8 Clever Codes

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Prof. Dr. Sebastian Wild

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Philipps-Universität Marburg
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Learning Outcomes

Unit 8: ***Clever Codes***

1. Know the principles and performance characteristics of *arithmetic coding*.
2. Judge the use of arithmetic coding in applications.
3. Understand the context of *error-prone communication*.
4. Understand concepts of *error-detecting codes* and *error-correcting codes*.
5. Know and understand *Hamming codes*, in particular (7,4) Hamming code.
6. Reason about the *suitability of a code* for an application.

Outline

8 Clever Codes

8.1 Arithmetic Coding

Stream Codes

- **Recall:** (binary) character encoding $E : \Sigma \to \{0,1\}^\star$
  - Huffman codes *optimal* for any given character frequencies
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- Stream codes instead compress entire **sequence** of characters
  - RLE and LZW are examples of stream codes $\rightsquigarrow$ can sometimes do better

- Two indicative examples

  1. **"Low entropy bits:"** $\Sigma = \{0,1\}$, highly skewed: $p_0 = 0.99$

  $\rightsquigarrow$ entropy $\mathcal{H}(\frac{1}{100}, \frac{99}{100}) \approx 0.08$ bits per character,
  Huffman code must use 1 bit per character!

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     - $\rightsquigarrow$ entropy $\mathcal{H}(\frac{1}{100}, \frac{99}{100}) \approx 0.08$ bits per character,
       Huffman code must use 1 bit per character!
     - $\rightsquigarrow$ "optimal" Huffman code gives 12-fold space increase over entropy!
       - Can certainly do better here (RLE!)
  2. **"Trits":** $\Sigma = \{0,1,2\}$, equally likely
     - $\rightsquigarrow$ entropy $\mathcal{H}(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = \lg(3) \approx 1.58$ bits per character,
       Huffman code uses average of $\frac{1}{3} \cdot 1 + \frac{2}{3} \cdot 2 = \frac{5}{3} \approx 1.67$

- Can we do better?

A Decent Hack: Block Codes

- Huffman on trits wastes ≈ 0.0817 bits per character and over $5\,\%$ of space
- A simple trick can reduce this substantially!
  - treat 5 trits as one "supercharacter", e. g., `21101`
  - $\rightsquigarrow$ $3^5 = 243$ possible combinations
  - $\rightsquigarrow$ encode these using 8 bits (with $2^8 = 256$ possible combinations)
  - entropy $\lg(3^5) \approx 7.92$ bits, so less than $0.1\,\%$ wasted space!

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- We can even use a Huffman code for the supercharacters to handle nonuniformity!
- For the low-entropy bits, could use 3 bits
  - $\rightsquigarrow$ probabilities:
    `000` : 0.97
    `001`, `010`, `100` : 0.0098
    `011`, `101`, `110` : 0.000099
    `111`: 0.000001
  - $\rightsquigarrow$ with Huffman code, 1.06 bits per superchar of 3 input bits
  - $\rightsquigarrow$ almost factor 3 better; can improve with larger blocks!

Block Codes – A Panacea?

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- For general case, need to *communicate* the supercharacter encoding
  - Blocks of k characters need $\Omega(\sigma^k)$ space for code
  - Huffman code has to be part of coded message
  - ⇝ Can only sensibly use block codes for small σ and k

There is no such thing as a free lunch . . .

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except in isolated lucky cases

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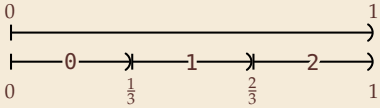
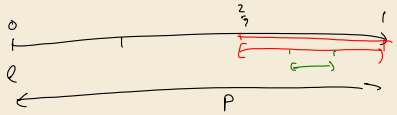
- ***Arithmetic Coding:***
  0. Maintain $[\ell, \ell + p) \subseteq [0, 1)$; initially $\ell = 0$, $p = 1$
  1. Zoom into subinterval for each character
  2. Output dyadic encoding of final interval
- ***Step 1:*** "Zoom" for each character (trit) in $S[0..n)$:
  - Of the current subinterval $[\ell, \ell + p)$, take first, second or last third depending whether $S[i] =$ `0`, `1`, resp. `2`:
    $\ell := \ell + S[i] \cdot \frac{1}{3} \cdot p$
    $p := p \cdot \frac{1}{3}$





$S[0] = 2 \quad \rightsquigarrow \quad \ell = \frac{2}{3} \quad p = \frac{1}{3}$

$S[1] = 1 \quad \rightsquigarrow \quad \ell = \frac{2}{3} + \frac{1}{9} \quad p = \frac{1}{3} \cdot \frac{1}{3}$

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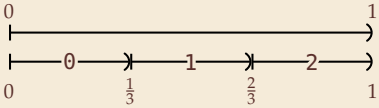

$[\ell, \ell+p)$ not "nice" to encode

- ***Step 2:*** Dyadic encoding
  - Find smallest m so that $\exists x \in \mathbb{N}_0$ with $\left[\frac{x}{2^m}, \frac{x+1}{2^m}\right) \subseteq [\ell, \ell + p)$
  - Output x in binary using m bits.

