7 Text Compression

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Learning Outcomes

Unit 7: ***Text Compression***

1. Understand the necessity for encodings and know *ASCII* and *UTF-8 character encodings*.
2. Understand (qualitatively) the *limits of compressibility*.
3. Know and understand the algorithms (encoding and decoding) for *Huffman codes*, *RLE*, *Elias codes*, *LZW*, *MTF*, and *BWT*, including their *properties* like running time complexity.
4. Select and *adapt* (slightly) a *compression* pipeline for a specific type of data.

Outline

7 Text Compression

7.1 Context

Overview

- Unit 6 & 13: How to *work* with strings
  - finding substrings
  - finding approximate matches ⇝ Unit ~~1~~3 3
  - finding repeated parts ⇝ Unit ~~1~~3 3
  - ...
  - assumed character array (random access)!

- Unit 7 & 8: How to *store/transmit* strings
  - computer memory: must be binary
  - how to compress strings (save space)
  - how to robustly transmit over noisy channels ⇝ Unit 8

Clicker Question

What compression methods do you know?

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Terminology

- **source text:** string $S \in \Sigma_S^\star$ to be stored / transmitted
  Σ_S is some alphabet
- **coded text:** encoded data $C \in \Sigma_C^\star$ that is actually stored / transmitted
  usually use $\Sigma_C = \{0, 1\}$
- **encoding:** algorithm mapping source texts to coded texts $S \rightsquigarrow C$
- **decoding:** algorithm mapping coded texts back to original source text $C \rightsquigarrow S$

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- **Lossy vs. Lossless**
  - **lossy compression** can only decode **approximately**;
    the exact source text S is lost
  - **lossless compression** always decodes S exactly

- For media files, lossy, logical compression is useful (e. g. JPEG, MPEG)

- We will concentrate on *lossless* compression algorithms.
  These techniques can be used for any application.

What is a good encoding scheme?

- Depending on the application, goals can be
  - efficiency of encoding/decoding
  - resilience to errors/noise in transmission
  - security (encryption)
  - integrity (detect modifications made by third parties)
  - size

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  - integrity (detect modifications made by third parties)
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- Focus in this unit: **size** of coded text

  Encoding schemes that (try to) minimize the size of coded texts perform *data compression*.

- We will measure the *compression ratio*: $\dfrac{|C| \cdot \lg|\Sigma_C|}{|S| \cdot \lg|\Sigma_S|} \overset{\Sigma_C=\{0,1\}}{=} \dfrac{|C|}{|S| \cdot \lg|\Sigma_S|}$

  < 1 means successful compression

  $= 1$ means no compression

  > 1 means "compression" made it bigger!? (yes, that happens . . .)

Clicker Question

Do you know what uncomputable/undecidable problems (halting problem, Post's correspondence problem, . . .) are?

A Sure, I could explain what it is.

B Heard that in a lecture, but don't quite remember

C No, never heard of it

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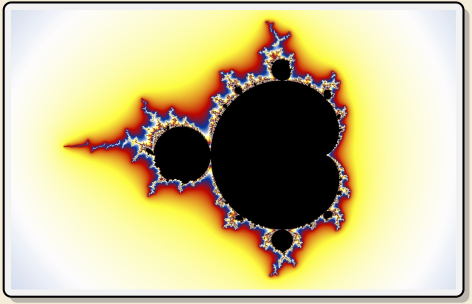
Limits of algorithmic compression

Is this image compressible?

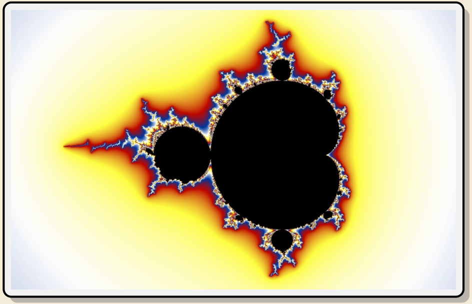


Limits of algorithmic compression

Is this image compressible?

visualization of Mandelbrot set

- Clearly a complex shape!
- Will not compress (too) well using, say, PNG.
- but:
  - completely defined by mathematical formula
  - ⇝ **can be generated by a very small program!**



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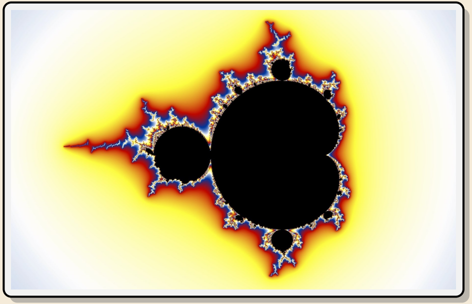

$\rightsquigarrow$ ***Kolmogorov complexity***

- C = *any program* that outputs S

  self-extracting archives!

  needs fixed machine model, but compilers transfer results
- Kolmogorov complexity = length of smallest such program

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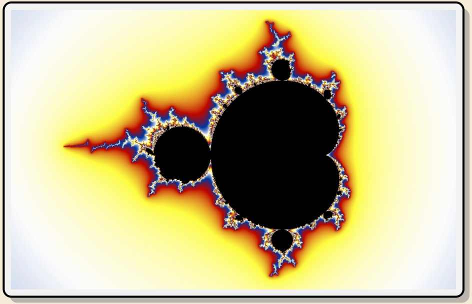

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⇝ ***Kolmogorov complexity***

- C = *any program* that outputs S

  self-extracting archives!

  needs fixed machine model, but compilers transfer results
- Kolmogorov complexity = length of smallest such program
- **Problem:** finding smallest such program is *uncomputable*.
- ⇝ No optimal encoding algorithm is possible!
- ⇝ must be inventive to get efficient methods

What makes data compressible?

- Lossless compression methods mainly exploit two types of redundancies in source texts:

  1. **uneven character frequencies**
     some characters occur more often than others $\rightarrow$ Part I
  2. **repetitive texts**
     different parts in the text are (almost) identical $\rightarrow$ Part II

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  1. **uneven character frequencies**
     some characters occur more often than others → Part I
  2. **repetitive texts**
     different parts in the text are (almost) identical → Part II

There is no such thing as a free lunch!

Not *everything* is compressible (→ tutorials)

⇝ focus on versatile methods that often work

Part I

Exploiting character frequencies

7.2 Character Encodings

Character encodings

- Simplest form of encoding: Encode each source character individually

$\rightsquigarrow$ encoding function $E : \Sigma_S \to \Sigma_C^\star$

  - typically, $|\Sigma_S| \gg |\Sigma_C|$, so need several bits per character
  - for $c \in \Sigma_S$, we call $E(c)$ the *codeword* of c

- **fixed-length code:** $|E(c)|$ is the same for all $c \in \Sigma_{\cancel{C}S}$
- **variable-length code:** not all codewords of same length

Fixed-length codes

- fixed-length codes are the simplest type of character encodings
- Example: **ASCII** (American Standard Code for Information Interchange, 1963)

<table>
<tr><td>0000000 NUL</td><td>0010000 DLE</td><td>0100000</td><td>0110000 0</td><td>1000000 @</td><td>1010000 P</td><td>1100000 ‘</td><td>1110000 p</td></tr>
<tr><td>0000001 SOH</td><td>0010001 DC1</td><td>0100001 !</td><td>0110001 1</td><td>1000001 A</td><td>1010001 Q</td><td>1100001 a</td><td>1110001 q</td></tr>
<tr><td>0000010 STX</td><td>0010010 DC2</td><td>0100010 "</td><td>0110010 2</td><td>1000010 B</td><td>1010010 R</td><td>1100010 b</td><td>1110010 r</td></tr>
<tr><td>0000011 ETX</td><td>0010011 DC3</td><td>0100011 #</td><td>0110011 3</td><td>1000011 C</td><td>1010011 S</td><td>1100011 c</td><td>1110011 s</td></tr>
<tr><td>0000100 EOT</td><td>0010100 DC4</td><td>0100100 $</td><td>0110100 4</td><td>1000100 D</td><td>1010100 T</td><td>1100100 d</td><td>1110100 t</td></tr>
<tr><td>0000101 ENQ</td><td>0010101 NAK</td><td>0100101 %</td><td>0110101 5</td><td>1000101 E</td><td>1010101 U</td><td>1100101 e</td><td>1110101 u</td></tr>
<tr><td>0000110 ACK</td><td>0010110 SYN</td><td>0100110 &</td><td>0110110 6</td><td>1000110 F</td><td>1010110 V</td><td>1100110 f</td><td>1110110 v</td></tr>
<tr><td>0000111 BEL</td><td>0010111 ETB</td><td>0100111 ’</td><td>0110111 7</td><td>1000111 G</td><td>1010111 W</td><td>1100111 g</td><td>1110111 w</td></tr>
<tr><td>0001000 BS</td><td>0011000 CAN</td><td>0101000 (</td><td>0111000 8</td><td>1001000 H</td><td>1011000 X</td><td>1101000 h</td><td>1111000 x</td></tr>
<tr><td>0001001 HT</td><td>0011001 EM</td><td>0101001)</td><td>0111001 9</td><td>1001001 I</td><td>1011001 Y</td><td>1101001 i</td><td>1111001 y</td></tr>
<tr><td>0001010 LF</td><td>0011010 SUB</td><td>0101010 *</td><td>0111010 :</td><td>1001010 J</td><td>1011010 Z</td><td>1101010 j</td><td>1111010 z</td></tr>
<tr><td>0001011 VT</td><td>0011011 ESC</td><td>0101011 +</td><td>0111011 ;</td><td>1001011 K</td><td>1011011 [</td><td>1101011 k</td><td>1111011 {</td></tr>
<tr><td>0001100 FF</td><td>0011100 FS</td><td>0101100 ,</td><td>0111100 <</td><td>1001100 L</td><td>1011100 \</td><td>1101100 l</td><td>1111100 |</td></tr>
<tr><td>0001101 CR</td><td>0011101 GS</td><td>0101101 -</td><td>0111101 =</td><td>1001101 M</td><td>1011101]</td><td>1101101 m</td><td>1111101 }</td></tr>
<tr><td>0001110 SO</td><td>0011110 RS</td><td>0101110 .</td><td>0111110 ></td><td>1001110 N</td><td>1011110 ^</td><td>1101110 n</td><td>1111110 ~</td></tr>
<tr><td>0001111 SI</td><td>0011111 US</td><td>0101111 /</td><td>0111111 ?</td><td>1001111 O</td><td>1011111 _</td><td>1101111 o</td><td>1111111 DEL</td></tr>
</table>

- 7 bit per character
- just enough for English letters and a few symbols (plus control characters)

Fixed-length codes – Discussion

👍 Encoding & Decoding as fast as it gets

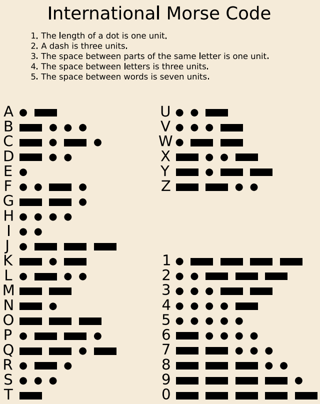
👎 Unless all characters equally likely, it wastes a lot of space

👎 inflexible (how to support adding a new character?)

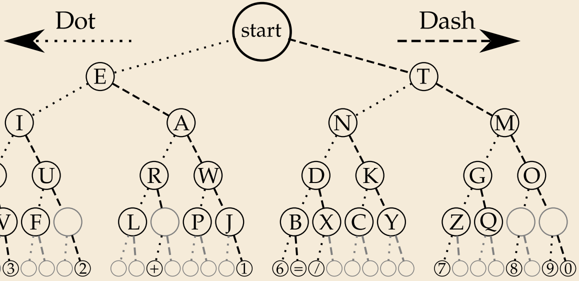
Variable-length codes

- to gain more flexibility, have to allow different lengths for codewords

- actually an old idea: **Morse Code**



https://commons.wikimedia.org/wiki/File:International_Morse_Code.svg



https://commons.wikimedia.org/wiki/File:Morse-code-tree.svg

Clicker Question

How many characters are there in the alphabet of the coded text in Morse Code, i. e., what is $|\Sigma_C|$?

A 1

B 2

C 3

D 4

E 26

F 36

G 256

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- A ~~1~~
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- C 3 ✓
- D ~~4~~
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- F ~~36~~
- G ~~256~~

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Variable-length codes – UTF-8

- Modern example: UTF-8 encoding of Unicode:
  default encoding for text-files, XML, HTML since 2009
  - Encodes any Unicode character (154 998 as of Nov 2024, and counting)
  - uses 1 – 4 bytes (codeword lengths: 8, 16, 24, or 32 bits)
  - Every ASCII character is encoded in 1 byte with leading bit `0`, followed by the 7 bits for ASCII
  - Non-ASCII characters start with 1 – 4 `1`s indicating the total number of bytes, followed by a `0` and 3–5 bits.
    The remaining bytes each start with `10` followed by 6 bits.

Char. number range (hexadecimal)	UTF-8 octet sequence (binary)
`0000 0000 – 0000 007F`	`0xxxxxxx`
`0000 0080 – 0000 07FF`	`110xxxxx 10xxxxxx`
`0000 0800 – 0000 FFFF`	`1110xxxx 10xxxxxx 10xxxxxx`
`0001 0000 – 0010 FFFF`	`11110xxx 10xxxxxx 10xxxxxx 10xxxxxx`

👍 For English text, most characters use only 8 bit,
but we can include any Unicode character, as well. 🤓

Pitfall in variable-length codes

- Suppose we have the following code:

c	a	n	b	s
$E(c)$	0	10	110	100

- Happily encode text S = banana with the coded text $C = \underbrace{\texttt{110}}_{\texttt{b}}\underbrace{\texttt{0}}_{\texttt{a}}\underbrace{\texttt{10}}_{\texttt{n}}\underbrace{\texttt{0}}_{\texttt{a}}\underbrace{\texttt{10}}_{\texttt{n}}\underbrace{\texttt{0}}_{\texttt{a}}$

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↯ C = 1100100100 decodes **both** to banana and to bass: $\underbrace{\texttt{110}}_{\texttt{b}}\underbrace{\texttt{0}}_{\texttt{a}}\underbrace{\texttt{100}}_{\texttt{s}}\underbrace{\texttt{100}}_{\texttt{s}}$

⇝ not a valid code . . . (cannot tolerate ambiguity)

but how should we have known?

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$E(\texttt{n}) = \texttt{10}$ is a (proper) **prefix** of $E(\texttt{s}) = \texttt{100}$

⇝ Leaves decoder wondering whether to stop after reading 10 or continue!

⇝ Usually require a ***prefix-free*** code: No codeword is a prefix of another.

prefix-free $\Longrightarrow$ instantaneously decodable $\Longrightarrow$ uniquely decodable

Code tries

- From now on only consider prefix-free codes E: $E(c)$ is not a proper prefix of $E(c')$ for any $c, c' \in \Sigma_S$.

