$}}$

$\boxed{\texttt{ban}}\boxed{\texttt{\$\$\$}}$

$\boxed{\texttt{\$\$\$}}$

$\rightsquigarrow$ suffixes of $T^{\square}$ $\leftrightsquigarrow$ $T_0, T_3, T_6, T_9, \ldots$

- $T' = T[1..n)^{\square}\,\boxed{\texttt{\$\$\$}}\,T[2..n)^{\square}\,\boxed{\texttt{\$\$\$}} \quad \leftrightsquigarrow \quad T_i$ with $i \not\equiv 0 \pmod 3$.

$\rightsquigarrow$ Can call suffix sorting recursively on T' and map result to $R_{1,2}$

DC3 / Skew algorithm – Fix alphabet explosion

- Still does not quite work!
  - Each recursive step *cubes* σ by using triples!
  - $\rightsquigarrow$ (Eventually) cannot use linear-time sorting anymore!

- But: Have at most $\frac{2}{3}n$ different triples $\boxed{\texttt{abc}}$ in T'!

$\rightsquigarrow$ Before recursion:

1. Sort all occurring triples. (using counting sort in $O(n)$)
2. Replace them by their *rank* (in Σ).

$\rightsquigarrow$ Maintains $\sigma \le n$ without affecting order of suffixes.

DC3 / Skew algorithm – Step 3. Example

$$T' \;=\; T[1..n)^{\square}\,\boxed{\texttt{\$\$\$}}\,T[2..n)^{\square}\,\boxed{\texttt{\$\$\$}}$$

- T = `hannahbansbananasman$` T_2 = `nnahbansbananasman$`

  T' = $\boxed{\texttt{ann}}\boxed{\texttt{ahb}}\boxed{\texttt{ans}}\boxed{\texttt{ban}}\boxed{\texttt{ana}}\boxed{\texttt{sma}}\boxed{\texttt{n\$\$}}\ \boxed{\texttt{\$\$\$}}\ \boxed{\texttt{nna}}\boxed{\texttt{hba}}\boxed{\texttt{nsb}}\boxed{\texttt{ana}}\boxed{\texttt{nas}}\boxed{\texttt{man}}\boxed{\texttt{\$\$\$}}$

- Occurring triples:

  $\boxed{\texttt{ann}}\boxed{\texttt{ahb}}\boxed{\texttt{ans}}\boxed{\texttt{ban}}\boxed{\texttt{ana}}\boxed{\texttt{sma}}\boxed{\texttt{n\$\$}}\ \boxed{\texttt{\$\$\$}}\ \boxed{\texttt{nna}}\boxed{\texttt{hba}}\boxed{\texttt{nsb}}\qquad\boxed{\texttt{nas}}\boxed{\texttt{man}}$

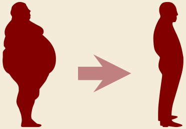
- Sorted triples with ranks:

Rank	00	01	02	03	04	05	06	07	08	09	10	11	12
Triple	$$$	ahb	ana	ann	ans	ban	hba	man	n$$	nas	nna	nsb	sma

- T' = $\boxed{\texttt{ann}}\boxed{\texttt{ahb}}\boxed{\texttt{ans}}\boxed{\texttt{ban}}\boxed{\texttt{ana}}\boxed{\texttt{sma}}\boxed{\texttt{n\$\$}}\ \boxed{\texttt{\$\$\$}}\ \boxed{\texttt{nna}}\boxed{\texttt{hba}}\boxed{\texttt{nsb}}\boxed{\texttt{ana}}\boxed{\texttt{nas}}\boxed{\texttt{man}}\boxed{\texttt{\$\$\$}}$

  T'' = $\boxed{\texttt{03}}\ \boxed{\texttt{01}}\ \boxed{\texttt{04}}\ \boxed{\texttt{05}}\ \boxed{\texttt{02}}\ \boxed{\texttt{12}}\ \boxed{\texttt{08}}\quad\boxed{\texttt{00}}\quad\boxed{\texttt{10}}\ \boxed{\texttt{06}}\ \boxed{\texttt{11}}\ \boxed{\texttt{02}}\ \boxed{\texttt{09}}\ \boxed{\texttt{07}}\ \boxed{\texttt{00}}$