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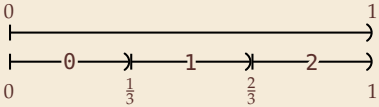

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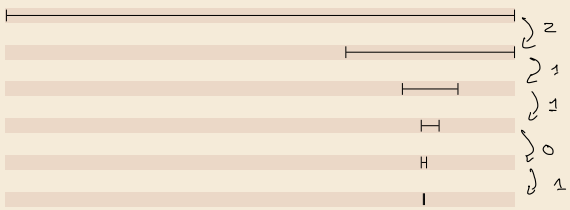
$\rightsquigarrow$ Encode n trits in $n \lg(3) + 2$ bits(!) without cheating

Arithmetic Coding – Encode Trits Example

- $S[0..n) = \texttt{21101}$ $\quad (n = 5)$
- **Step 1:** Zoom into subintervals

Iteration	ℓ	p	Interval (rounded)
0	0	1	$[0.00000, 1.00000)$
1	$\frac{2}{3}$	$\frac{1}{3}$	$[0.66667, 1.00000)$
2	$\frac{7}{9}$	$\frac{1}{9}$	$[0.77778, 0.88889)$
3	$\frac{22}{27}$	$\frac{1}{27}$	$[0.81482, 0.85185)$
4	$\frac{66}{81}$	$\frac{1}{81}$	$[0.81482, 0.82716)$
5	$\frac{199}{243}$	$\frac{1}{243}$	$[0.81893, 0.82305)$



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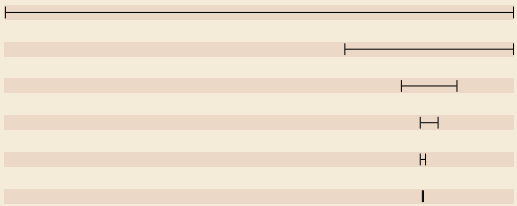

- **Step 2:** Dyadic encoding for interval $[\ell, \ell + p) = \left[\frac{199}{243}, \frac{200}{243}\right)$ $\quad 2^{-m} \leq p$
  - Must have $m \geq \lg(1/p) > 7$

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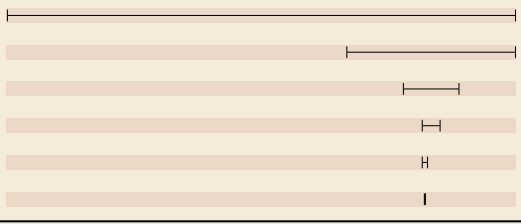

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    - $m = 8$: smallest $x/2^m \geq \frac{199}{243}$ is $x = 210$, but $[210/256, 211/256) \approx [0.82031, 0.8242) \not\subset [\ell, \ell + p)$

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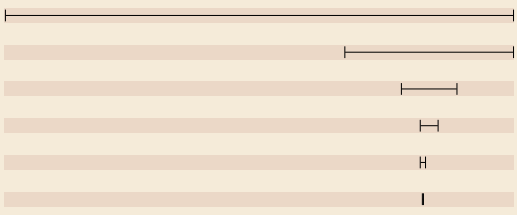

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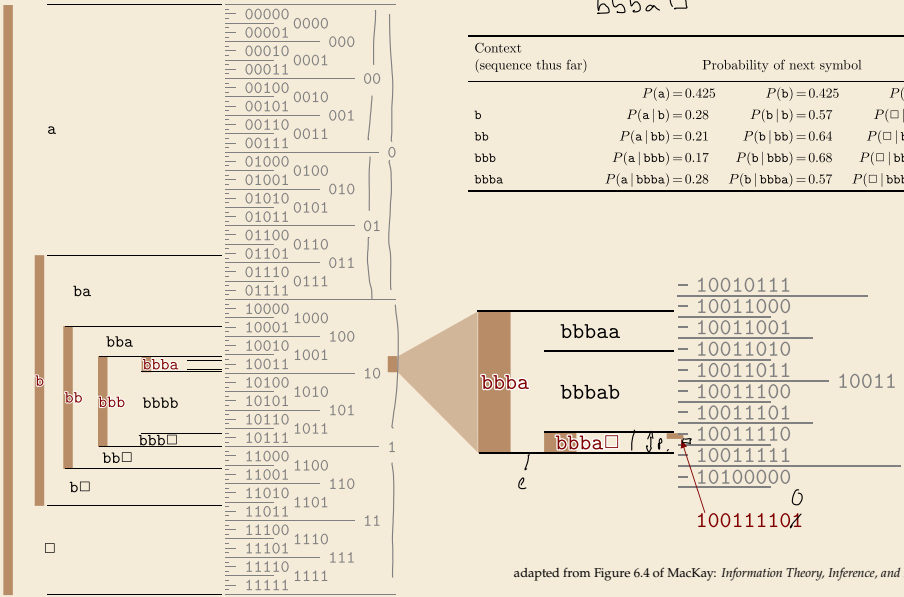
  - $\rightsquigarrow$ Output $x = 420$ in binary with $m = 9$ digits: $\texttt{110100100}$