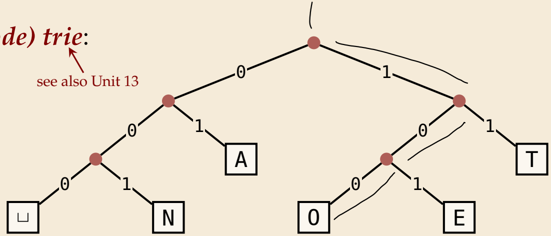
- **Example:**

c	A	E	N	O	T	␣
$E(c)$	01	101	001	100	11	000

Any prefix-free code corresponds to a ***(code) trie***:

see also Unit 13

- binary tree
- one **leaf** for each characters of Σ_S
- path from root to leave = codeword
  left child = 0; right child = 1



- Example for using the code trie:
  - Encode AN␣ANT
  - Decode 111000001010111
    T O

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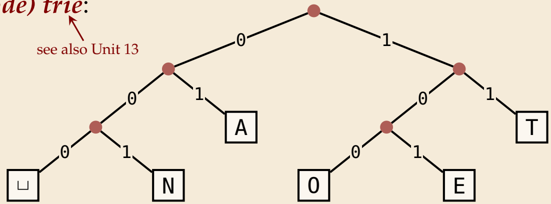

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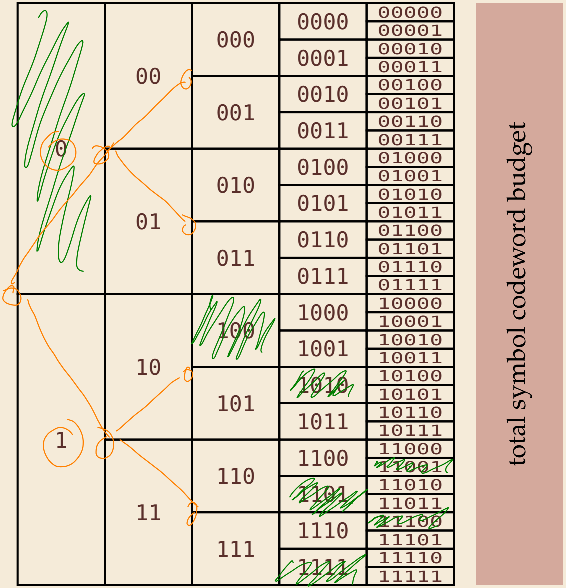
  - Encode AN␣ANT $\longrightarrow$ 010010000100111
  - Decode 111000001010111 $\longrightarrow$ TO␣EAT

The Codeword Supermarket



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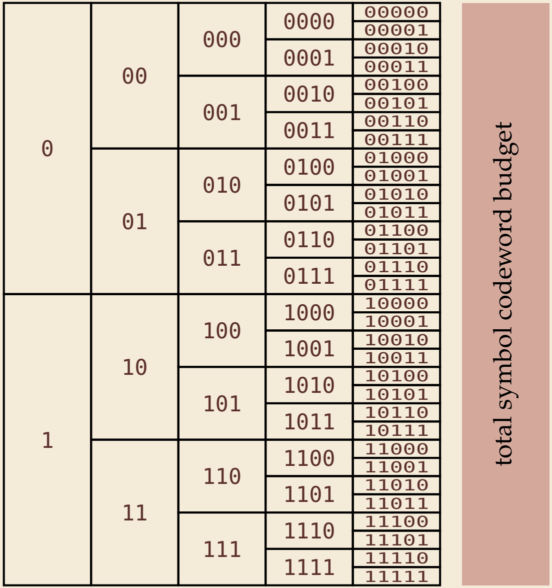

- Can "spend" at most budget of 1 across all codewords
  - Codeword with ℓ bits costs $2^{-\ell}$
- *Kraft-McMillan inequality:* any uniquely decodable code with codeword lengths $\ell_1, \ldots, \ell_\sigma$ satisfies

$$\sum_{i=1}^{\sigma} 2^{-\ell_i} \leq 1$$

and for any such lengths there is a prefix-free code

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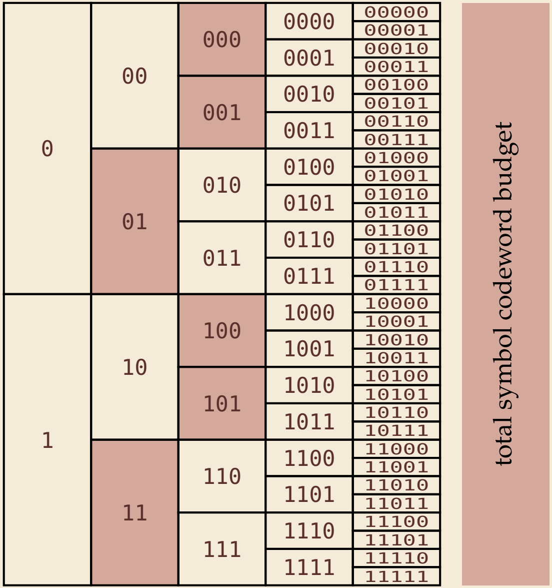

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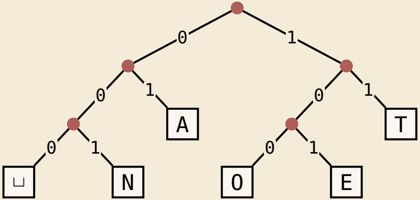

Who decodes the decoder?

- Depending on the application, we have to **store/transmit** the **used code**!
- We distinguish:
  - **fixed coding:** code agreed upon in advance, not transmitted (e. g., Morse, UTF-8)
  - **static coding:** code depends on message, but stays same for entire message; it must be transmitted (e. g., Huffman codes $\rightarrow$ next)
  - **adaptive coding:** code depends on message and changes during encoding; implicitly stored withing the message (e. g., LZW $\rightarrow$ below)

7.3 Huffman Codes

Character frequencies

- **Goal:** Find character encoding that produces short coded text
- Convention here: fix $\Sigma_C = \{0, 1\}$ (binary codes), abbreviate $\Sigma = \Sigma_S$,

- **Observation:** Some letters occur more often than others.

Typical English prose:

e	12.70%	**d**	4.25%	**p**	1.93%
t	9.06%	**l**	4.03%	**b**	1.49%
a	8.17%	**c**	2.78%	**v**	0.98%
o	7.51%	**u**	2.76%	**k**	0.77%
i	6.97%	**m**	2.41%	**j**	0.15%
n	6.75%	**w**	2.36%	**x**	0.15%
s	6.33%	**f**	2.23%	**q**	0.10%
h	6.09%	**g**	2.02%	**z**	0.07%
r	5.99%	**y**	1.97%		

$\rightsquigarrow$ Want shorter codes for more frequent characters!

Huffman coding

e. g. frequencies / probabilities

- **Given:** Σ and weights $w : \Sigma \to \mathbb{R}_{\geq 0}$
- **Goal:** prefix-free code E (= code trie) for Σ that minimizes coded text length

  i. e., a code trie minimizing $\displaystyle\sum_{c\in\Sigma} w(c) \cdot |E(c)|$

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- Let's abbreviate $|S|_c$ = #occurrences of c in S
- If we use $w(c) = |S|_c$,
  this is the character encoding with smallest possible $|C|$

$\rightsquigarrow$ **best possible *character-wise* encoding**

- Quite ambitious! *Is this efficiently possible?*

Huffman's algorithm

- Actually, yes! A greedy/myopic approach succeeds here.

Huffman's algorithm:

1. Find two characters $\mathtt{a}$, $\mathtt{b}$ with lowest weights.
   - We will encode them with the same prefix, plus one distinguishing bit, i. e., $E(\mathtt{a}) = u\mathtt{0}$ and $E(\mathtt{b}) = u\mathtt{1}$ for a bitstring $u \in \{0,1\}^\star$ (u to be determined)
2. (Conceptually) replace $\mathtt{a}$ and $\mathtt{b}$ by a single character "$\boxed{\mathtt{ab}}$" with $w(\boxed{\mathtt{ab}}) = w(\mathtt{a}) + w(\mathtt{b})$.
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- efficient implementation using a (min-oriented) ***priority queue***
  - start by inserting all characters with their weight as key
  - step 1 uses two `deleteMin` calls
  - step 2 inserts a new character with the sum of old weights as key

Huffman's algorithm – Example

- Example text: $S = \texttt{LOSSLESS}$ $\rightsquigarrow$ $\Sigma_S = \{\texttt{E}, \texttt{L}, \texttt{O}, \texttt{S}\}$
- Character frequencies: $\texttt{E} : 1,\quad \texttt{L} : 2,\quad \texttt{O} : 1,\quad \texttt{S} : 4$

1	2	1	4
$\boxed{\texttt{E}}$	$\boxed{\texttt{L}}$	$\boxed{\texttt{O}}$	$\boxed{\texttt{S}}$

Huffman's algorithm – Example

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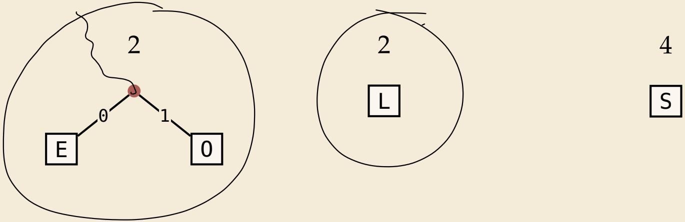

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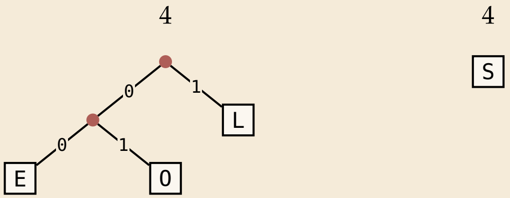

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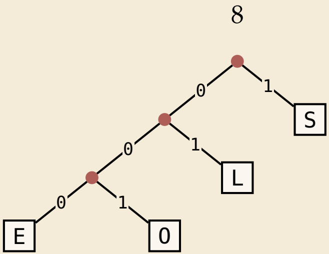

Huffman's algorithm – Example

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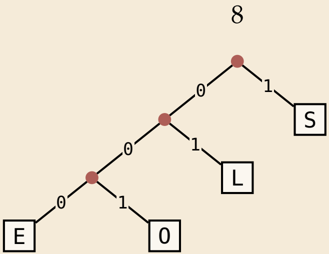

⇝ *Huffman tree* (code trie for Huffman code)

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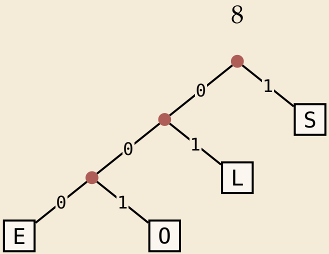

$\leadsto$ *Huffman tree* (code trie for Huffman code)

$\texttt{LOSSLESS} \to \texttt{01001110100011}$ compression ratio: $\frac{14}{8 \cdot \log 4} = \frac{14}{16} \approx 88\%$

Huffman tree – tie breaking

- The above procedure is ambiguous:
  - which characters to choose when weights are equal?
  - which subtree goes left, which goes right?

- For CS 566: always use the following rule:

> **1.** To break ties when selecting the two characters,
> first use the smallest letter according to the alphabetical order,
> or the tree containing the smallest alphabetical letter.
>
> **2.** When combining two trees of different values,
> place the lower-valued tree on the left (corresponding to a `0`-bit).
>
> **3.** When combining trees of equal value,
> place the one containing the smallest letter to the left.

⇝ practice in tutorials

Encoding with Huffman code

- The overall encoding procedure is as follows:
  - **Pass 1:** Count character frequencies in S
  - Construct Huffman code E (as above)
  - Store the Huffman code in C (details omitted)
  - **Pass 2:** Encode each character in S using E and append result to C
- Decoding works as follows:
  - Decode the Huffman code E from C. (details omitted)
  - Decode S character by character from C using the code trie.
- Note: Decoding is much simpler/faster!

Huffman code – Optimality

Theorem 7.1 (Optimality of Huffman's Algorithm)

Given Σ and $w : \Sigma \to \mathbb{R}_{\geq 0}$, Huffman's Algorithm computes codewords $E : \Sigma \to \{0,1\}^\star$ with minimal expected codeword length $\ell(E) = \sum_{c \in \Sigma} w(c) \cdot |E(c)|$ among all prefix-free codes for Σ. ◀

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IB: $\sigma = 2$ ✓

IH: $\forall \sigma' < \sigma$ Huffman's algorithm computes optimal PF code

Proof sketch: by induction over $\sigma = |\Sigma|$

- Given any optimal prefix-free code E^* (as its code trie).
- code trie $\rightsquigarrow$ $\exists$ two sibling leaves x, y at largest depth D

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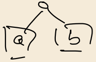

- swap characters in leaves to have two lowest-weight characters $\mathtt{a}$, $\mathtt{b}$ in x, y (that can only make ℓ smaller, so still optimal)
- any optimal code for $\Sigma' = \Sigma \setminus \{\mathtt{a}, \mathtt{b}\} \cup \{\boxed{\mathtt{ab}}\}$ yields optimal code for Σ by replacing leaf $\boxed{\mathtt{ab}}$ by internal node with children $\mathtt{a}$ and $\mathtt{b}$.

$\rightsquigarrow$ recursive call yields optimal code for Σ' by inductive hypothesis, so Huffman's algorithm finds optimal code for Σ.

7.4 Entropy

Entropy

Definition 7.2 (Entropy)

Given probabilities $p_1, \ldots, p_n$ (for outcomes $1, \ldots, n$ of a random variable), the *entropy* of the distribution is defined as

$$\mathcal{H}(p_1, \ldots, p_n) \;=\; -\sum_{i=1}^{n} p_i \lg p_i \;=\; \underline{\sum_{i=1}^{n} p_i \lg\left(\frac{1}{p_i}\right)}$$

$$0 \le \mathcal{H}(p_1, \ldots, p_n) \le \lg n = \mathcal{H}\left(\frac{1}{n}, \ldots, \frac{1}{n}\right)$$

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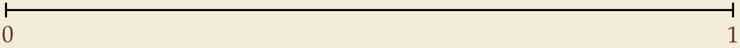
- entropy is a **measure** of **information** content of a distribution
  - *"20 Questions on* $[0,1)$*":* Land inside my interval by halving.