Suffix array – Discussion



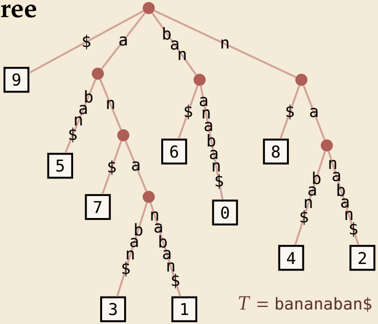
- 👍 sleek data structure compared to suffix tree
- 👍 simple and fast $O(n \log n)$ construction
- 👍 more involved but optimal $O(n)$ construction
- 👍 supports efficient string matching

- 👎 string matching takes $O(m \log n)$, not optimal $O(m)$
- 👎 Cannot use more advanced suffix tree features
  e. g., for longest repeated substrings

13.8 The LCP Array

String depths of internal nodes

- Recall algorithm for longest repeated substring in **suffix tree**
  1. Compute *string depth* of nodes
  2. Find *path label to node* with maximal string depth



T = bananaban$

- Can we do this using **suffix *arrays***?

- Yes, by **enhancing** the suffix array with the ***LCP array***!

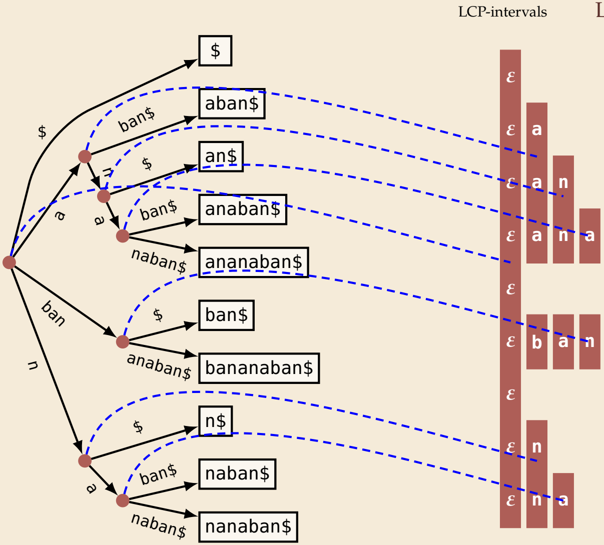
  $\mathrm{LCP}[1..n]$

  $\mathrm{LCP}[r] \;=\; \mathrm{LCP}(T_{L[r]}, T_{L[r-1]})$

length of longest common prefix of suffixes of rank r and $r-1$

$\rightsquigarrow$ longest repeated substring $=$ find maximum in $\mathrm{LCP}[1..n]$

LCP array and internal nodes



LCP[1..n]		L[0..n]
	$	9
0		
	aban$	5
1	a	
	an$	7
2	an	
	anaban$	3
3	ana	
	ananaban$	1
0		
	ban$	6
3	ban	
	bananaban$	0
0		
	n$	8
1	n	
	naban$	4
2	na	
	nanaban$	2

⇝ Leaf array $L[0..n]$ plus LCP array LCP$[1..n]$ encode full tree!

13.9 LCP Array Construction

LCP array construction

- computing LCP$[1..n]$ naively too expensive
  - each value could take $\Theta(n)$ time
  - 👎 $\Theta(n^2)$ in total

- but: seeing one large (= costly) LCP value ⇝ can find another large one!