Versatility of Arithmetic Coding – Adaptive Model

bbba□

Context (sequence thus far)	Probability of next symbol		
	$P(\mathtt{a}) = 0.425$	$P(\mathtt{b}) = 0.425$	$P(\square) = 0.15$
b	$P(\mathtt{a} \mid \mathtt{b}) = 0.28$	$P(\mathtt{b} \mid \mathtt{b}) = 0.57$	$P(\square \mid \mathtt{b}) = 0.15$
bb	$P(\mathtt{a} \mid \mathtt{bb}) = 0.21$	$P(\mathtt{b} \mid \mathtt{bb}) = 0.64$	$P(\square \mid \mathtt{bb}) = 0.15$
bbb	$P(\mathtt{a} \mid \mathtt{bbb}) = 0.17$	$P(\mathtt{b} \mid \mathtt{bbb}) = 0.68$	$P(\square \mid \mathtt{bbb}) = 0.15$
bbba	$P(\mathtt{a} \mid \mathtt{bbba}) = 0.28$	$P(\mathtt{b} \mid \mathtt{bbba}) = 0.57$	$P(\square \mid \mathtt{bbba}) = 0.15$



adapted from Figure 6.4 of MacKay: *Information Theory, Inference, and Learning Algorithms* 2003

Arithmetic Coding – General framework

- Note: Arithmetic coder *doesn't care* if probabilities or even σ change all the time!
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General stochastic sequence:
Sequence of random variables $X_0, X_1, X_2, \ldots$ such that

1. $X_i \in [0..U_i) \cup \{\$\}$ (We use $ to signal "end of text")
2. $\mathbb{P}[X_i = j] = P_{ij}$
3. both U_i and P_{ij} are random variables as they *depend* on $X_0, \ldots X_{i-1}$, but conditioned on $X_0, \ldots, X_{i-1}$, they are fixed and known:
   $P_{ij} = P_{ij}(X_0, \ldots, X_{i-1}) = \mathbb{P}[X_i = j \mid X_0, \ldots, X_{i-1}]$
   $U_i = U_i(X_0, \ldots, X_{i-1}) = \max\{j : P_{ij}(X_0, \ldots, X_{i-1}) > 0\}$

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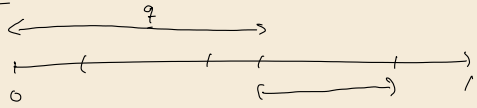
- Can model arbitrary dependencies on previous outcomes
- Assume here that random process is known by both encoder and decoder (fixed coding) otherwise extra space needed to encode model!

Arithmetic Coding – Encoding

1 **procedure** arithmeticEncode($X_0, \ldots, X_n$):
2 *// Assume model U_i and P_{ij} are fixed.*
3 *// Assume $X_i \in [0..U_i)$ for $i < n$ and $X_n = \$$*
4 *// Step 1: Interval zooming*
5 $\ell := 0;\ \ p := 1$
6 **for** $i := 0, \ldots, n-1$ **do**
7 $q := \sum_{j=0}^{X_i - 1} P_{ij};$
8 $\ell := \ell + q \cdot p;\ \ p := p \cdot P_{i,X_i}$
9 **end for**
10 $q := 1 - P_{n,\$}$ *// encode \$ as last character*
11 $\ell := \ell + q \cdot p;\ \ p := p \cdot P_{n,\$}$
12 *// Step 2: Dyadic encoding*
13 $m := \lceil \lg(1/p) \rceil - 1$
14 **do**
15 $m := m + 1;\ \ x := \lceil \ell \cdot 2^m \rceil$
16 **while** $(x+1)/2^m > \ell + p$
17 **return** x **in** binary using m bits



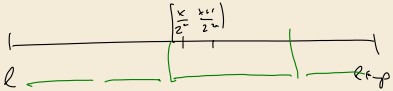
$q = P_{i,0} + P_{i,1} + \cdots + P_{i,X_i-1}$

Arithmetic Coding – Decoding

```
1  procedure arithmeticDecode(C[0..m)):
2      // Assume model U_i and P_ij are fixed.
3      // C[0..m) bit string produced by arithmeticEncode
4      x = Σ_{i=0}^{m-1} C[i] · 2^{m-1-i}  // final interval [x/2^m, (x+1)/2^m)
5      ℓ := 0;  p := 1;  i := 0
6      while true
7          c := 0;  q := 0 // Decode next character c
8          while ℓ + q · p < x/2^m // Iterate through characters until final interval
9              if c == U_i + 1 // reached $
10                 X[i] := $
11                 return X[0..i]
12             else
13                 q := q + P_{i,c};  c := c + 1
14         end while
15         c := c − 1;  q := q − P_{i,c} // we overshot by 1
16         X[i] := c
17         ℓ := ℓ + q · p;  p := p · P_{i,c}
18         i := i + 1
19     end for
```

Example: adaptive model on $\Sigma = \{a, \$\}$

$$P[S[i] = a \mid S[0..i)] = \frac{|S[0..i)|_a + 1}{i + 2}$$



8.2 Practical Arithmetic Coding

Arithmetic Coding – Numerics

- As implemented above, p usually gets smaller by a constant factor with *each character*
  - $\rightsquigarrow$ p gets exponentially small in n!
    - ℓ does not get smaller in absolute terms, but we need it to ever higher accuracy