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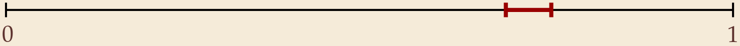

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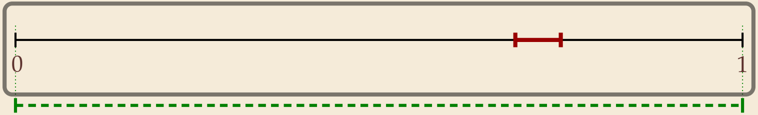

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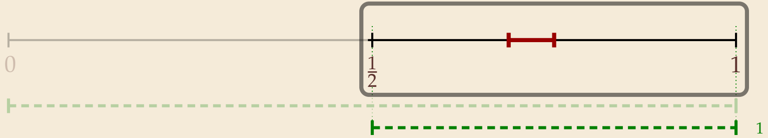

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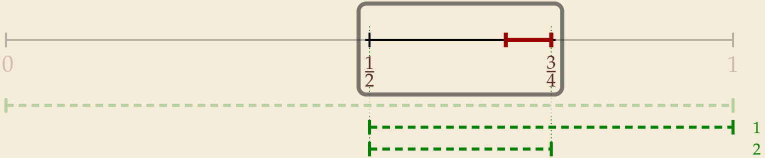

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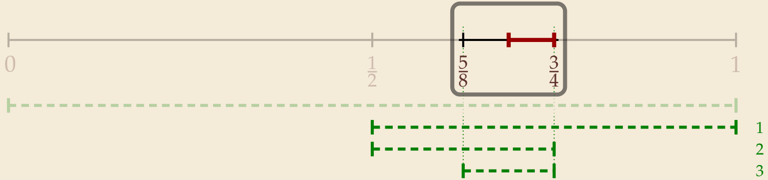

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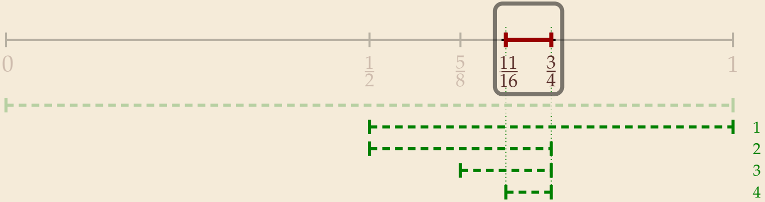

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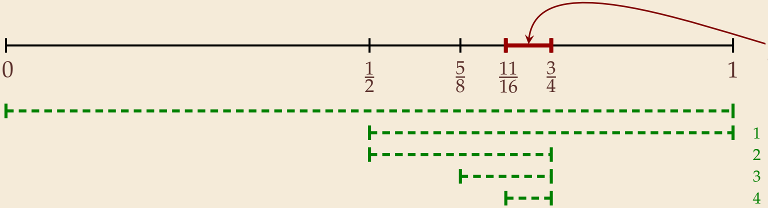

$p_i = \frac{3}{4} - \frac{11}{16} = \frac{1}{16}$

$\rightsquigarrow \lg(1/p_i) = 4$

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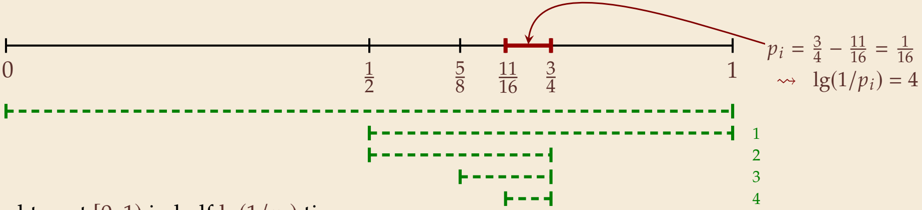


$\rightsquigarrow$ Need to cut $[0, 1)$ in half $\lg(1/p_i)$ times

- more precisely: the $\underbrace{\text{expected}}$ number of bits (Yes/No questions) required to nail down the random value

Entropy and Huffman codes

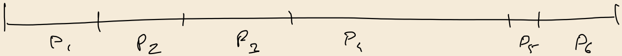
- would ideally encode value i using $\lg(1/p_i)$ bits

  not always possible; cannot use codeword of 1.5 bits . . .

not as length of single codeword that is; but can be possible *on average*!

positive example:

all $p_i = 2^{-k_i}$



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Theorem 7.3 (Entropy bounds for Huffman codes)

For any probabilities $p_1, \ldots, p_\sigma$ for $\Sigma = \{a_1, \ldots, a_\sigma\}$, the Huffman code E for Σ with weights $p(a_i) = p_i$ satisfies $\boxed{\mathcal{H} \;\le\; \ell(E) \;\le\; \mathcal{H} + 1}$ where $\mathcal{H} = \mathcal{H}(p_1, \ldots, p_\sigma)$. ◀

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Proof sketch:

- $\ell(E) \ge \mathcal{H}$

  Any prefix-free code E induces weights $q_i = 2^{-|E(a_i)|}$.

  By *Kraft's Inequality*, we have $q_1 + \cdots + q_\sigma \le 1$.

50% 0 50%

in general $p_i \neq q_i$

$\sigma = 2$ $p_1 = \varepsilon$

$p_2 = 1 - \varepsilon$

$q_1 = q_2 = \frac{1}{2}$

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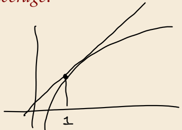

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$$\boxed{\ln(x) \le x - 1 \quad x > 0}$$

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  Any prefix-free code E induces weights $q_i = 2^{-|E(a_i)|}$.
  By *Kraft's Inequality*, we have $q_1 + \cdots + q_\sigma \le 1$.
  Hence we can apply *Gibb's Inequality* to get

$$\mathcal{H} \;=\; \underbrace{\sum_{i=1}^{\sigma} p_i \lg\left(\frac{1}{p_i}\right) \;\le\; \sum_{i=1}^{\sigma} p_i \lg\left(\frac{1}{\color{red}{q_i}}\right)}_{\sum p_i = 1, \quad \sum q_i \le 1} \;=\; \ell(E).$$

$$0 \overset{!}{\geq} \sum p_i \ln\left(\frac{1}{p_i}\right) - \sum p_i \ln\left(\frac{1}{q_i}\right)$$

$$= \sum p_i \ln\left(\frac{q_i}{p_i}\right)$$

$$\le \sum p_i \left(q_i/p_i - 1\right)$$

$$= \underbrace{\sum q_i}_{\le 1} - \underbrace{\sum p_i}_{1} \le 0$$

Entropy and Huffman codes [2]

Proof sketch (continued):

- $\ell(E) \leq \mathcal{H} + 1$
  Set $q_i = 2^{-\lceil \lg(1/p_i) \rceil}$. We have $\displaystyle\sum_{i=1}^{\sigma} p_i \lg\left(\frac{1}{q_i}\right) = \sum_{i=1}^{\sigma} p_i \lceil \lg(1/p_i) \rceil \leq \mathcal{H} + 1.$

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  We construct a code E' for Σ with $|E'(a_i)| \le \lg(1/q_i)$ as follows;
  w.l.o.g. assume $q_1 \le q_2 \le \cdots \le q_\sigma$

  - If $\sigma = 2$, E' uses a single bit each.
    Here, $q_i \le 1/2$, so $\lg(1/q_i) \ge 1 = |E'(a_i)|$ ✓

Entropy and Huffman codes [2]

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    If $q_1 = q_2$, this is like Huffman; otherwise, q_1 is a unique smallest value and
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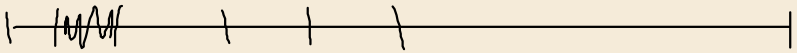

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    $\underline{q_2 + q_2} + \cdots + q_\sigma \le 1$.

    By the inductive hypothesis, we have $\left|E'(\boxed{a_1 a_2})\right| \le \lg\left(\dfrac{1}{2q_2}\right) = \lg\left(\dfrac{1}{q_2}\right) - 1.$

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    By construction, $|E'(a_1)| = |E'(a_2)| = \left|E'(\boxed{a_1 a_2})\right| + 1$, so $|E'(a_1)| \leq \lg(\frac{1}{q_1})$ and $|E'(a_2)| \leq \lg(\frac{1}{q_2})$.

  By optimality of E, we have $\displaystyle \ell(E) \leq \ell(E') \leq \sum_{i=1}^{\sigma} p_i \lg\left(\frac{1}{q_i}\right) \leq \mathcal{H} + 1.$

Clicker Question

When does Huffman coding yield more efficient compression than a fixed-length character encoding?

- A always
- B when $\mathcal{H} \approx \lg(\sigma)$
- C when $\mathcal{H} < \lg(\sigma)$
- D when $\mathcal{H} < \lg(\sigma) - 1$
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→ `sli.do/cs566`

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  ... but we only have a string S! (nothing random about it!)

use relative frequencies: $p_i = \dfrac{|S|_{a_i}}{|S|} = \dfrac{\text{\#occurences of } a_i \text{ in string } S}{\text{length of } S}$

- Recall: For $S[0..n)$ over $\Sigma = \{a_1, \ldots, a_\sigma\}$,
  length of Huffman-coded text is

$$|C| = \sum_{i=1}^{\sigma} |S|_{a_i} \cdot |E(a_i)| = n \sum_{i=1}^{\sigma} \overbrace{\boxed{\frac{|S|_{a_i}}{n}}}^{=p_i} \cdot |E(a_i)| = \underline{n\ell(E)}$$

⇝ Theorem 7.3 tells us rather precisely how well Huffman compresses:
$\mathcal{H}_0(S) \cdot n \le |C| \le (\mathcal{H}_0(S) + 1)n$

zero-th order empirical entropy

- $\mathcal{H}_0(S) = \mathcal{H}\left(\frac{|S|_{a_1}}{n}, \ldots, \frac{|S|_{a_\sigma}}{n}\right) = \sum_{i=1}^{\sigma} \frac{n}{|S|_{a_i}} \log_2\left(\frac{|S|_{a_i}}{n}\right)$ is called the *empirical entropy* of S

Huffman coding – Discussion

- running time complexity: $\underline{O(\sigma \log \sigma)}$ to construct code
  - build PQ + $\sigma \cdot$ (2 deleteMins and 1 insert)
  - can do $\Theta(\sigma)$ time when characters already sorted by weight
  - time for encoding text (after Huffman code done): $O(n + |C|)$
- many variations in use (tie-breaking rules, estimated frequencies, adaptive encoding, ...)

Huffman coding – Discussion

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  - time for encoding text (after Huffman code done): $O(n + |C|)$
- many variations in use (tie-breaking rules, estimated frequencies, adaptive encoding, ...)

👍 optimal prefix-free character encoding

👍 very fast decoding

👎 needs 2 passes over source text for encoding
  - one-pass variants possible, but more complicated

👎 have to store code alongside with coded text

Part II

Compressing repetitive texts

Beyond Character Encoding

- Many "natural" texts show repetitive redundancy

```
All work and no play makes Jack a dull boy.  All work and no play makes Jack a dull
boy.  All work and no play makes Jack a dull boy.  All work and no play makes Jack
a dull boy.  All work and no play makes Jack a dull boy.  All work and no play makes
Jack a dull boy.  All work and no play makes Jack a dull boy.  All work and no play
makes Jack a dull boy.  All work and no play makes Jack a dull boy.  All work and no
play makes Jack a dull boy.  All work and no play makes Jack a dull boy.  All work and
no play makes Jack a dull boy.  All work and no play makes Jack a dull boy.  All work
and no play makes Jack a dull boy.  All work and no play makes Jack a dull boy.  All
work and no play makes Jack a dull boy.  All work and no play makes Jack a dull boy.
```

- character-by-character encoding will **not** capture such repetitions
  - ⇝ Huffman won't compression this very much

Beyond Character Encoding

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work and no play makes Jack a dull boy.  All work and no play makes Jack a dull boy.
```

- character-by-character encoding will **not** capture such repetitions
  - ⇝ Huffman won't compression this very much

⇝ Have to encode whole *phrases* of S by a single codeword

7.5 Run-Length Encoding

Run-Length encoding

- simplest form of repetition: ***runs*** of characters

  same character repeated

```
00000000000000000000000000000000000000000
00000000000000000000000000000000000000000
00000000000000000000000000000000000000000
00010110010000000001111100000000011111000
00111111111000000111111100000001111111000
00111101101000001110000000000011100000000
00110000000000011100000000000111000000000
00110000000000111000000000001110000000000
00110000000000111000000000001110000000000
00110110000000110011111000001100111110000
00111111110000111111111100001111111111000
00111011111000111110011110001111100111100
00000000011100111000000111001110000001110
00000000011100111000000110001110000001100
00000000001100011000000111000110000001110
00000000001100111000000110001110000001100
00000000011100011000000111000110000001110
00000000011000011100001110000111000011100
00110111111000001111111100000011111111000
01111111110000000111111000000001111110000
00010110000000000010010000000000100100000
00000000000000000000000000000000000000000
00000000000000000000000000000000000000000
```

- here: only consider $\Sigma_S = \{0, 1\}$ (work on a binary representation)
  - can be extended for larger alphabets

Run-Length encoding

- simplest form of repetition: ***runs*** of characters

  same character repeated

```
0000000000000000000000000000000000000000
0000000000000000000000000000000000000000
0000000000000000000000000000000000000000
0001011001000000000111110000000011111000
0011111111100000011111110000011111111000
0011110110100000011100000000011100000000
0011000000000000111000000000111000000000
0011000000000011100000000001110000000000
0011000000000011100000000001110000000000
0011011000000000001111110001100111110000
0011111111100011111111110001111111111000
0011101111110001111001110011111001111100
0000000000111001100000000011100000001110
0000000000111001100000000011100000001100
0000000000011000100000000001100000001110
0000000000011001100000000011100000001100
0000000000111000100000000001100000001110
0000000000110000110000010000111000011100
0011011111110000011111110000011111111000
0111111111110000011111110000011111110000
0001011000000000000100100000000100100000
0000000000000000000000000000000000000000
0000000000000000000000000000000000000000
```

- here: only consider $\Sigma_S = \{0, 1\}$ (work on a binary representation)
  - can be extended for larger alphabets

⇝ **run-length encoding (RLE)**:
use runs as phrases: $S = \underbrace{\texttt{00000}}\ \underbrace{\texttt{111}}\ \underbrace{\texttt{0000}}$

Run-Length encoding

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  same character repeated

