
# 9 Graph Algorithms

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CS566 (Wintersemester 2024/25)
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# Learning Outcomes

**Unit 9:** ***Graph Algorithms***

1. Know basic terminology from graph theory, including types of graphs.
2. Know adjacency matrix and adjacency list representations and their performance characteristica.
3. Know graph-traversal based algorithm, including efficient implementations.
4. Be able to proof correctness of graph-traversal-based algorithms.
5. Know algorithms for maximum flows in networks.
6. Be able to model new algorithmic problems as graph problems.

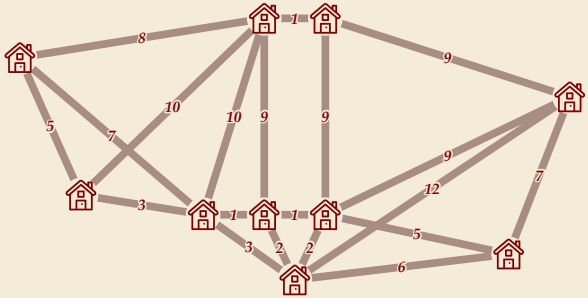
## Outline

# 9 Graph Algorithms

# 9.1 Introduction & Definitions

# Graphs in real life

- a graph is an abstraction of *entities* with their (pairwise) *relationships*
- abundant examples in real life (often called network there)
  - social networks: e. g. persons and their friendships, . . . *Five/Six? degrees of separation*
  - physical networks: cities and highways, roads networks, power grids etc., the Internet, . . .
  - content networks: world wide web, ontologies, . . .
  - . . .




Many More examples, e. g., in Sedgewick & Wayne's videos:

`https://www.coursera.org/learn/algorithms-part2`

# Flavors of Graphs

- Since graphs are used to model so many different entities and relations, they come in several variants

| **Property** | **Yes** | **No** |
|---|---|---|
| edges are one-way | *directed* graph (*digraph*) | *undirected* graph |
| $\leq 1$ edge between $u$ and $v$ | *simple* graph | *multigraph* / with *parallel* edges |
| edges can lead from $v$ to $v$ | with *loops* (Schlinge, Schlaufe) | (loop-free) |
| edges have weights | *(edge-) weighted* graph | *unweighted* graph |

🙃 any combination of the above can make sense . . .

- Synonyms:
  - **vertex** („Knoten“) = node = point = „Ecke”
  - **edge** („Kante“) = arc = line = relation = arrow = „Pfeil“
  - **graph** = network

# Graph Theory

- default: unweighted, undirected, loop-free & simple graphs
- *Graph* $G = (V, E)$ with
  - $V$ a finite of *vertices*
  - $E \subseteq [V]^2$ a set of *edges*, which are 2-subsets of $V$: $[V]^2 = \{e : e \subseteq V \wedge |e| = 2\}$

$e = \{u, v\}$

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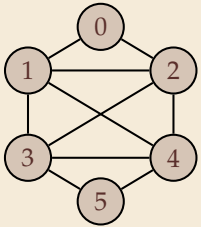
**Example**

$$\begin{aligned} V &= \{0,1,2,3,4,5\} \\ E &= \big\{\{0,1\},\{1,2\},\{1,4\},\{1,3\},\{0,2\}, \\ &\quad \{2,4\},\{2,3\},\{3,4\},\{3,5\},\{4,5\}\big\}. \end{aligned}$$

**Graphical representation**




like so . . .

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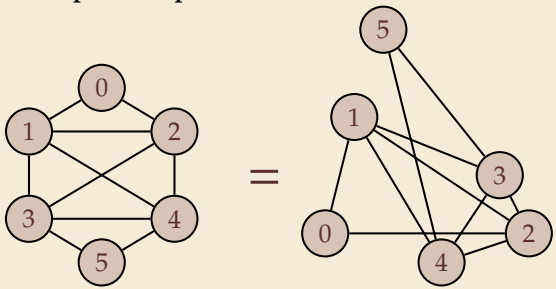
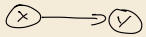



like so . . . . . or so

(same graph)