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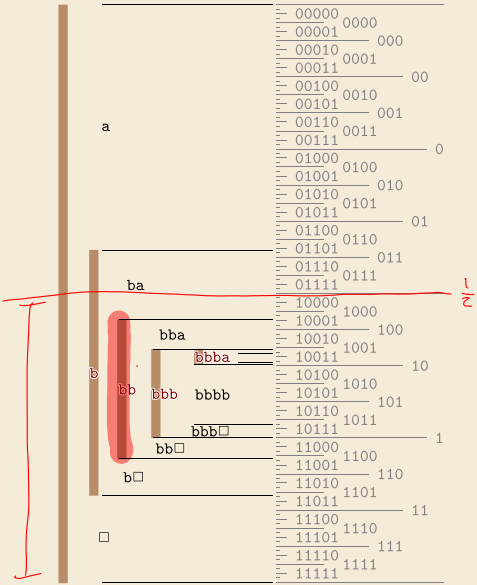
- *With a clever trick, this can be avoided!*
  - If $[\ell, \ell + p) \subseteq [0, \frac{1}{2})$, we know:
    - Our final x with $\left[\frac{x}{2^m}, \frac{x+1}{2^m}\right) \subseteq [\ell, \ell + p)$ *must start with a* `0`*-bit*!
    - $\rightsquigarrow$ Output a `0` and renormalize interval: $\ell := 2\ell;\ p := 2p$

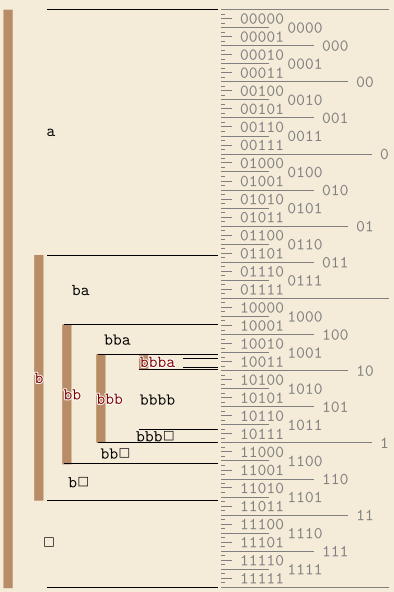


Arithmetic Coding – Numerics

- As implemented above, p usually gets smaller by a constant factor with *each character*
  - $\rightsquigarrow$ p gets exponentially small in n!
  - ℓ does not get smaller in absolute terms, but we need it to ever higher accuracy

$\rightsquigarrow$ requires $\Omega(n)$ bit precision and exact arithmetic!

- *With a clever trick, this can be avoided!*
  - If $[\ell, \ell + p) \subseteq [0, \frac{1}{2})$, we know:
    - Our final x with $\left[\frac{x}{2^m}, \frac{x+1}{2^m}\right) \subseteq [\ell, \ell + p)$ *must start with a* `0`*-bit*!
    - $\rightsquigarrow$ Output a `0` and renormalize interval: $\ell := 2\ell;\ p := 2p$
  - If $[\ell, \ell + p) \subseteq [\frac{1}{2}, 1)$, similarly:
    - Output `1` and renormalize: $\ell := \ell - \frac{1}{2}$, $\ell := 2\ell;\ p := 2p$



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$$\rightsquigarrow\ p = \left(\frac{1}{3}\right)^n, \quad \ell = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots = \sum_{i=1}^{n} 3^{-i} = \frac{1}{2} - \frac{3^{-n}}{2}$$

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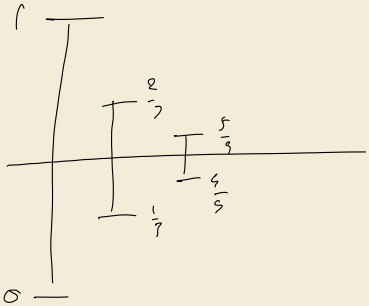

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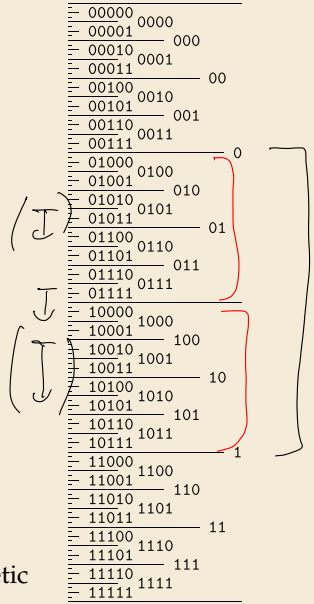
But: If $[\ell, \ell + p) \subseteq [\frac{1}{4}, \frac{3}{4})$, next **two** bits are either `01` or `10`

- Remember an *"outstanding opposite bit"* (increment counter)
- Renormalize:
  $\ell := \ell - \frac{1}{4}$
  $\ell := 2\ell;\ p := 2p$

$\rightsquigarrow$ ℓ and p remain in range of P_{ij}

$\rightsquigarrow$ round P_{ij} to integer multiple of 2^{-F} $\rightsquigarrow$ fixed-precision arithmetic



Fixed Precision Arithmetic Encode

Detailed code from Moffat, Neal, Witten, *Arithmetic Coding Revisited*, ACM Trans. Inf. Sys. 1998

Note: L is our ℓ, R is our p, $b \le w$ is #bits for variables

arithmetic_encode(l, h, t)

/* Arithmetically encode the range $[l/t, h/t)$ using low-precision arithmetic. The state variables R and L are modified to reflect the new range, and then renormalized to restore the initial and final invariants $2^{b-2} < R \le 2^{b-1}$, $0 \le L < 2^b - 2^{b-2}$, and $L + R \le 2^b$ */