```
0000000000000000000000000000000000000000
0000000000000000000000000000000000000000
0000000000000000000000000000000000000000
0001011001000000000111110000000011111000
0011111111100000011111100000001111111000
0011110110100000111000000000011100000000
0011000000000001110000000000111000000000
0011000000000011100000000001110000000000
0011000000000011100000000001110000000000
0011011000000011001111100001100111110000
0011111111000011111111100001111111111000
0011101111100011111001111001111100111100
0000000001110011100000111001110000001110
0000000001110011100000110001110000001100
0000000000110001100000111000110000001110
0000000000110011100000110001110000001100
0000000001110001100000111000110000001110
0000000001100011100001110000111000011100
0011011111100000111111110000011111111000
0111111111000000011111100000001111110000
0001011000000000001001000000000100100000
0000000000000000000000000000000000000000
0000000000000000000000000000000000000000
```

- here: only consider $\Sigma_S = \{0, 1\}$ (work on a binary representation)
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⇝ **run-length encoding (RLE)**:
use runs as phrases: $S = \underbrace{\texttt{00000}}\ \underbrace{\texttt{111}}\ \underbrace{\texttt{0000}}$

⇝ We have to store
  - the first bit of S (either `0` or `1`)
  - the length of each subsequent run
  - Note: don't have to store bit for later runs since they must alternate.

- Example becomes: $0, 5, 3, 4$

Run-Length encoding

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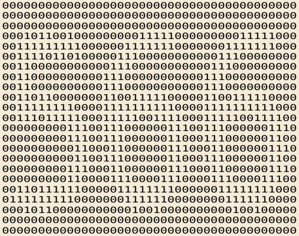
  same character repeated



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- the length of each subsequent run
- Note: don't have to store bit for later runs since they must alternate.

- Example becomes: $0, 5, 3, 4$

- **Question**: How to encode a run length k in binary? (k can be arbitrarily large!)

Clicker Question

How would you encode a string that can we arbitrarily long?

→ sli.do/cs566

Elias codes

- Need a *prefix-free encoding* for $\mathbb{N} = \{1, 2, 3, \ldots, \}$
  - must allow arbitrarily large integers
  - must know when to stop reading

Elias codes

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  $7 \mapsto$ `00000001` $3 \mapsto$ `0001` $0 \mapsto$ `1` $30 \mapsto$ `0000000000000000000000000000001`

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  Much too long
  - (wasn't the whole point of RLE to get rid of long runs??)

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- Refinement: ***Elias gamma code***
  - Store the **length** ℓ of the binary representation in **unary**
  - Followed by the binary digits themselves

Elias codes

$\mathbb{N}_{\geq 1}$

- Need a *prefix-free encoding* for $\cancel{\mathbb{N}} = \{1, 2, 3, \ldots, \}$
  - must allow arbitrarily large integers
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- But that's simple! Just use ***unary* encoding**!
  $7 \mapsto$ `00000001` $3 \mapsto$ `0001` $0 \mapsto$ `1` $30 \mapsto$ `0000000000000000000000000000001`

  Much too long
  - (wasn't the whole point of RLE to get rid of long runs??)

- Refinement: ***Elias gamma code***
  - Store the **length** ℓ of the binary representation in **unary**
  - Followed by the binary digits themselves
  - little tricks:
    - always have $\ell \geq 1$, so store $\ell - 1$ instead
    - binary representation always starts with `1` $\rightsquigarrow$ don't need terminating `1` in unary

  $\rightsquigarrow$ Elias gamma code = $\ell - 1$ zeros, followed by binary representation

Examples: $1 \mapsto$ `1`, $3 \mapsto$ `011`, $5 \mapsto$ `00101`, $30 \mapsto$ `000011110`

Clicker Question

Decode the **first** number in Elias gamma code (at the beginning) of the following bitstream:

$$000\underbrace{1101}_{13}11011100110.$$

→ *sli.do/cs566*

Run-length encoding – Examples

- Encoding:
  S = `11111110010000000000000000000011111111111`

  C = `1`

- Decoding:
  C = `00001101001001010`

  S =

Run-length encoding – Examples

- Encoding:
  S = 11111110010000000000000000000011111111111
  k = 7
  C = 100111
- Decoding:
  C = 00001101001001010

  S =

Run-length encoding – Examples

- Encoding:
  $S =$ `1111111`**`00`**`10000000000000000000011111111111`
  $k = 2$
  $C =$ `100111`**`010`**

- Decoding:
  $C =$ `00001101001001010`

  $S =$

Run-length encoding – Examples

- Encoding:

  S = `11111110010000000000000000000011111111111`

  $k = 1$

  C = `1001110101`

- Decoding:

  C = `00001101001001010`

  S =

Run-length encoding – Examples

- Encoding:
  $S =$ `111111100100000000000000000000001111111111`
  $k = 20$
  $C =$ `1001110101000010100`

- Decoding:
  $C =$ `00001101001001010`

  $S =$

Run-length encoding – Examples

- Encoding:
  $S =$ `111111100100000000000000000000` `11111111111`
  $k = 11$
  $C =$ `1001110101000010100` `0001011`

- Decoding:
  $C =$ `00001101001001010`

  $S =$

Run-length encoding – Examples

- Encoding:
  S = `11111110010000000000000000000011111111111`

  C = `10011101010000101000001011`

  Compression ratio: $26/41 \approx 63\%$

- Decoding:
  C = `00001101001001010`

  S =

Run-length encoding – Examples

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Run-length encoding – Examples

- Encoding:
  $S =$ `11111110010000000000000000000011111111111`

  $C =$ `10011101010000101000001011`

  Compression ratio: $26/41 \approx 63\%$

- Decoding:
  $C =$ ~~`0`~~`0001101001001010`
  $b = 0$

  $S =$

Run-length encoding – Examples

- Encoding:

  $S =$ `11111110010000000000000000000011111111111`

  $C =$ `10011101010000101000001011`

  Compression ratio: $26/41 \approx 63\%$

- Decoding:

  $C =$ `0000110100100101 0`

  $b = 0$

  $\ell = 3 + 1$

  $S =$

Run-length encoding – Examples

- Encoding:

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- Decoding:

  $C =$ `0000110100100101 0`

  $b = 0$

  $\ell = 3 + 1$

  $k = 13$

  $S =$ `0000000000000`

Run-length encoding – Examples

- Encoding:

  $S =$ `11111110010000000000000000000011111111111`

  $C =$ `10011101010000101000001011`

  Compression ratio: $26/41 \approx 63\%$

- Decoding:

  $C =$ `00001101`**`00`**`1001010`

  $b = 1$

  $\ell = 2 + 1$

  $k =$

  $S =$ `0000000000000`

Run-length encoding – Examples

- Encoding:

  S = `11111110010000000000000000000011111111111`

  C = `10011101010000101000001011`

  Compression ratio: $26/41 \approx 63\%$

- Decoding:

  C = `00001101001001010`

  $b = 1$

  $\ell = 2 + 1$

  $k = 4$

  S = `00000000000001111`

Run-length encoding – Examples

- Encoding:

  S = `11111110010000000000000000000011111111111`

  C = `10011101010000101000001011`

  Compression ratio: $26/41 \approx 63\%$

- Decoding:

  C = `00001101001001010`

  $b = 0$

  $\ell = 0 + 1$

  $k =$

  S = `00000000000001111`

Run-length encoding – Examples

- Encoding:

  $S =$ `11111110010000000000000000000011111111111`

  $C =$ `10011101010000101000001011`

  Compression ratio: $26/41 \approx 63\%$

- Decoding:

  $C =$ `00001101001001010`

  $b = 0$

  $\ell = 0 + 1$

  $k = 1$

  $S =$ `000000000000011110`

Run-length encoding – Examples

- Encoding:

  S = `11111110010000000000000000000011111111111`

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  $\ell = 1 + 1$

  $k =$

  S = `000000000000011110`

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- Encoding:
  S = `11111110010000000000000000000011111111111`

  C = `10011101010000101000001011`

  Compression ratio: $26/41 \approx 63\%$

- Decoding:
  C = `00001101001001010`
  $b = 1$
  $\ell = 1 + 1$
  $k = 2$
  S = `000000000000001111011`

Run-length encoding – Discussion

- extensions to larger alphabets possible (must store next character then)
- used in some image formats (e. g. TIFF)

Run-length encoding – Discussion

- extensions to larger alphabets possible (must store next character then)
- used in some image formats (e. g. TIFF)

👍 fairly simple and fast

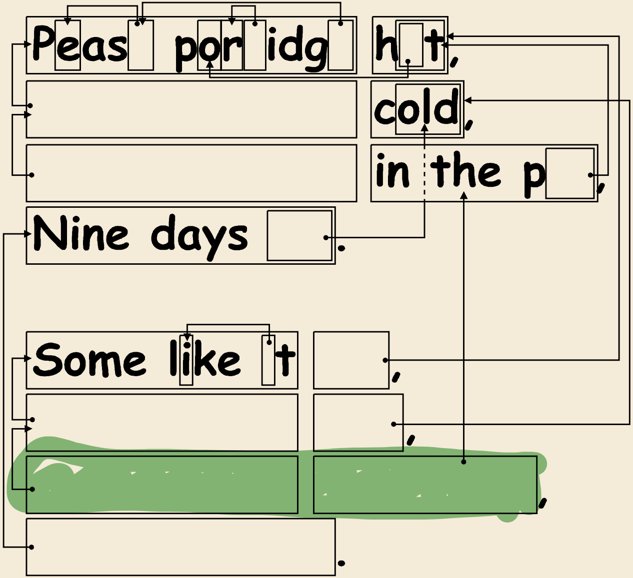
👍 can compress n bits to $\Theta(\log n)$!
for extreme case of constant number of runs

👎 negligible compression for many common types of data

- No compression until run lengths $k \geq 6$
- **expansion** for run length $k = 2$ or 6

7.6 Lempel-Ziv-Welch

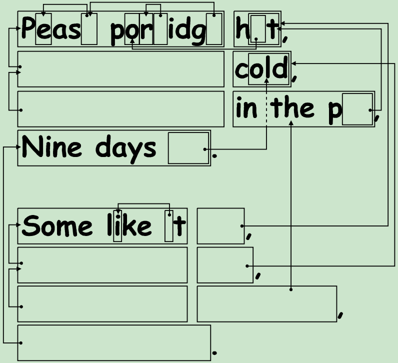
Warmup



https://classic.csunplugged.org/text-compression/

https://www.flickr.com/photos/quintanaroo/2742726346

Clicker Question



What is the **second-to-last line** of the poem to the left?

→ sli.do/cs566

Lempel-Ziv Compression

- Huffman and RLE mostly take advantage of frequent or repeated *single characters*.
- **Observation**: Certain *substrings* are much more frequent than others.
  - in English text: the, be, to, of, and, a, in, that, have, I
  - in HTML: "`<a href`", "`<img src`", "`<br/>`"

Lempel-Ziv Compression

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  - in English text: the, be, to, of, and, a, in, that, have, I
  - in HTML: “`<a href`”, “`<img src`”, “`<br/>`”
- **Lempel-Ziv** stands for family of *adaptive* compression algorithms.
  - **Idea:** store repeated parts by reference!

  ⇝ each codeword refers to
    - either a single character in Σ_S,
    - or a *substring* of S (that both encoder and decoder have seen before).

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  ⇝ each codeword refers to
    - either a single character in Σ_S,
    - or a *substring* of S (that both encoder and decoder have seen before).
  - Variants of Lempel-Ziv compression
    - **"LZ77"** Original version (sliding window, overlapping phrases)
      Derivatives: LZSS, LZFG, LZRW, LZP, DEFLATE, ...
      DEFLATE used in (`pk`)`zip`, `gzip`, `PNG`
    - **"LZ78"** Second version (whole-phrase references)
      Derivatives: LZW, LZMW, LZAP, LZY, ...
      LZW used in `compress`, `GIF`

Lempel-Ziv-Welch

- here: ***Lempel-Ziv-Welch (LZW)*** (arguably the "cleanest" variant of Lempel-Ziv)
- *variable-to-fixed* **encoding**
  - all codewords have k bits (typical: $k = 12$) $\rightsquigarrow$ fixed-length
  - but they represent a variable portion of the source text!

Lempel-Ziv-Welch

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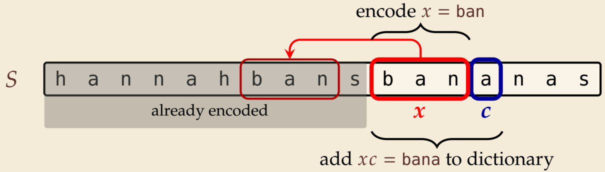
- *variable-to-fixed* **encoding**
  - all codewords have k bits (typical: $k = 12$) $\rightsquigarrow$ fixed-length
  - but they represent a variable portion of the source text!
- maintain a **dictionary** D with 2^k entries $\rightsquigarrow$ codewords = indices in dictionary
  - initially, first $|\Sigma_S|$ entries encode single characters (rest is empty)
  - **add** a new entry to D **after each step**:
  - **Encoding:** after encoding a substring x of S,
    add xc to D where c is the character that follows x in S.



$\rightsquigarrow$ new codeword in D

  - D actually stores codewords for x and c, not the expanded string

LZW encoding – Example

Input: `YO!␣YOU!␣YOUR␣YOYO!`

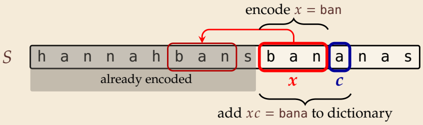
Σ_S = ASCII character set (0–127)

$C =$

$D =$

Code	String
...	
32	␣
33	!
...	
79	O
...	
82	R
...	
85	U
...	
89	Y
...	

Code	String
128	
129	
130	
131	
132	
133	
134	
135	
136	
137	
138	
139	



LZW encoding – Example

Input: `YO!␣YOU!␣YOUR␣YOYO!`

Σ_S = ASCII character set (0–127)

Y
$C = 89$

$D =$

Code	String
. . .	
32	␣
33	!
. . .	
79	O
. . .	
82	R
. . .	
85	U
. . .	
89	Y
. . .	

Code	String
128	
129	
130	
131	
132	
133	
134	
135	
136	
137	
138	
139	

encode x = ban

S h a n n a h b a n s b a n a n a s

already encoded x c

add xc = bana to dictionary

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...	
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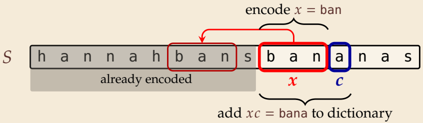
...	
89	Y
...	

Code	String
128	YO
129	
130	
131	
132	
133	
134	
135	
136	
137	
138	
139	



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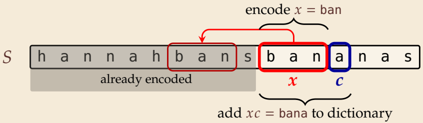
Σ_S = ASCII character set (0–127)

	Y	O
$C =$	89	79

$D =$

Code	String
...	
32	␣
33	!
...	
79	O
...	
82	R
...	
85	U
...	
89	Y
...	

Code	String
128	YO
129	
130	
131	
132	
133	
134	
135	
136	
137	
138	
139	



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Code	String
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encode x = ban

S `h a n n a h b a n s b a n a n a s`

already encoded

x c

add xc = bana to dictionary

LZW encoding – Example

Input: `YO!␣YOU!␣YOUR␣YOYO!`

Σ_S = ASCII character set (0–127)

	Y	O	!
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$D =$

Code	String
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32	␣
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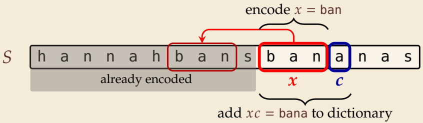
138	
139	



LZW encoding – Example

Input: `YO!␣YOU!␣YOUR␣YOYO!`

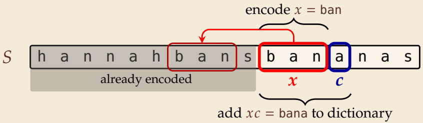
Σ_S = ASCII character set (0–127)

	Y	O	!
$C =$	89	79	33

$D =$

Code	String
...	
32	␣
33	!
...	
79	O
...	
82	R
...	
85	U
...	
89	Y
...	

Code	String
128	YO
129	O!
130	!␣
131	
132	
133	
134	
135	
136	
137	
138	
139	



LZW encoding – Example

Input: `YO!␣YOU!␣YOUR␣YOYO!`

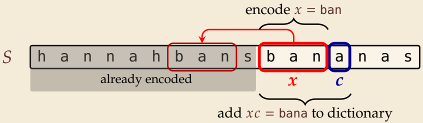
Σ_S = ASCII character set (0–127)

	Y	O	!	␣
$C =$	89	79	33	32

$D =$

Code	String
...	
32	␣
33	!
...	
79	O
...	
82	R
...	
85	U
...	
89	Y
...	

Code	String
128	YO
129	O!
130	!␣
131	
132	
133	
134	
135	
136	
137	
138	
139	



LZW encoding – Example

Input: `YO!␣YOU!␣YOUR␣YOYO!`

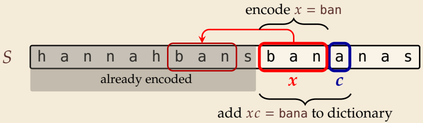
Σ_S = ASCII character set (0–127)

	Y	O	!	␣
$C =$	89	79	33	32

$D =$

Code	String
...	
32	␣
33	!
...	
79	O
...	
82	R
...	
85	U
...	
89	Y
...	

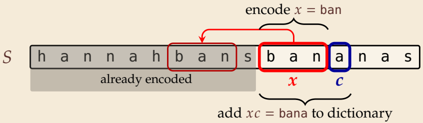
Code	String
128	YO
129	O!
130	!␣
131	␣Y
132	
133	
134	
135	
136	
137	
138	
139	



LZW encoding – Example

Input: `YO!␣YOU!␣YOUR␣YOYO!` Σ_S = ASCII character set (0–127)

	Y	O	!	␣	YO
$C =$	89	79	33	32	128



$D =$

Code	String
...	
32	␣
33	!
...	
79	O
...	
82	R
...	
85	U
...	
89	Y
...	

Code	String
128	YO
129	O!
130	!␣
131	␣Y
132	
133	
134	
135	
136	
137	
138	
139	

LZW encoding – Example

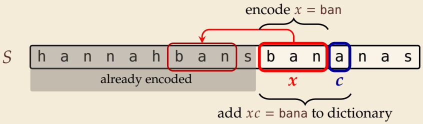
Input: YO!␣YOU!␣YOUR␣YOYO! $\qquad \Sigma_S$ = ASCII character set (0–127)

	Y	O	!	␣	YO
C =	89	79	33	32	128

D =

Code	String
...	
32	␣
33	!
...	
79	O
...	
82	R
...	
85	U
...	
89	Y
...	

Code	String
128	YO
129	O!
130	!␣
131	␣Y
132	YOU
133	
134	
135	
136	
137	
138	
139	



LZW encoding – Example

Input: `YO!␣YOU!␣YOUR␣YOYO!`

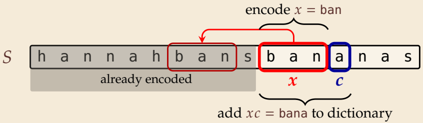
Σ_S = ASCII character set (0–127)

	Y	O	!	␣	YO	U
$C =$	89	79	33	32	128	85

$D =$

Code	String
...	
32	␣
33	!
...	
79	O
...	
82	R
...	
85	U
...	
89	Y
...	

Code	String
128	YO
129	O!
130	!␣
131	␣Y
132	YOU
133	
134	
135	
136	
137	
138	
139	



LZW encoding – Example

Input: `YO!␣YOU!␣YOUR␣YOYO!`

Σ_S = ASCII character set (0–127)

	Y	O	!	␣	YO	U
$C =$	89	79	33	32	128	85

$D =$

Code	String
...	
32	␣
33	!
...	
79	O
...	
82	R
...	
85	U
...	
89	Y
...	

Code	String
128	YO
129	O!
130	!␣
131	␣Y
132	YOU
133	U!
134	
135	
136	
137	
138	
139	

encode x = ban

S `h a n n a h b a n s b a n a n a s`

already encoded x c

add xc = bana to dictionary

LZW encoding – Example

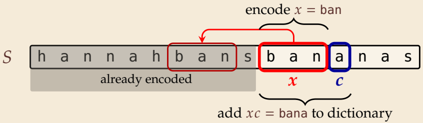
Input: `YO!␣YOU!␣YOUR␣YOYO!` Σ_S = ASCII character set (0–127)

	Y	O	!	␣	YO	U	!␣
$C =$	89	79	33	32	128	85	130

$D =$

Code	String
...	
32	␣
33	!
...	
79	O
...	
82	R
...	
85	U
...	
89	Y
...	

Code	String
128	YO
129	O!
130	!␣
131	␣Y
132	YOU
133	U!
134	
135	
136	
137	
138	
139	



LZW encoding – Example

Input: `YO!␣YOU!␣YOUR␣YOYO!`

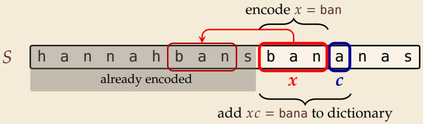
Σ_S = ASCII character set (0–127)

	Y	O	!	␣	YO	U	!␣
$C =$	89	79	33	32	128	85	130

$D =$

Code	String
...	
32	␣
33	!
...	
79	O
...	
82	R
...	
85	U
...	
89	Y
...	

Code	String
128	YO
129	O!
130	!␣
131	␣Y
132	YOU
133	U!
134	!␣Y
135	
136	
137	
138	
139	



LZW encoding – Example

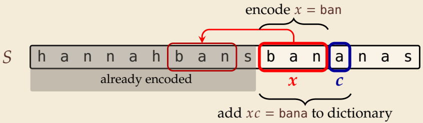
Input: `YO!␣YOU!␣YOUR␣YOYO!` Σ_S = ASCII character set (0–127)

	Y	O	!	␣	YO	U	!␣	YOU
$C =$	89	79	33	32	128	85	130	132

$D =$

Code	String
...	
32	␣
33	!
...	
79	O
...	
82	R
...	
85	U
...	
89	Y
...	

Code	String
128	YO
129	O!
130	!␣
131	␣Y
132	YOU
133	U!
134	!␣Y
135	
136	
137	
138	
139	



LZW encoding – Example

Input: `YO!␣YOU!␣YOUR␣YOYO!`

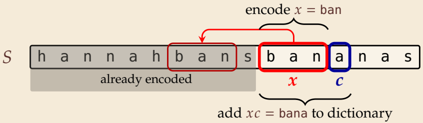
Σ_S = ASCII character set (0–127)

	Y	O	!	␣	YO	U	!␣	YOU
C =	89	79	33	32	128	85	130	132

D =

Code	String
...	
32	␣
33	!
...	
79	O
...	
82	R
...	
85	U
...	
89	Y
...	

Code	String
128	YO
129	O!
130	!␣
131	␣Y
132	YOU
133	U!
134	!␣Y
135	YOUR
136	
137	
138	
139	



LZW encoding – Example

Input: `YO!␣YOU!␣YOUR␣YOYO!`

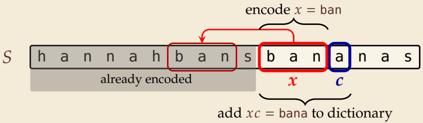
Σ_S = ASCII character set (0–127)

	Y	O	!	␣	YO	U	!␣	YOU	R
C =	89	79	33	32	128	85	130	132	82

D =

Code	String
...	
32	␣
33	!
...	
79	O
...	
82	R
...	
85	U
...	
89	Y
...	

Code	String
128	YO
129	O!
130	!␣
131	␣Y
132	YOU
133	U!
134	!␣Y
135	YOUR
136	
137	
138	
139	



LZW encoding – Example

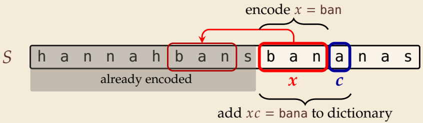
Input: `YO!␣YOU!␣YOUR␣YOYO!` $\qquad \Sigma_S$ = ASCII character set (0–127)

	Y	O	!	␣	YO	U	!␣	YOU	R
C =	89	79	33	32	128	85	130	132	82

D =

Code	String
...	
32	␣
33	!
...	
79	O
...	
82	R
...	
85	U
...	
89	Y
...	

Code	String
128	YO
129	O!
130	!␣
131	␣Y
132	YOU
133	U!
134	!␣Y
135	YOUR
136	R␣
137	
138	
139	



LZW encoding – Example

Input: `YO!␣YOU!␣YOUR␣YOYO!`

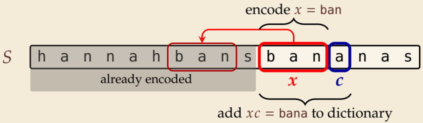
Σ_S = ASCII character set (0–127)

	Y	O	!	␣	YO	U	!␣	YOU	R	␣Y
C =	89	79	33	32	128	85	130	132	82	131

D =

Code	String
...	
32	␣
33	!
...	
79	O
...	
82	R
...	
85	U
...	
89	Y
...	

Code	String
128	YO
129	O!
130	!␣
131	␣Y
132	YOU
133	U!
134	!␣Y
135	YOUR
136	R␣
137	
138	
139	



LZW encoding – Example

Input: `YO!␣YOU!␣YOUR␣YOYO!`

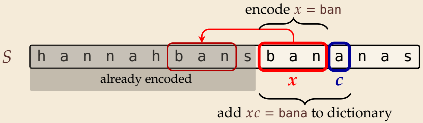
Σ_S = ASCII character set (0–127)

	Y	O	!	␣	YO	U	!␣	YOU	R	␣Y
C =	89	79	33	32	128	85	130	132	82	131

D =

Code	String
...	
32	␣
33	!
...	
79	O
...	
82	R
...	
85	U
...	
89	Y
...	

Code	String
128	YO
129	O!
130	!␣
131	␣Y
132	YOU
133	U!
134	!␣Y
135	YOUR
136	R␣
137	␣YO
138	
139	



LZW encoding – Example

Input: `YO!␣YOU!␣YOUR␣YOYO!`

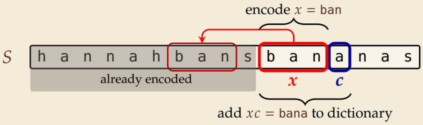
Σ_S = ASCII character set (0–127)

	Y	O	!	␣	YO	U	!␣	YOU	R	␣Y	O
$C =$	89	79	33	32	128	85	130	132	82	131	79

$D =$

Code	String
...	
32	␣
33	!
...	
79	O
...	
82	R
...	
85	U
...	
89	Y
...	

Code	String
128	YO
129	O!
130	!␣
131	␣Y
132	YOU
133	U!
134	!␣Y
135	YOUR
136	R␣
137	␣YO
138	
139	



LZW encoding – Example

Input: `YO!␣YOU!␣YOUR␣YOYO!`

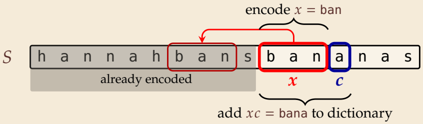
Σ_S = ASCII character set (0–127)

	Y	O	!	␣	YO	U	!␣	YOU	R	␣Y	O
$C =$	89	79	33	32	128	85	130	132	82	131	79

$D =$

Code	String
...	
32	␣
33	!
...	
79	O
...	
82	R
...	
85	U
...	
89	Y
...	

Code	String
128	YO
129	O!
130	!␣
131	␣Y
132	YOU
133	U!
134	!␣Y
135	YOUR
136	R␣
137	␣YO
138	OY
139	



LZW encoding – Example

Input: `YO!␣YOU!␣YOUR␣YOYO!`

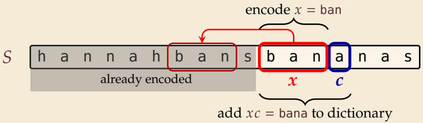
Σ_S = ASCII character set (0–127)

	Y	O	!	␣	YO	U	!␣	YOU	R	␣Y	O	YO
$C =$	89	79	33	32	128	85	130	132	82	131	79	128

$D =$

Code	String
...	
32	␣
33	!
...	
79	O
...	
82	R
...	
85	U
...	
89	Y
...	

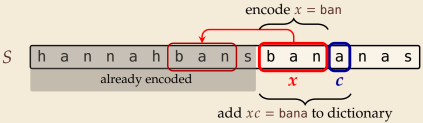
Code	String
128	YO
129	O!
130	!␣
131	␣Y
132	YOU
133	U!
134	!␣Y
135	YOUR
136	R␣
137	␣YO
138	OY
139	



LZW encoding – Example

Input: `YO!␣YOU!␣YOUR␣YOYO!` $\qquad\qquad \Sigma_S$ = ASCII character set (0–127)

	Y	O	!	␣	YO	U	!␣	YOU	R	␣Y	O	YO
C =	89	79	33	32	128	85	130	132	82	131	79	128



D =

Code	String
. . .	
32	␣
33	!
. . .	
79	O
. . .	
82	R
. . .	
85	U
. . .	
89	Y
. . .	

Code	String
128	YO
129	O!
130	!␣
131	␣Y
132	YOU
133	U!
134	!␣Y
135	YOUR
136	R␣
137	␣YO
138	OY
139	YO!

LZW encoding – Example

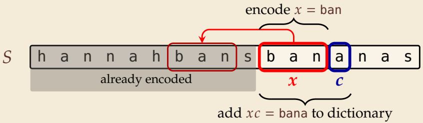
Input: `YO!␣YOU!␣YOUR␣YOYO!` Σ_S = ASCII character set (0–127)

	Y	O	!	␣	YO	U	!␣	YOU	R	␣Y	O	YO	!
C =	89	79	33	32	128	85	130	132	82	131	79	128	33

D =

Code	String
. . .	
32	␣
33	!
. . .	
79	O
. . .	
82	R
. . .	
85	U
. . .	
89	Y
. . .	

Code	String
128	YO
129	O!
130	!␣
131	␣Y
132	YOU
133	U!
134	!␣Y
135	YOUR
136	R␣
137	␣YO
138	OY
139	YO!



LZW encoding – Code

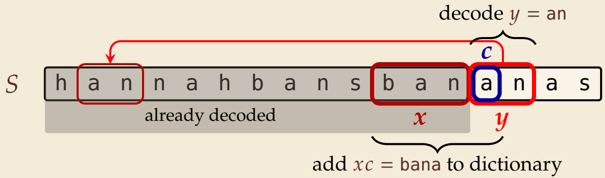
```
procedure LZWencode(S[0..n))
    x := ε // previous phrase, initially empty
    C := ε // output, initially empty
    D := dictionary, initialized with codes for c ∈ Σ_S // stored as trie (⇝ Unit 12 13)
    k := |Σ_S| // next free codeword
    for i := 0, . . . , n − 1 do
        c := S[i]
        if D.containsKey(xc) then
            x := xc
        else
            C := C · D.get(x) // append codeword for x
            D.put(xc, k) // add xc to D, assigning next free codeword
            k := k + 1;  x := c
    end for
    C := C · D.get(x)
    return C
```

7.7 Lempel-Ziv-Welch Decoding

LZW decoding

- Decoder has to replay the process of growing the dictionary!

⇝ **Decoding:**
after decoding a substring y of S, add xc to D,
where x is previously encoded/decoded substring of S,
and $c = y[0]$ (first character of y)



⇝ Note: only start adding to D after *second* substring of S is decoded

LZW decoding – Example

- Same idea: build dictionary while reading string.
- Example: 67 65 78 32 66 129 133

$D =$

Code #	String
...	
32	␣
...	
...	
65	A
66	B
67	C
...	
78	N
...	
83	S
...	

input	decodes to	Code #	String (human)	String (computer)

LZW decoding – Example

- Same idea: build dictionary while reading string.
- Example: 67 65 78 32 66 129 133

$D =$

Code #	String
...	
32	␣
...	
...	
65	A
66	B
67	C
...	
78	N
...	
83	S
...	

input	decodes to	Code #	String (human)	String (computer)
67	C			

LZW decoding – Example

- Same idea: build dictionary while reading string.
- Example: 67 65 78 32 66 129 133

$D =$

Code #	String
...	
32	␣
...	
...	
65	A
66	B
67	C
...	
78	N
...	
83	S
...	

input	decodes to	Code #	String (human)	String (computer)
67	C			
65	A	128	CA	67, A

LZW decoding – Example

- Same idea: build dictionary while reading string.
- Example: 67 65 78 32 66 129 133

$D =$

Code #	String
. . .	
32	␣
. . .	
. . .	
65	A
66	B
67	C
. . .	
78	N
. . .	
83	S
. . .	

input	decodes to	Code #	String (human)	String (computer)
67	C			
65	A	128	CA	67, A
78	N	129	AN	65, N

LZW decoding – Example

- Same idea: build dictionary while reading string.
- Example: 67 65 78 32 66 129 133

$D =$

Code #	String
...	
32	␣
...	
...	
65	A
66	B
67	C
...	
78	N
...	
83	S
...	

input	decodes to	Code #	String (human)	String (computer)
67	C			
65	A	128	CA	67, A
78	N	129	AN	65, N
32	␣	130	N␣	78, ␣

LZW decoding – Example

- Same idea: build dictionary while reading string.
- Example: 67 65 78 32 66 129 133

$D =$

Code #	String
. . .	
32	␣
. . .	
. . .	
65	A
66	B
67	C
. . .	
78	N
. . .	
83	S
. . .	

input	decodes to	Code #	String (human)	String (computer)
67	C			
65	A	128	CA	67, A
78	N	129	AN	65, N
32	␣	130	N␣	78, ␣
66	B	131	␣B	32, B

LZW decoding – Example

- Same idea: build dictionary while reading string.
- Example: 67 65 78 32 66 129 133

$D =$

Code #	String
...	
32	␣
...	
...	
65	A
66	B
67	C
...	
78	N
...	
83	S
...	

input	decodes to	Code #	String (human)	String (computer)
67	C			
65	A	128	CA	67, A
78	N	129	AN	65, N
32	␣	130	N␣	78, ␣
66	B	131	␣B	32, B
129	AN	132	BA	66, A

LZW decoding – Example

- Same idea: build dictionary while reading string.
- Example: 67 65 78 32 66 129 133

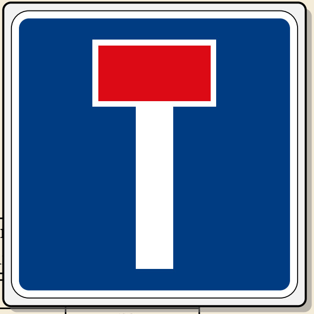
$D =$

Code #	String
. . .	
32	␣
. . .	
. . .	
65	A
66	B
67	C
. . .	
78	N
. . .	
83	S
. . .	

input	decodes to	Code #	String (human)	String (computer)
67	C			
65	A	128	CA	67, A
78	N	129	AN	65, N
32	␣	130	N␣	78, ␣
66	B	131	␣B	32, B
129	AN	132	BA	66, A
133	???	133		

LZW decoding – Example

- Same idea: build dictionary while reading string.
- Example: 67 65 78 32 66 129 133



$D =$

Code #	String
...	
32	␣
...	
...	
65	A
66	B
67	C
...	
78	N
...	
83	S
...	

input	decodes to	Code #	St (hu	
67	C			
65	A	128	CA	67, A
78	N	129	AN	65, N
32	␣	130	N␣	78, ␣
66	B	131	␣B	32, B
129	AN	132	BA	66, A
133	???	133		

LZW decoding – Bootstrapping

- example: Want to decode 133, but not yet in dictionary!

⚠ decoder is "one step behind" in creating dictionary

LZW decoding – Bootstrapping

- example: Want to decode 133, but not yet in dictionary!

⚠ decoder is "one step behind" in creating dictionary

⇝ problem occurs if *we want to use a code* that we are *just about to build*.

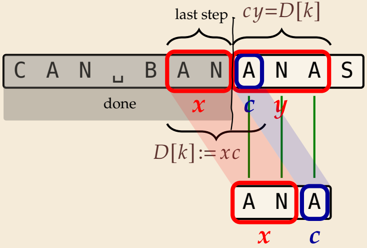
LZW decoding – Bootstrapping

- example: Want to decode 133, but not yet in dictionary!

⚠ decoder is "one step behind" in creating dictionary

⇝ problem occurs if *we want to use a code* that we are *just about to build*.

- *But then we actually know what is going on!*
  - Situation: decode using k in the step that will define k.
  - decoder knows last phrase x, needs phrase $y = D[k] = xc$.



1. en/decode x.

2. store $D[k] := xc$

3. next phrase y equals $D[k]$

⇝ $D[k] = xc = \underline{x \cdot x[0]}$ (all known)

LZW decoding – Code