- Example: T = `Buffalo␣buffalo␣buffalo␣buffalo$`
  - first few suffixes in sorted order:

$T_{L[0]}$ = `$`
$T_{L[1]}$ = `alo␣buffalo$`
$T_{L[2]}$ = `alo␣buffalo␣buffalo$`
`alo␣buffalo␣buffalo` ⇝ LCP[3] = **19**
$T_{L[3]}$ = `alo␣buffalo␣buffalo␣buffalo$`

⇝ **Removing first character** from $T_{L[2]}$ and $T_{L[3]}$ gives two new suffixes:

$T_{L[?]}$ = `lo␣buffalo␣buffalo$`
`lo␣buffalo␣buffalo` ⇝ LCP[**?**] = **18**
$T_{L[?]}$ = `lo␣buffalo␣buffalo␣buffalo$`
↑ unclear where...

⚠ Shortened suffixes might *not* be *adjacent* in sorted order!
⇝ no LCP entry for them!

Kasai’s algorithm – Example

- Kasai et al. used above observation systematically
- Key idea: *compute* LCP values in ***text order***
- Dropping first character of adjacent suffixes might not lead to *adjacent* shorter suffixes, but LCP entry can only be *longer*.

i	$R[i]$	T_i
0	6th	bananaban$
1	4th	ananaban$
2	9th	nanaban$
3	3th	anaban$
4	8th	naban$
5	1th	aban$
6	5th	ban$
7	2th	an$
8	7th	n$
9	0th	$

r	$L[r]$	$T_{L[r]}$	LCP$[r]$
0	9	$	–
1	5	aban$	0
2	7	an$	1
3	3	anaban$	2
4	1	ananaban$	3
5	6	ban$	0
6	0	bananaban$	3
7	8	n$	0
8	4	naban$	1
9	2	nanaban$	2

Kasai's algorithm – Code

```
procedure computeLCP($T[0..n]$, $L[0..n]$, $R[0..n]$):
    // Assume $T[n] = \$$, $L$ and $R$ are suffix array and inverse
    $\ell := 0$
    for $i := 0, \ldots, n-1$ // Consider $T_i$ now
        $r := R[i]$
        // compute $\mathrm{LCP}[r]$; note that $r > 0$ since $R[n] = 0$
        $i_{-1} := L[r-1]$
        while $T[i+\ell] == T[i_{-1}+\ell]$ do
            $\ell := \ell + 1$
        $\mathrm{LCP}[r] := \ell$
        $\ell := \max\{\ell - 1, 0\}$
    return $\mathrm{LCP}[1..n]$
```

- remember length ℓ of induced common prefix
- use L to get start index of suffixes

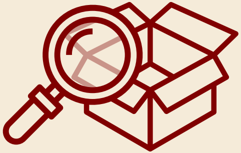
Analysis:

- dominant operation: character comparisons
- separately count those with outcomes "=" resp. "≠"
- each ≠ ends iteration of for-loop
  $\rightsquigarrow \le n$ cmps
- each = implies increment of ℓ, but $\ell \le n$ and decremented $\le n$ times
  $\rightsquigarrow \le 2n$ cmps

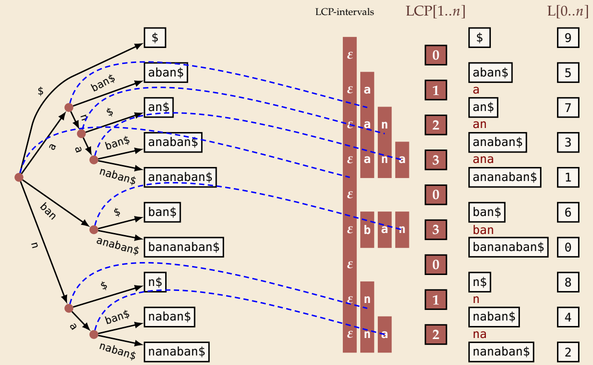
$\rightsquigarrow$ $\Theta(n)$ overall time

Back to suffix trees

We can finally look into the black box of linear-time suffix-array construction!



1. Compute suffix array for T.
2. Compute LCP array for T.
3. Construct $\mathcal{T}$ from suffix array and LCP array.



Conclusion

- *(Enhanced) Suffix Arrays* are the modern version of suffix trees

👎 can be harder to reason about

👍 can support same algorithms as suffix trees

👍 but use much less space

👍 simpler linear-time construction