(1) Set $r \leftarrow R$ div t

(2) Set $L \leftarrow L + r$ times l

(3) If $h < t$ then
    set $R \leftarrow r$ times $(h - l)$
    else
    set $R \leftarrow R - r$ times l

(4) While $R \le 2^{b-2}$ do
    Use Algorithm ENCODER RENORMALIZATION (Figure 7) to renormalize R, adjust L, and output one bit

Fixed Precision Renormalize

In *arithmetic_encode*()

/* Reestablish the invariant on R, namely that $2^{b-2} < R \leq 2^{b-1}$. Each doubling of R corresponds to the output of one bit, either of known value, or of value opposite to the value of the next bit actually output */

(4) While $R \leq 2^{b-2}$ do

If $L + R \leq 2^{b-1}$ then

bit_plus_follow(0)

else if $2^{b-1} \leq L$ then

bit_plus_follow(1)

Set $L \leftarrow L - 2^{b-1}$

else

Set *bits_outstanding* $\leftarrow$ *bits_outstanding* $+ 1$ and $L \leftarrow L - 2^{b-2}$

Set $L \leftarrow 2L$ and $R \leftarrow 2R$

bit_plus_follow(x)

/* Write the bit x (value 0 or 1) to the output bit stream, plus any outstanding following bits, which are known to be of opposite polarity */

(1) *write_one_bit*(x).

(2) While *bits_outstanding* > 0 do

write_one_bit($1 - x$)

Set *bits_outstanding* $\leftarrow$ *bits_outstanding* $- 1$

Fixed Precision Arithmetic Decode

Functions *decode_target* and *arithmetic_decode* to be called alternatingly.

decode_target(t)

/* Returns an integer *target*, $0 \leq target < t$ that is guaranteed to lie in the range $[l, h)$ that was used at the corresponding call to ***arithmetic_encode***() */

(1) Set $r \leftarrow R$ div t

(2) Return $(\min\{t-1, D \text{ div } r\})$

arithmetic_decode(l, h, t)

/* Adjusts the decoder's state variables R and D to reflect the changes made in the encoder during the corresponding call to ***arithmetic_encode***(). Note that, compared with Algorithm CACM CODER (Figure 6), the transformation $D = V - L$ is used. It is also assumed that r has been set by a prior call to ***decode_target***() */

(1) Set $D \leftarrow D - r$ times l

(2) If $h < t$ then

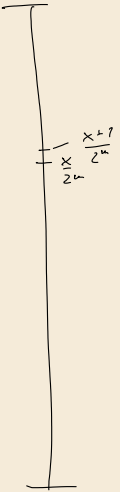
set $R \leftarrow r$ times $(h - l)$

else

set $R \leftarrow R - r$ times l

(3) While $R \leq 2^{b-2}$ do

Set $R \leftarrow 2R$ and $D \leftarrow 2D +$ ***read_one_bit***()



Arithmetic Coding Discussion

- 👎 Subtle code (⇝ libraries!)
- 👎 Typically slower to encode/decode than Huffman codes
- 👎 Encoded bits can be produced/consumed in bursts
- 👍 Extremely versatile w. r. t. random process
- 👍 Almost optimal space usage / compression
- 👍 Widely used (instead of Huffman) in JPEG, zip variants, . . .

8.3 Error Correcting Codes

Noisy Communication

- most forms of communication are “noisy”
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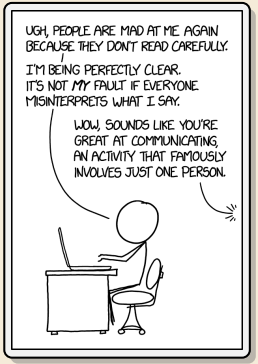
⇝ We can

1. **detect errors** "This sentence has aao pi dgsdho gioasghds."
2. **correct** (some) **errors** "Tiny errs ar corrrected automaticly."
   (sometimes too eagerly as in the Chinese Whispers / Telephone)



Noisy Channels

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- We can aim at
  1. **error detection** ⇝ can request a re-transmit
  2. **error correction** ⇝ avoid re-transmit for common types of errors
- This will require *redundancy*: sending *more* bits than plain message
  - ⇝ **goal:** robust code with lowest redundancy

that's the opposite of compression!

Clicker Question

What do you think, how many extra bits do we need to **detect** a **single bit error** in a message of 100 bits?

→ *sli.do/cs566*

Clicker Question

What do you think, how many extra bits do we need to **correct** a **single bit error** in a message of 100 bits?