```
procedure LZWdecode(C[0..m))
    D := dictionary [0..2^d) → Σ_S^+, initialized with codes for c ∈ Σ_S // stored as array
    k := |Σ_S| // next unused codeword
    q := C[0] // first codeword
    y := D[q] // lookup meaning of q in D
    S := y // output, initially first phrase
    for j := 1, . . . , m − 1 do
        x := y // remember last decoded phrase
        q := C[j] // next codeword
        if q == k then
            y := x · x[0] // bootstrap case
        else
            y := D[q]
        S := S · y // append decoded phrase
        D[k] := x · y[0] // store new phrase
        k := k + 1
    end for
    return S
```

LZW decoding – Example continued

- Example: 67 65 78 32 66 129 133 83

$D =$

Code #	String
. . .	
32	␣
. . .	
. . .	
65	A
66	B
67	C
. . .	
78	N
. . .	
83	S
. . .	

input	decodes to	Code #	String (human)	String (computer)
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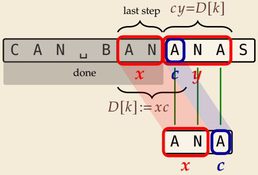
78	N	129	AN	65, N
32	␣	130	N␣	78, ␣
66	B	131	␣B	32, B
129	AN	132	BA	66, A
133	ANA	133	ANA	129, A



1. en/decode x.

2. store $D[k] := xc$

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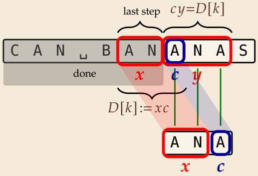
LZW decoding – Example continued

- Example: 67 65 78 32 66 129 133 83

$D =$

Code #	String
…	
32	␣
…	
…	
65	A
66	B
67	C
…	
78	N
…	
83	S
…	

input	decodes to	Code #	String (human)	String (computer)
67	C			
65	A	128	CA	67, A
78	N	129	AN	65, N
32	␣	130	N␣	78, ␣
66	B	131	␣B	32, B
129	AN	132	BA	66, A
133	ANA	133	ANA	129, A
83	S	134	ANAS	133, S



1. en/decode x.

2. store $D[k] := xc$

3. next phrase y equals $D[k]$
   $\rightsquigarrow$ $D[k] = xc = x \cdot x[0]$ (all known)

Clicker Question

How many phrases will LZW create on $S = \texttt{a}^n$, a run of n copies of as?

- A $\sim n$
- B $\sim n/2$
- C $\sim n/4$
- D $\Theta(n/\log n)$
- E $\Theta(\sqrt{n})$
- F $\Theta(\log n)$
- G $\Theta(\log\log n)$
- H 2
- I 1

→ sli.do/cs566

Clicker Question

a aa aaa a a a a

How many phrases will LZW create on $S = \mathtt{a}^n$, a run of n copies of as?

- A $\cancel{\sim n}$
- B $\cancel{\sim n/2}$
- C $\cancel{\sim n/4}$
- D $\cancel{\Theta(n/\log n)}$
- E $\Theta(\sqrt{n})$ ✓
- F $\cancel{\Theta(\log n)}$
- G $\cancel{\Theta(\log \log n)}$
- H $\cancel{2}$
- I $\cancel{1}$

→ sli.do/cs566

LZW – Discussion

- As presented, LZW uses coded alphabet $\Sigma_C = [0..2^d)$.
  - $\rightsquigarrow$ use another encoding for code numbers $\mapsto$ binary, e. g., Huffman
- need a rule when dictionary is full; different options:
  - increment d $\rightsquigarrow$ longer codewords
  - "flush" dictionary and start from scratch $\rightsquigarrow$ limits extra space usage
  - often: reserve a codeword to trigger flush at any time
- encoding and decoding both run in linear time (assuming $|\Sigma_S|$ constant)

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  - often: reserve a codeword to trigger flush at any time
- encoding and decoding both run in linear time (assuming $|\Sigma_S|$ constant)

- 👍 fast encoding & decoding
- 👍 works in streaming model (no random access, no backtrack on input needed)
- 👍 significant compression for many types of data

- 👎 captures only local repetitions (with bounded dictionary)

Compression summary

Huffman codes	Run-length encoding	Lempel-Ziv-Welch
fixed-to-variable	variable-to-variable	variable-to-fixed
2-pass	1-pass	1-pass
must send dictionary	can be worse than ASCII	can be worse than ASCII
60% compression on English text	bad on text	45% compression on English text
optimal binary character encopding	good on long runs (e.g., pictures)	good on English text
rarely used directly	rarely used directly	frequently used
part of pkzip, JPEG, MP3	fax machines, old picture-formats	GIF, part of PDF, Unix compress

Part III

Text Transforms

Text transformations

- compression is effective if we have one the following:
  - long runs $\rightsquigarrow$ RLE
  - frequently used characters $\rightsquigarrow$ Huffman
  - many (locally) repeated substrings $\rightsquigarrow$ LZW

Text transformations

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  - many (locally) repeated substrings $\rightsquigarrow$ LZW
- but methods can be frustratingly "blind" to other "obvious" redundancies
  - LZW: repetition too distant $\lightning$ dictionary already flushed
  - Huffman: changing probabilities (local clusters) $\lightning$ averaged out globally
  - RLE: run of alternating pairs of characters $\lightning$ not a run

Text transformations

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- but methods can be frustratingly “blind” to other “obvious” redundancies
  - LZW: repetition too distant ϟ dictionary already flushed
  - Huffman: changing probabilities (local clusters) ϟ averaged out globally
  - RLE: run of alternating pairs of characters ϟ not a run
- Enter: **text transformations**
  - invertible functions of text
  - do not by themselves reduce the space usage
  - but help compressors “see” existing redundancy

$\rightsquigarrow$ use as pre-/postprocessing in a compression pipeline

7.8 Move-to-Front Transformation

Move to Front

- *Move to Front (MTF)* is a heuristic for *self-adjusting linked lists*
  - unsorted linked list of objects
  - whenever an element is accessed, it is moved to the front of the list (leaving the relative order of other elements unchanged)

  ⇝ list "learns" probabilities of access to objects
  makes access to frequently requested objects cheaper

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- Here: use such a list for storing source alphabet Σ_S
  - to encode c, access it in list
  - encode c using its (old) **position in list**
  - then apply MTF to the list
  - $\rightsquigarrow$ codewords are integers, i. e., $\Sigma_C = [0..\sigma)$

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  - encode c using its (old) **position in list**
  - then apply MTF to the list
  - $\rightsquigarrow$ codewords are integers, i. e., $\Sigma_C = [0..\sigma)$

$\rightsquigarrow$ clusters of few characters $\rightsquigarrow$ many small numbers

MTF – Code

- **Transform (encode):**

```
procedure MTF−encode(S[0..n))
    L := list containing Σ_S (sorted order)
    C := ε
    for i := 0, ..., n − 1 do
        c := S[i]
        p := position of c in L
        C := C · p
        Move c to front of L
    end for
    return C
```

- **Inverse transform (decode):**

```
procedure MTF−decode(C[0..m))
    L := list containing Σ_S (sorted order)
    S := ε
    for j := 0, ..., m − 1 do
        p := C[j]
        c := character at position p in L
        S := S · c
        Move c to front of L
    end for
    return S
```

- Important: encoding and decoding produce same accesses to list

MTF – Example

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z

$S =$ I N E F F I C I E N C I E S

$C =$

MTF – Example

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z

S = I N E F F I C I E N C I E S

C = 8

MTF – Example

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
I	A	B	C	D	E	F	G	H	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z

$S =$ I N E F F I C I E N C I E S

$C =$ 8 13

MTF – Example

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
N	I	A	B	C	D	E	F	G	H	J	K	L	M	O	P	Q	R	S	T	U	V	W	X	Y	Z

$S =$ I N E F F I C I E N C I E S

$C =$ 8 13 6

MTF – Example

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
E	N	I	A	B	C	D	F	G	H	J	K	L	M	O	P	Q	R	S	T	U	V	W	X	Y	Z

$S =$ `I N E F F I C I E N C I E S`

$C = 8\ 13\ 6\ 7$

MTF – Example

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
F	E	N	I	A	B	C	D	G	H	J	K	L	M	O	P	Q	R	S	T	U	V	W	X	Y	Z

$S =$ I N E F F I C I E N C I E S

$C =$ 8 13 6 7 0

MTF – Example

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
F	E	N	I	A	B	C	D	G	H	J	K	L	M	O	P	Q	R	S	T	U	V	W	X	Y	Z

$S =$ I N E F F I C I E N C I E S

$C =$ 8 13 6 7 0 3

MTF – Example

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
I	F	E	N	A	B	C	D	G	H	J	K	L	M	O	P	Q	R	S	T	U	V	W	X	Y	Z

$S =$ I N E F F I C I E N C I E S

$C =$ 8 13 6 7 0 3 6

MTF – Example

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
C	I	F	E	N	A	B	D	G	H	J	K	L	M	O	P	Q	R	S	T	U	V	W	X	Y	Z

$S =$ I N E F F I C I E N C I E S

$C =$ 8 13 6 7 0 3 6 1

MTF – Example

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
S	E	I	C	N	F	A	B	D	G	H	J	K	L	M	O	P	Q	R	T	U	V	W	X	Y	Z

$S =$ I N E F F I C I E N C I E S

$C =$ 8 13 6 7 0 3 6 1 3 4 3 3 3 18

- What does a run in S encode to in C?
- What does a run in C mean about the source S?

MTF – Discussion

- MTF itself does not compress text (if we store codewords with fixed length)
  - $\rightsquigarrow$ used as part of longer pipeline
- Intuitively effect:
  MTF converts locally low empirical entropy to globally low empirical entropy(!)
  - $\rightsquigarrow$ makes Huffman coding much more effective!
  - cheaper option: Elias gamma code
    - $\rightsquigarrow$ smaller numbers gets shorter codewords
      works well for text with small "local effective" alphabet

👎 many natural texts do not have locally low empirical entropy

👍 but we can often make it so … stay tuned ($\rightarrow$ BWT)

7.9 Burrows-Wheeler Transform

Burrows-Wheeler Transform

- Burrows-Wheeler Transform (BWT) is a sophisticated text-transformation technique.
  - coded text has same letters as source, just in a different order
  - But: coded text is (typically) more compressible (local char frequencies)

Burrows-Wheeler Transform

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  - coded text has same letters as source, just in a different order
  - But: coded text is (typically) more compressible (local char frequencies)
- Encoding algorithm needs **all** of S (no streaming possible).

  $\rightsquigarrow$ BWT is a *block compression method*.

Burrows-Wheeler Transform

- Burrows-Wheeler Transform (BWT) is a sophisticated text-transformation technique.
  - coded text has same letters as source, just in a different order
  - But: coded text is (typically) more compressible (local char frequencies)

- Encoding algorithm needs **all** of S (no streaming possible).
  - ⇝ BWT is a *block compression method*.

- BWT followed by MTF, RLE, and Huffman is the algorithm used by the `bzip2` program. achieves best compression on English text of any algorithm we have seen:

```
4047392 bible.txt         # original
1191071 bible.txt.gz      # gzip      (0.2s)
 888604 bible.txt.7z      # 7z        (2s)
 845635 bible.txt.bz2     # bzip2     (0.3s)

 632634 bible.txt.paq8l   # paq8l -8  (6min!)
```

Hutter Prize

BWT – Definitions

- *cyclic shift* of a string: $T =$ `time␣flies␣quickly␣` `flies␣quickly␣time␣`

⇝ cyclic shift

BWT – Definitions

- *cyclic shift* of a string: $T =$ `time␣flies␣quickly␣` `flies␣quickly␣time␣`

$\rightsquigarrow$ cyclic shift

- add *end-of-word character* $\$$ to S (always assumed in this section!)

$\rightsquigarrow$ can **recover** original string

BWT – Definitions

- *cyclic shift* of a string: T = `time␣flies␣quickly␣`   `flies␣quickly␣time␣`

⇝ cyclic shift

- add *end-of-word character* `$` to S (always assumed in this section!)

⇝ can **recover** original string

- The Burrows-Wheeler Transform proceeds in three steps:

  0. Append end-of-word character `$` to S.

  1. Place *all cyclic shifts* of S in a list L

  2. Sort the strings in L lexicographically

  3. B is the *list of trailing characters* (last column, top-down) of each string in L

BWT – Example

S = `alf␣eats␣alfalfa$`

1. Take all cyclic shifts of S