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8.4 Coding Theory

Block codes

- **model:**
  - want to send message $S \in \{0,1\}^\star$ (bitstream) across a *(communication) channel*
  - any bit transmitted through the channel might ***flip*** ($0 \to 1$ resp. $1 \to 0$)
    no other errors occur (no bits lost, duplicated, inserted, etc.)
  - instead of S, we send *encoded bitstream* $C \in \{0,1\}^\star$
    sender *encodes* S to C, receiver *decodes* C to S (hopefully)

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- all codes discussed here are ***block codes***
  - divide S into *messages* $m \in \{0,1\}^k$ of k bits each (k = *message length*)
  - encode each message (separately) as $C(m) \in \{0,1\}^n$ (n = *block length*, $n \geq k$)

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  $\rightsquigarrow$ can analyze everything block-wise

- between 0 and n bits might be flipped
  - how many flipped bits can we definitely **detect**? (invalid code)
  - how many flipped bits can we **correct** without retransmit? (i.e. decoding m still possible)

Clicker Question

What is the Hamming distance between `heart` and `beard`?

heart
beard

2

→ *sli.do/cs566*

Code distance

$m \neq m' \implies C(m) \neq C(m')$

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- define $\mathcal{C}$ = set of all codewords = $C(\{0,1\}^k)$

$\rightsquigarrow$ $\mathcal{C} \subseteq \{0,1\}^n$ $\quad$ $|\mathcal{C}| = 2^k$ out of 2^n n-bit strings are valid codewords

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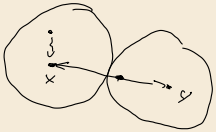

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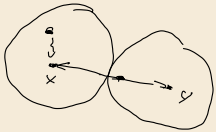

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Implications for codes

1. Need distance d to **detect** all errors flipping up to $d-1$ bits.
2. Need distance d to **correct** all errors flipping up to $\lfloor \frac{d-1}{2} \rfloor$ bits.

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- Main advantage of concept of code distance:
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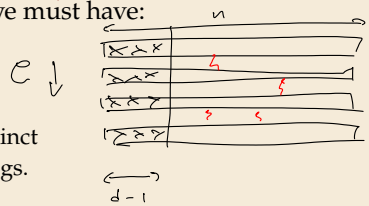
- ***Singleton bound:*** $2^k \leq 2^{n-(d-1)} \rightsquigarrow n \geq k + d - 1$

  - *proof sketch:* We have 2^k codewords with distance d
    after deleting the first $d-1$ bits, all are still distinct
    but there are only $2^{n-(d-1)}$ such shorter bitstrings.



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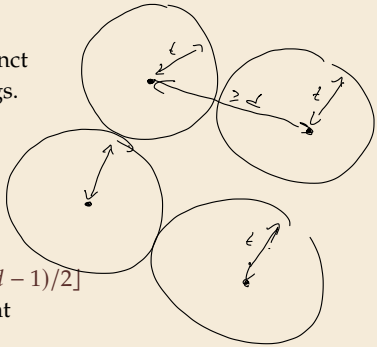
- ***Hamming bound:*** $2^k \leq \dfrac{2^n}{\sum_{f=0}^{\lfloor (d-1)/2 \rfloor} \binom{n}{f}}$
  - *proof idea:* consider "balls" of bitstrings around codewords
    count bitstrings with Hamming-distance $\leq t = \lfloor (d-1)/2 \rfloor$
    correcting t errors means all these balls are disjoint
    so $2^k \cdot$ ball size $\leq 2^n$

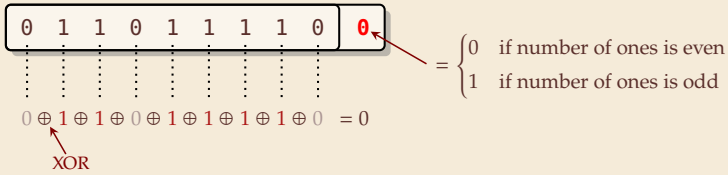


$\rightsquigarrow$ We will come back to these.

8.5 Hamming Codes

Parity Bit

- simplest possible error-detecting code: add a **parity bit**



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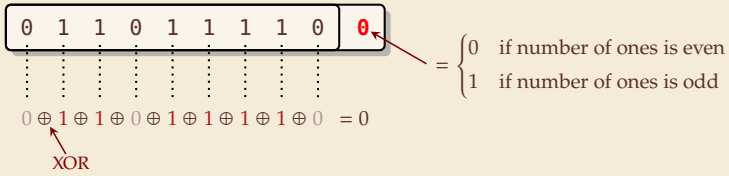

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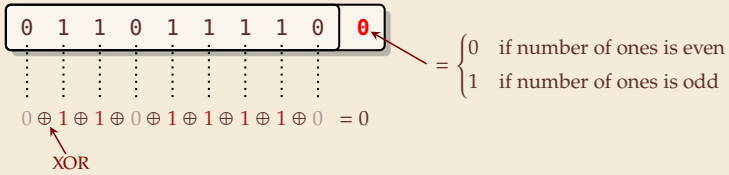


- ⇝ code distance 2
- can detect any single-bit error (actually, any odd number of flipped bits)
- used in many hardware (communication) protocols
  - PCI buses, serial buses
  - caches
  - early forms of main memory
- 👍 very simple and cheap
- 👎 cannot correct any errors

Clicker Question

What do you think, how many extra bits do we need to **detect** a **single bit error** in a message of 100 bits?

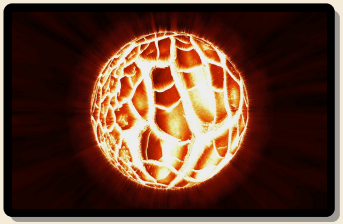
→ *sli.do/cs566*

Error-correcting codes

any downtime is expensive!