```
alf␣eats␣alfalfa$
lf␣eats␣alfalfa$a
f␣eats␣alfalfa$al
␣eats␣alfalfa$alf
eats␣alfalfa$alf␣
ats␣alfalfa$alf␣e
ts␣alfalfa$alf␣ea
s␣alfalfa$alf␣eat
␣alfalfa$alf␣eats
alfalfa$alf␣eats␣
lfalfa$alf␣eats␣a
falfa$alf␣eats␣al
alfa$alf␣eats␣alf
lfa$alf␣eats␣alfa
fa$alf␣eats␣alfal
a$alf␣eats␣alfalf
$alf␣eats␣alfalfa
```

$\overset{}{\rightsquigarrow}$ sort

BWT – Example

$\$ < c \in \Sigma$

S = alf␣eats␣alfalfa$

1. Take all cyclic shifts of S
2. Sort cyclic shifts

```
alf␣eats␣alfalfa$
lf␣eats␣alfalfa$a
f␣eats␣alfalfa$al
␣eats␣alfalfa$alf
eats␣alfalfa$alf␣
ats␣alfalfa$alf␣e
ts␣alfalfa$alf␣ea
s␣alfalfa$alf␣eat
␣alfalfa$alf␣eats
alfalfa$alf␣eats␣
lfalfa$alf␣eats␣a
falfa$alf␣eats␣al
alfa$alf␣eats␣alf
lfa$alf␣eats␣alfa
fa$alf␣eats␣alfal
a$alf␣eats␣alfalf
$alf␣eats␣alfalfa
```

⇝ sort

```
$alf␣eats␣alfalfa
␣alfalfa$alf␣eats
␣eats␣alfalfa$alf
a$alf␣eats␣alfalf
alf␣eats␣alfalfa$
alfa$alf␣eats␣alf
alfalfa$alf␣eats␣
ats␣alfalfa$alf␣e
eats␣alfalfa$alf␣
f␣eats␣alfalfa$al
fa$alf␣eats␣alfal
falfa$alf␣eats␣al
lf␣eats␣alfalfa$a
lfa$alf␣eats␣alfa
lfalfa$alf␣eats␣a
s␣alfalfa$alf␣eat
ts␣alfalfa$alf␣ea
```

BWT – Example

$S =$ `alf␣eats␣alfalfa$`

1. Take all cyclic shifts of S
2. Sort cyclic shifts
3. Extract last column

$B =$ `asff$f␣e␣lllaaata`

```
alf␣eats␣alfalfa$
lf␣eats␣alfalfa$a
f␣eats␣alfalfa$al
␣eats␣alfalfa$alf
eats␣alfalfa$alf␣
ats␣alfalfa$alf␣e
ts␣alfalfa$alf␣ea
s␣alfalfa$alf␣eat
␣alfalfa$alf␣eats
alfalfa$alf␣eats␣
lfalfa$alf␣eats␣a
falfa$alf␣eats␣al
alfa$alf␣eats␣alf
lfa$alf␣eats␣alfa
fa$alf␣eats␣alfal
a$alf␣eats␣alfalf
$alf␣eats␣alfalfa
```

sort

BWT

```
$alf␣eats␣alfalfa
␣alfalfa$alf␣eats
␣eats␣alfalfa$alf
a$alf␣eats␣alfalf
alf␣eats␣alfalfa$
alfa$alf␣eats␣alf
alfalfa$alf␣eats␣
ats␣alfalfa$alf␣e
eats␣alfalfa$alf␣
f␣eats␣alfalfa$al
fa$alf␣eats␣alfal
falfa$alf␣eats␣al
lf␣eats␣alfalfa$a
lfa$alf␣eats␣alfa
lfalfa$alf␣eats␣a
s␣alfalfa$alf␣eat
ts␣alfalfa$alf␣ea
```

BWT – Example

$S =$ `alf␣eats␣alfalfa$`

1. Take all cyclic shifts of S
2. Sort cyclic shifts
3. Extract last column

$B =$ `asff$f␣e␣lllaaata`

```
alf␣eats␣alfalfa$
lf␣eats␣alfalfa$a
f␣eats␣alfalfa$al
␣eats␣alfalfa$alf
eats␣alfalfa$alf␣
ats␣alfalfa$alf␣e
ts␣alfalfa$alf␣ea
s␣alfalfa$alf␣eat
␣alfalfa$alf␣eats
alfalfa$alf␣eats␣
lfalfa$alf␣eats␣a
falfa$alf␣eats␣al
alfa$alf␣eats␣alf
lfa$alf␣eats␣alfa
fa$alf␣eats␣alfal
a$alf␣eats␣alfalf
$alf␣eats␣alfalfa
```

$\rightsquigarrow$ sort

BWT ↓

```
$alf␣eats␣alfalfa
␣alfalfa$alf␣eats
␣eats␣alfalfa$alf
a$alf␣eats␣alfalf
alf␣eats␣alfalfa$
alfa$alf␣eats␣alf
alfalfa$alf␣eats␣
ats␣alfalfa$alf␣e
eats␣alfalfa$alf␣
f␣eats␣alfalfa$al
fa$alf␣eats␣alfal
falfa$alf␣eats␣al
lf␣eats␣alfalfa$a
lfa$alf␣eats␣alfa
lfalfa$alf␣eats␣a
s␣alfalfa$alf␣eat
ts␣alfalfa$alf␣ea
```

- BWT can be computed in $O(n)$ time!
  - totally non-obvious from definition (naive sorting could take $\Omega(n^2)$ time in worst case!)
  - will use one of the most sophisticated algorithms we cover $\rightsquigarrow$ Unit 13!

BWT – Properties

Why does BWT help for compression?

- sorting *groups* characters *by what follows*
  - Example: `lf` always preceded by `a`
  - more generally: BWT can be partitioned into letters following a given context

⇝ repeated substring in S ⇝ *runs* in B

- Example: `alf` ⇝ run of `a`s
- picked up by RLE

(formally: low higher-order empirical entropy)

⇝ *If S allows predicting symbols from context, B has locally low entropy of characters.*

- that makes MTF effective!

	r	$\downarrow L[r]$	
`alf␣eats␣alfalfa$`	0	`$alf␣eats␣alfalfa`	16
`lf␣eats␣alfalfa$a`	1	`␣alfalfa$alf␣eats`	8
`f␣eats␣alfalfa$al`	2	`␣eats␣alfalfa$alf`	3
`␣eats␣alfalfa$alf`	3	`a$alf␣eats␣alfalf`	15
`eats␣alfalfa$alf␣`	4	`alf␣eats␣alfalfa$`	0
`ats␣alfalfa$alf␣e`	5	`alfa$alf␣eats␣alf`	12
`ts␣alfalfa$alf␣ea`	6	`alfalfa$alf␣eats␣`	9
`s␣alfalfa$alf␣eat`	7	`ats␣alfalfa$alf␣e`	5
`␣alfalfa$alf␣eats`	8	`eats␣alfalfa$alf␣`	4
`alfalfa$alf␣eats␣`	9	`f␣eats␣alfalfa$al`	2
`lfalfa$alf␣eats␣a`	10	`fa$alf␣eats␣alfal`	14
`falfa$alf␣eats␣al`	11	`falfa$alf␣eats␣al`	11
`alfa$alf␣eats␣alf`	12	`lf␣eats␣alfalfa$a`	1
`lfa$alf␣eats␣alfa`	13	`lfa$alf␣eats␣alfa`	13
`fa$alf␣eats␣alfal`	14	`lfalfa$alf␣eats␣a`	10
`a$alf␣eats␣alfalf`	15	`s␣alfalfa$alf␣eat`	7
`$alf␣eats␣alfalfa`	16	`ts␣alfalfa$alf␣ea`	6

A Bigger Example

```
have␣had␣hadnt␣hasnt␣havent␣has␣what$
ave␣had␣hadnt␣hasnt␣havent␣has␣what$h
ve␣had␣hadnt␣hasnt␣havent␣has␣what$ha
e␣had␣hadnt␣hasnt␣havent␣has␣what$hav
␣had␣hadnt␣hasnt␣havent␣has␣what$have
had␣hadnt␣hasnt␣havent␣has␣what$have␣
ad␣hadnt␣hasnt␣havent␣has␣what$have␣h
d␣hadnt␣hasnt␣havent␣has␣what$have␣ha
␣hadnt␣hasnt␣havent␣has␣what$have␣had
hadnt␣hasnt␣havent␣has␣what$have␣had␣
adnt␣hasnt␣havent␣has␣what$have␣had␣h
dnt␣hasnt␣havent␣has␣what$have␣had␣ha
nt␣hasnt␣havent␣has␣what$have␣had␣had
t␣hasnt␣havent␣has␣what$have␣had␣hadn
␣hasnt␣havent␣has␣what$have␣had␣hadnt
hasnt␣havent␣has␣what$have␣had␣hadnt␣
asnt␣havent␣has␣what$have␣had␣hadnt␣h
snt␣havent␣has␣what$have␣had␣hadnt␣ha
nt␣havent␣has␣what$have␣had␣hadnt␣has
t␣havent␣has␣what$have␣had␣hadnt␣hasn
␣havent␣has␣what$have␣had␣hadnt␣hasnt
havent␣has␣what$have␣had␣hadnt␣hasnt␣
avent␣has␣what$have␣had␣hadnt␣hasnt␣h
vent␣has␣what$have␣had␣hadnt␣hasnt␣ha
ent␣has␣what$have␣had␣hadnt␣hasnt␣hav
nt␣has␣what$have␣had␣hadnt␣hasnt␣have
t␣has␣what$have␣had␣hadnt␣hasnt␣haven
␣has␣what$have␣had␣hadnt␣hasnt␣havent
has␣what$have␣had␣hadnt␣hasnt␣havent␣
as␣what$have␣had␣hadnt␣hasnt␣havent␣h
s␣what$have␣had␣hadnt␣hasnt␣havent␣ha
␣what$have␣had␣hadnt␣hasnt␣havent␣has
what$have␣had␣hadnt␣hasnt␣havent␣has␣
hat$have␣had␣hadnt␣hasnt␣havent␣has␣w
at$have␣had␣hadnt␣hasnt␣havent␣has␣wh
t$have␣had␣hadnt␣hasnt␣havent␣has␣wha
$have␣had␣hadnt␣hasnt␣havent␣has␣what
```

```
$have␣had␣hadnt␣hasnt␣havent␣has␣what
␣had␣hadnt␣hasnt␣havent␣has␣what$have
␣hadnt␣hasnt␣havent␣has␣what$have␣had
␣has␣what$have␣had␣hadnt␣hasnt␣havent
␣hasnt␣havent␣has␣what$have␣had␣hadnt
␣havent␣has␣what$have␣had␣hadnt␣hasnt
␣what$have␣had␣hadnt␣hasnt␣havent␣has
ad␣hadnt␣hasnt␣havent␣has␣what$have␣h
adnt␣hasnt␣havent␣has␣what$have␣had␣h
as␣what$have␣had␣hadnt␣hasnt␣havent␣h
asnt␣havent␣has␣what$have␣had␣hadnt␣h
at$have␣had␣hadnt␣hasnt␣havent␣has␣wh
ave␣had␣hadnt␣hasnt␣havent␣has␣what$h
avent␣has␣what$have␣had␣hadnt␣hasnt␣h
d␣hadnt␣hasnt␣havent␣has␣what$have␣ha
dnt␣hasnt␣havent␣has␣what$have␣had␣ha
e␣had␣hadnt␣hasnt␣havent␣has␣what$hav
ent␣has␣what$have␣had␣hadnt␣hasnt␣hav
had␣hadnt␣hasnt␣havent␣has␣what$have␣
hadnt␣hasnt␣havent␣has␣what$have␣had␣
has␣what$have␣had␣hadnt␣hasnt␣havent␣
hasnt␣havent␣has␣what$have␣had␣hadnt␣
hat$have␣had␣hadnt␣hasnt␣havent␣has␣w
have␣had␣hadnt␣hasnt␣havent␣has␣what$
havent␣has␣what$have␣had␣hadnt␣hasnt␣
nt␣has␣what$have␣had␣hadnt␣hasnt␣have
nt␣hasnt␣havent␣has␣what$have␣had␣had
nt␣havent␣has␣what$have␣had␣hadnt␣has
s␣what$have␣had␣hadnt␣hasnt␣havent␣ha
snt␣havent␣has␣what$have␣had␣hadnt␣ha
t$have␣had␣hadnt␣hasnt␣havent␣has␣wha
t␣has␣what$have␣had␣hadnt␣hasnt␣haven
t␣hasnt␣havent␣has␣what$have␣had␣hadn
t␣havent␣has␣what$have␣had␣hadnt␣hasn
ve␣had␣hadnt␣hasnt␣havent␣has␣what$ha
vent␣has␣what$have␣had␣hadnt␣hasnt␣ha
what$have␣had␣hadnt␣hasnt␣havent␣has␣
```

$T =$ `h a v e ␣ h a d ␣ h a d n t ␣ h a s n t ␣ h a v e n t ␣ h a s ␣ w h a t $`

$B =$ `t e d t t t s h h h h h h h a a v v ␣ ␣ ␣ ␣ w $ ␣ e d s a a a n n n a a ␣`

$MTF(B) =$ 8 5 5 2 **0** **0** 8 7 **0** **0** **0** **0** **0** **0** 7 **0** 9 **0** 8 **0** **0** **0** 10 9 2 9 9 8 7 **0** **0** 10 **0** **0** 1 **0** 5

A Bigger Example

For T some English text, $MTF(B)$ has typically around 50% zeroes!

```
have␣had␣hadnt␣hasnt␣havent␣has␣what$
ave␣had␣hadnt␣hasnt␣havent␣has␣what$h
ve␣had␣hadnt␣hasnt␣havent␣has␣what$ha
e␣had␣hadnt␣hasnt␣havent␣has␣what$hav
␣had␣hadnt␣hasnt␣havent␣has␣what$have
had␣hadnt␣hasnt␣havent␣has␣what$have␣
ad␣hadnt␣hasnt␣havent␣has␣what$have␣h
d␣hadnt␣hasnt␣havent␣has␣what$have␣ha
␣hadnt␣hasnt␣havent␣has␣what$have␣had
hadnt␣hasnt␣havent␣has␣what$have␣had␣
adnt␣hasnt␣havent␣has␣what$have␣had␣h
dnt␣hasnt␣havent␣has␣what$have␣had␣ha
nt␣hasnt␣havent␣has␣what$have␣had␣had
t␣hasnt␣havent␣has␣what$have␣had␣hadn
␣hasnt␣havent␣has␣what$have␣had␣hadnt
hasnt␣havent␣has␣what$have␣had␣hadnt␣
asnt␣havent␣has␣what$have␣had␣hadnt␣h
snt␣havent␣has␣what$have␣had␣hadnt␣ha
nt␣havent␣has␣what$have␣had␣hadnt␣has
t␣havent␣has␣what$have␣had␣hadnt␣hasn
␣havent␣has␣what$have␣had␣hadnt␣hasnt
havent␣has␣what$have␣had␣hadnt␣hasnt␣
avent␣has␣what$have␣had␣hadnt␣hasnt␣h
vent␣has␣what$have␣had␣hadnt␣hasnt␣ha
ent␣has␣what$have␣had␣hadnt␣hasnt␣hav
nt␣has␣what$have␣had␣hadnt␣hasnt␣have
t␣has␣what$have␣had␣hadnt␣hasnt␣haven
␣has␣what$have␣had␣hadnt␣hasnt␣havent
has␣what$have␣had␣hadnt␣hasnt␣havent␣
as␣what$have␣had␣hadnt␣hasnt␣havent␣h
s␣what$have␣had␣hadnt␣hasnt␣havent␣ha
␣what$have␣had␣hadnt␣hasnt␣havent␣has
what$have␣had␣hadnt␣hasnt␣havent␣has␣
hat$have␣had␣hadnt␣hasnt␣havent␣has␣w
at$have␣had␣hadnt␣hasnt␣havent␣has␣wh
t$have␣had␣hadnt␣hasnt␣havent␣has␣wha
$have␣had␣hadnt␣hasnt␣havent␣has␣what
```

```
$have␣had␣hadnt␣hasnt␣havent␣has␣what
␣had␣hadnt␣hasnt␣havent␣has␣what$have
␣hadnt␣hasnt␣havent␣has␣what$have␣had
␣has␣what$have␣had␣hadnt␣hasnt␣havent
␣hasnt␣havent␣has␣what$have␣had␣hadnt
␣havent␣has␣what$have␣had␣hadnt␣hasnt
␣what$have␣had␣hadnt␣hasnt␣havent␣has
ad␣hadnt␣hasnt␣havent␣has␣what$have␣h
adnt␣hasnt␣havent␣has␣what$have␣had␣h
as␣what$have␣had␣hadnt␣hasnt␣havent␣h
asnt␣havent␣has␣what$have␣had␣hadnt␣h
at$have␣had␣hadnt␣hasnt␣havent␣has␣wh
ave␣had␣hadnt␣hasnt␣havent␣has␣what$h
avent␣has␣what$have␣had␣hadnt␣hasnt␣h
d␣hadnt␣hasnt␣havent␣has␣what$have␣ha
dnt␣hasnt␣havent␣has␣what$have␣had␣ha
e␣had␣hadnt␣hasnt␣havent␣has␣what$hav
ent␣has␣what$have␣had␣hadnt␣hasnt␣hav
had␣hadnt␣hasnt␣havent␣has␣what$have␣
hadnt␣hasnt␣havent␣has␣what$have␣had␣
has␣what$have␣had␣hadnt␣hasnt␣havent␣
hasnt␣havent␣has␣what$have␣had␣hadnt␣
hat$have␣had␣hadnt␣hasnt␣havent␣has␣w
have␣had␣hadnt␣hasnt␣havent␣has␣what$
havent␣has␣what$have␣had␣hadnt␣hasnt␣
nt␣has␣what$have␣had␣hadnt␣hasnt␣have
nt␣hasnt␣havent␣has␣what$have␣had␣had
nt␣havent␣has␣what$have␣had␣hadnt␣has
s␣what$have␣had␣hadnt␣hasnt␣havent␣ha
snt␣havent␣has␣what$have␣had␣hadnt␣ha
t$have␣had␣hadnt␣hasnt␣havent␣has␣wha
t␣has␣what$have␣had␣hadnt␣hasnt␣haven
t␣hasnt␣havent␣has␣what$have␣had␣hadn
t␣havent␣has␣what$have␣had␣hadnt␣hasn
ve␣had␣hadnt␣hasnt␣havent␣has␣what$ha
vent␣has␣what$have␣had␣hadnt␣hasnt␣ha
what$have␣had␣hadnt␣hasnt␣havent␣has␣
```

T = h a v e ␣ h a d ␣ h a d n t ␣ h a s n t ␣ h a v e n t ␣ h a s ␣ w h a t $

B = t e d t t t s h h h h h h h a a v v ␣ ␣ ␣ ␣ w $ ␣ e d s a a a n n n a a ␣

$MTF(B)$ = 8 5 5 2 **0** **0** 8 7 **0** **0** **0** **0** **0** **0** 7 **0** 9 **0** 8 **0** **0** **0** 10 9 2 9 9 8 7 **0** **0** 10 **0** **0** 1 **0** 5

Clicker Question

Consider T = `have␣had␣hadnt␣hasnt␣havent␣has␣what$`.
The BWT is B = `tedtttshhhhhhhaavv␣␣␣␣w$␣edsaaannnaa␣`.
How can we explain the long run of `h`s in B?

- A `h` is the most frequent character
- B `h` always appears at the beginning of a word
- C almost all words start with `h`
- D `h` is always followed by `a`
- E all `a`s are preceded by `h`
- F `h` is the 4th character in the alphabet

→ *sli.do/cs566*

Clicker Question

Consider T = `have␣had␣hadnt␣hasnt␣havent␣has␣what$`.
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How can we explain the long run of `h`s in B?

- **A** ~~`h` is the most frequent character~~
- **B** ~~`h` always appears at the beginning of a word~~
- **C** ~~almost all words start with `h`~~
- **D** ~~`h` is always followed by `a`~~
- **E** all `a`s are preceded by `h` ✓
- **F** ~~`h` is the 4th character in the alphabet~~

→ sli.do/cs566

Run-length BWT Compression

- amazingly, just run-length compressing the BWT is already powerful!
- r = number of runs in BWT
- $r = O(z \log^2(n))$, z number of LZ77 phrases
  proven in 2019 (!)

Example:

$S = \texttt{alf␣eats␣alfalfa\$}$

$B = \texttt{asff\$f␣e␣lllaaata}$

$RL(B) = \begin{bmatrix}\texttt{a}\\1\end{bmatrix}\begin{bmatrix}\texttt{s}\\1\end{bmatrix}\begin{bmatrix}\texttt{f}\\2\end{bmatrix}\begin{bmatrix}\texttt{\$}\\1\end{bmatrix}\begin{bmatrix}\texttt{f}\\1\end{bmatrix}\begin{bmatrix}\texttt{␣}\\1\end{bmatrix}\begin{bmatrix}\texttt{e}\\1\end{bmatrix}\begin{bmatrix}\texttt{␣}\\1\end{bmatrix}\begin{bmatrix}\texttt{l}\\3\end{bmatrix}\begin{bmatrix}\texttt{a}\\3\end{bmatrix}\begin{bmatrix}\texttt{t}\\1\end{bmatrix}\begin{bmatrix}\texttt{a}\\1\end{bmatrix}$

$\rightsquigarrow$ $r = |RL(B)| = 12$; $n = 17$

Larger Example:

$S = \texttt{have␣had␣hadnt␣hasnt␣havent␣has␣what\$}$

$\rightsquigarrow$ $r = 19$; $n = 36$

7.10 Inverse BWT

Inverse BWT

- Great, can compute BWT efficiently and it helps compression. *But how can we decode it?*

not even obvious that
it is at all invertible!

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- **"Magic" solution:**

  1. Create array $D[0..n]$ of pairs: $D[r] = (B[r], r)$.
  2. Sort D *stably* with respect to *first entry*.
  3. Use D as linked list with $(\text{char}, \text{next entry})$

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Example:
$B =$ `ard$rcaaaabb`
$S =$

	D
0	(a, 0)
1	(r, 1)
2	(d, 2)
3	($, 3)
4	(r, 4)
5	(c, 5)
6	(a, 6)
7	(a, 7)
8	(a, 8)
9	(a, 9)
10	(b, 10)
11	(b, 11)

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Example:

$B =$ `ard$rcaaaabb`

$S =$

	D
0	(a, 0)
1	(r, 1)
2	(d, 2)
3	($, 3)
4	(r, 4)
5	(c, 5)
6	(a, 6)
7	(a, 7)
8	(a, 8)
9	(a, 9)
10	(b, 10)
11	(b, 11)

	sorted D (char, next)
0	($, 3)
1	(a, 0)
2	(a, 6)
3	(a, 7)
4	(a, 8)
5	(a, 9)
6	(b, 10)
7	(b, 11)
8	(c, 5)
9	(d, 2)
10	(r, 1)
11	(r, 4)

Inverse BWT

- Great, can compute BWT efficiently and it helps compression. *But how can we decode it?*

not even obvious that it is at all invertible!

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  1. Create array $D[0..n]$ of pairs: $D[r] = (B[r], r)$.
  2. Sort D *stably* with respect to *first entry*.
  3. Use D as linked list with (char, next entry)

Example:

B = `ard$rcaaaabb`

S = `a`

	D		sorted D (char, next)
0	(a, 0)	0	($, 3)
1	(r, 1)	1	(a, 0)
2	(d, 2)	2	(a, 6)
3	($, 3)	3	(a, 7)
4	(r, 4)	4	(a, 8)
5	(c, 5)	5	(a, 9)
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9	(a, 9)	9	(d, 2)
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Example:

$B =$ `ard$rcaaaabb`

$S =$ `ab`

	D		sorted D (char, next)
0	(a, 0)	0	($, 3)
1	(r, 1)	1	(a, 0)
2	(d, 2)	2	(a, 6)
3	($, 3)	3	(a, 7)
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Example:

B = `ard$rcaaaabb`

S = `abr`

	D		sorted D (char, next)
0	(a, 0)	0	($, 3)
1	(r, 1)	1	(a, 0)
2	(d, 2)	2	(a, 6)
3	($, 3)	3	(a, 7)
4	(r, 4)	4	(a, 8)
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Example:

B = `ard$rcaaaabb`

S = `abra`

	D		sorted D
			char next
0	(a, 0)	0	($, 3)
1	(r, 1)	1	(a, 0)
2	(d, 2)	2	(a, 6)
3	($, 3)	3	(a, 7)
4	(r, 4)	4	(a, 8)
5	(c, 5)	5	(a, 9)
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Example:

$B =$ `ard$rcaaaabb`

$S =$ `abrac`

	D		sorted D (char, next)
0	(a, 0)	0	($, 3)
1	(r, 1)	1	(a, 0)
2	(d, 2)	2	(a, 6)
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Example:

B = `ard$rcaaaabb`

S = `abraca`

	D		sorted D (char, next)
0	(a, 0)	0	($, 3)
1	(r, 1)	1	(a, 0)
2	(d, 2)	2	(a, 6)
3	($, 3)	3	(a, 7)
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  3. Use D as linked list with (char, next entry)

Example:

B = ard$rcaaaabb

S = abracad

	D		sorted D (char, next)
0	(a, 0)	0	($, 3)
1	(r, 1)	1	(a, 0)
2	(d, 2)	2	(a, 6)
3	($, 3)	3	(a, 7)
4	(r, 4)	4	(a, 8)
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  3. Use D as linked list with (char, next entry)

Example:

$B =$ `ard$rcaaaabb`

$S =$ `abracada`

	D		sorted D (char, next)
0	(a, 0)	0	($, 3)
1	(r, 1)	1	(a, 0)
2	(d, 2)	2	(a, 6)
3	($, 3)	3	(a, 7)
4	(r, 4)	4	(a, 8)
5	(c, 5)	5	(a, 9)
6	(a, 6)	6	(b, 10)
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  3. Use D as linked list with (char, next entry)

Example:

$B =$ `ard$rcaaaabb`

$S =$ `abracadab`

	D	sorted D (char, next)
0	(a, 0)	($, 3)
1	(r, 1)	(a, 0)
2	(d, 2)	(a, 6)
3	($, 3)	(a, 7)
4	(r, 4)	(a, 8)
5	(c, 5)	(a, 9)
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Example:

B = `ard$rcaaaabb`

S = `abracadabr`

	D		sorted D (char, next)
0	(a, 0)	0	($, 3)
1	(r, 1)	1	(a, 0)
2	(d, 2)	2	(a, 6)
3	($, 3)	3	(a, 7)
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Example:

B = `ard$rcaaaabb`

S = `abracadabra`

	D		sorted D (char, next)
0	(a, 0)	0	($, 3)
1	(r, 1)	1	(a, 0)
2	(d, 2)	2	(a, 6)
3	($, 3)	3	(a, 7)
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$B =$ `ard$rcaaaabb`

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	D	sorted D (char, next)
0	(a, 0)	($, 3)
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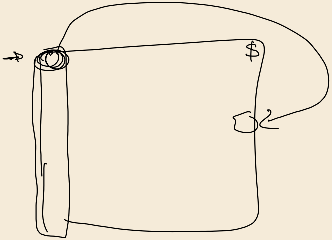
Inverse BWT – The magic revealed

- Inverse BWT very easy to compute:
  - only sort individual characters in B (not suffixes)
  - $\rightsquigarrow$ $O(n)$ with counting sort
- *but why does this work!?*

Inverse BWT – The magic revealed

- Inverse BWT very easy to compute:
  - only sort individual characters in B (not suffixes)
  - ⇝ $O(n)$ with counting sort
- *but why does this work!?*
- decode char by char
  - can find unique \$ ⇝ starting row
- to get next char, we need
  - (i) char in *first* column of *current row*
  - (ii) find row with that char's copy in BWT
  - ⇝ then we can walk through and decode



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- decode char by char
  - can find unique \$ $\rightsquigarrow$ starting row
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  (i) char in *first* column of *current row*
  (ii) find row with that char's copy in BWT

  $\rightsquigarrow$ then we can walk through and decode
- for (i): first column $=$ characters of B in sorted order ✓

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  - only sort individual characters in B (not suffixes)
  - $\rightsquigarrow$ $O(n)$ with counting sort
- *but why does this work!?*
- decode char by char
  - can find unique \$ $\rightsquigarrow$ starting row
- to get next char, we need
  - (i) char in *first* column of *current row*
  - (ii) find row with that char's copy in BWT
  - $\rightsquigarrow$ then we can walk through and decode
- for (i): first column = characters of B in sorted order ✓
- for (ii): relative order of same character stays same:
  ith `a` in first column = ith `a` in BWT
  - $\rightsquigarrow$ stably sorting $(B[r], r)$ by first entry enough ✓

r	$L[r]$	$T_{L[r]}$	$B[r]$
0	9	`$bananaba`	**n**
1	5	`aban$bana`	**n**
2	7	`an$banana`	**b**
3	3	`anaban$ba`	**n**
4	1	`ananaban$`	**b**
5	6	`ban$banan`	**a**
6	0	`bananaban`	**$**
7	8	`n$bananab`	**a**
8	4	`naban$ban`	**a**
9	2	`nanaban$b`	**a**

BWT – Discussion

- Running time: $\Theta(n)$
  - **encoding** uses suffix sorting
  - decoding only needs counting sort
  - $\rightsquigarrow$ decoding much simpler & faster (but same Θ-class)

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👎 typically slower than other methods

👎 need access to entire text (or apply to blocks independently)

👍 BWT-MTF-RLE-Huffman (bzip2) pipeline tends to have best compression

Summary of Compression Methods

Huffman	Variable-width, single-character (optimal in this case)
RLE	Variable-width, multiple-character encoding
LZW	Adaptive, fixed-width, multiple-character encoding Augments dictionary with repeated substrings
MTF	Adaptive, transforms to smaller integers should be followed by variable-width integer encoding
BWT	Block compression method, should be followed by MTF