- typical application: heavy-duty server RAM
  - bits can randomly flip (e. g., by cosmic rays)
  - individually very unlikely,
    but in always-on server with lots of RAM, it happens!

    https://blogs.oracle.com/linux/attack-of-the-cosmic-rays-v2



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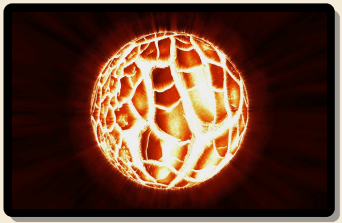

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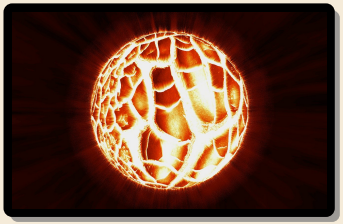

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- Yes! store every bit *three times!*
  - upon read, do majority vote
  - if only one bit flipped, the other two (correct) will still win

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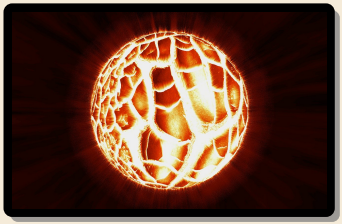

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You want WHAT!?!

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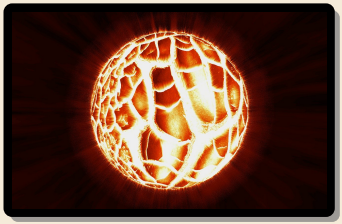

Can we **correct** a bit error without knowing where it occurred? How?

- Yes! store every bit *three times!* 010 → 0
  - upon read, do majority vote (Mehrheit)
  - if only one bit flipped, the other two (correct) will still win
  - *triples* the cost!

You want WHAT!?!

instead of 200% (!)

Can do it with 11% extra memory!

How to locate errors?

- **Idea:** Use several parity bits
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$n = 7$

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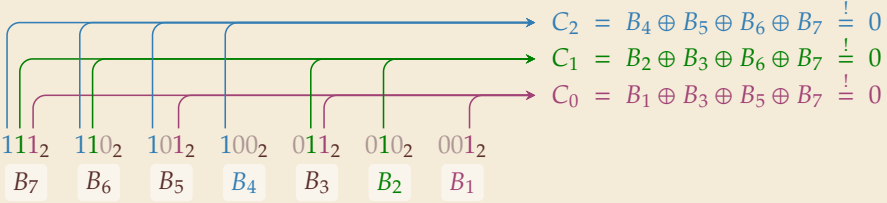

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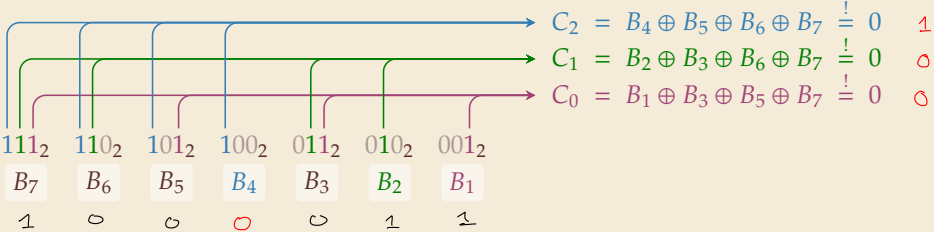

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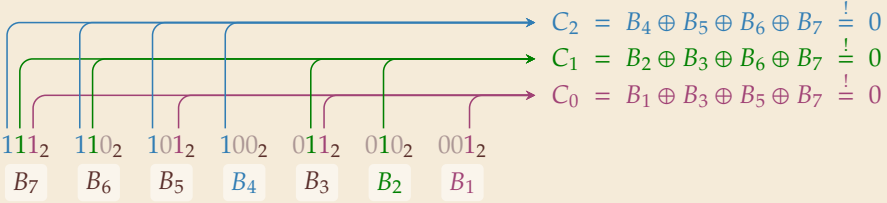

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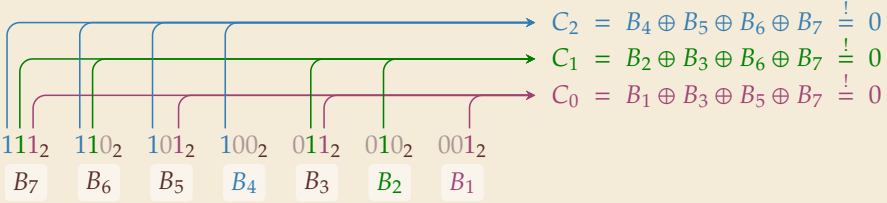

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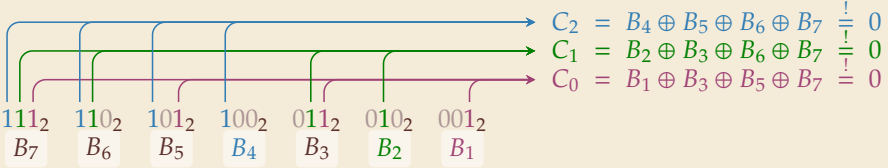
(7, 4) Hamming Code

- *How can we turn this into a code?*



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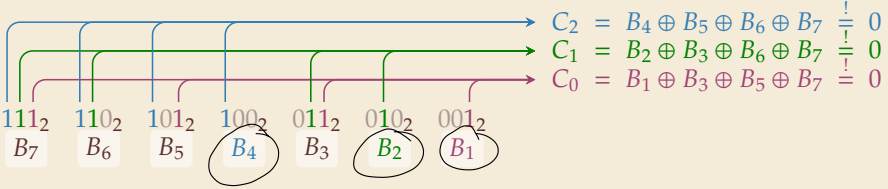

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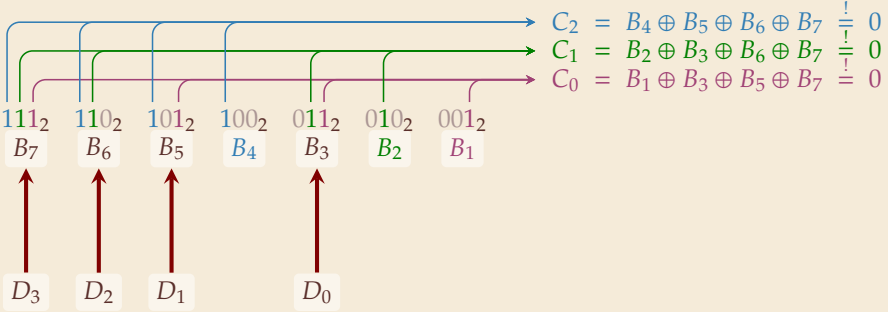

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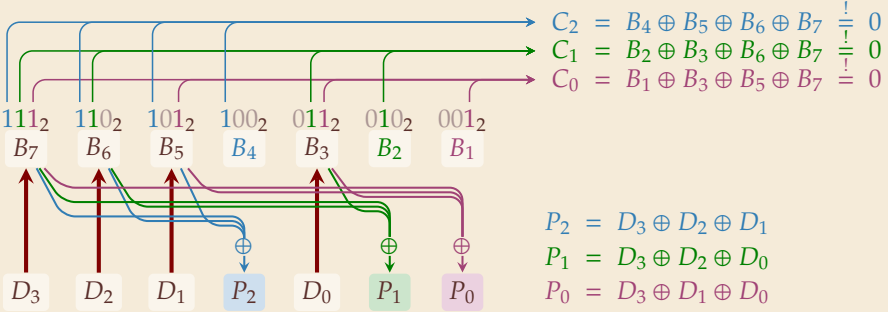

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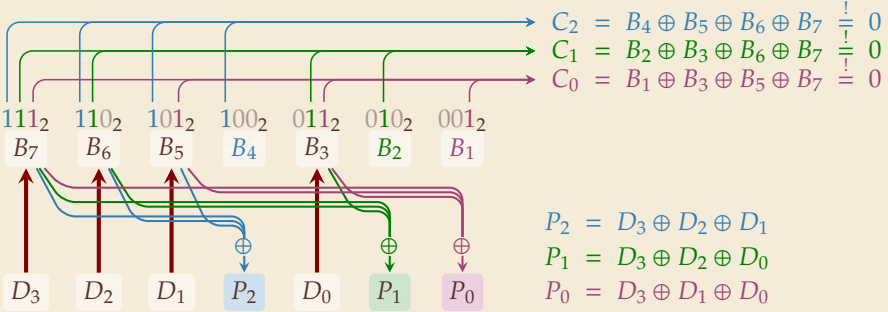

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(7, 4) Hamming Code – Decoding

- **(7, 4) *Hamming Code* – Decoding**
  1. **Given:** block $B_7B_6B_5B_4B_3B_2B_1$ of length $n = 7$
  2. compute C (as above)
  3. if $C = 0$ no (detectable) error occurred
     otherwise, flip B_C (the Cth bit was twisted)
  4. return 4-bit message $B_7B_6B_5B_3$

Clicker Question

What is the code distance of $(7,4)$ Hamming code?

A 0

B 1

C 2

D 3

E 4

F 5

G 6

H ≥ 7

→ *sli.do/cs566*

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(7, 4) Hamming Code – Properties

- **Hamming bound:**
  - 2^4 valid 7-bit codewords (on per message)
  - any of the 7 single-bit errors corrected towards valid codeword
  - $\rightsquigarrow$ each codeword covers 8 of all possible 7-bit strings
  - $2^4 \cdot 2^3 = 2^7 \rightsquigarrow$ exactly cover space of 7-bit strings

codeword

7 · single bit error

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- distance $d = 3$
- can *correct* any 1-bit error
- How about 2-bit errors?
  - We can *detect* that *something* went wrong.
  - **But:** above decoder mistakes it for a (different!) 1-bit error and "corrects" that
  - Variant: store one additional parity bit for entire block

  $\rightsquigarrow$ Can *detect* any 2-bit error, but *not correct* it.

Hamming Codes – General recipe

- construction can be generalized:
  - Start with $n = 2^\ell - 1$ bits for $\ell \in \mathbb{N}$ (we had $\ell = 3$)
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simple and efficient coding / decoding


fairly space-efficient

Outlook

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  = matches Hamming lower bound

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- For other scenarios, finding good codes is an active research area
  - information theory predicts that *almost all* randomly chosen codes are good(!)
  - but these are inefficient to decode

  $\rightsquigarrow$ clever tricks and constructions needed
  e. g. *low density parity check codes*