# Digraphs

- default digraph: unweighted, loop-free & simple
- *Digraph (directed graph)* $G = (V, E)$ with
  - $V$ a finite of *vertices*
  - $E \subseteq V^2 \setminus \{(v, v) : v \in V\}$ a set of *(directed) edges*,
    $V^2 = V \times V = \{(x, y) : x \in V \wedge y \in V\}$ 2-tuples / ordered pairs over $V$



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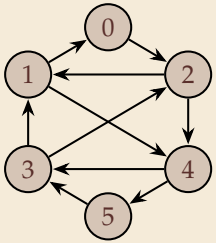
**Example**

$V = \{0, 1, 2, 3, 4, 5\}$

$E = \{(0, 2), (1, 0), (1, 4), (2, 1), (2, 4), (3, 1), (3, 2), (4, 3), (4, 5), (5, 3)\}$

**Graphical representation**



# Graph Terminology

**Undirected Graphs**

- $V(G)$ set of vertices, $E(G)$ set of edges
- write $\underline{uv}$ (or $vu$) for edge $\{u, v\}$
- edges *incident* at vertex $v$: $E(v)$
- $u$ and $v$ are *adjacent* iff $\{u, v\} \in E$,
- *neighborhood* $N(v) = \{w \in V : w \text{ adjacent to } v\}$
- *degree* $d(v) = |E(v)|$

**Directed Graphs** (where different)

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- *walk* $w$ of length $n$: sequence of vertices $w[0..n]$ with $\forall i \in [0..n) : w[i]w[i+1] \in E$
- *path* $p$ is a (vertex-) simple walk: without duplicate vertices except possibly its endpoints
- *edge-simple* walk: no edge used twice
- *cycle* $c$ is a closed path, i. e., $c[0] = c[n]$

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- $G$ is *connected*
  iff for all $u \neq v \in V$ there is a path from $u$ to $v$
- $G$ is *acyclic* iff $\nexists$ cycle (of length $n \geq 1$) in $G$

- *strongly connected* for digraphs
  (*weakly connected* = connected ignoring directions)

# Typical graph-processing problems

- **Path**: Is there a path between $s$ and $t$?
  **Shortest path**: What is the shortest path (distance) between $s$ and $t$?
- **Cycle**: Is there a cycle in the graph?
  **Euler tour**: Is there a cycle that uses each edge exactly once?
  **Hamilton(ian) cycle**: Is there a cycle that uses each vertex exactly once.
- **Connectivity**: Is there a way to connect all of the vertices?
  **MST**: What is the best way to connect all of the vertices?
  **Biconnectivity**: Is there a vertex whose removal disconnects the graph?
- **Planarity**: Can you draw the graph in the plane with no crossing edges?
- **Graph isomorphism**: Are two graphs the same up to renaming vertices?

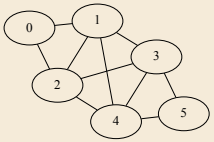
can vary a lot, despite superficial similarity of problems

**Challenge**: Which of these problems
can be computed in (near) linear time?
in reasonable polynomial time?
are intractable?

# Tools to work with graphs

- Convenient GUI to edit & draw graphs: *yEd live*
  `yworks.com/yed-live`
- *graphviz* cmdline utility to draw graphs
  - Simple text format for graphs: `DOT`

```
graph G {
     0 -- 2;     2 -- 4;
     1 -- 0;     2 -- 3;
     1 -- 4;     3 -- 4;
     1 -- 3;     3 -- 5;
     2 -- 1;     4 -- 5;
}
```




```
dot -Tpdf graph.dot -Kfdp > graph.pdf
```

- graphs are typically not built into programming languages, but libraries exist
  - e. g. part of *Google Guava* for Java
  - they usually allow arbitrary objects as vertices
  - aimed at ease of use

# 9.2 Graph Representations

# Graphs in Computer Memory

- We defined graphs in set-theoretic terms. . .
  but computers can't directly deal with sets efficiently

⇝ need to choose a *representation* for graphs.

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**Key Operations:**

- `isAdjacent(`$u$`,`$v$`)`
  Test whether $uv \in E$
- `adj(`$v$`)`
  Adjacency list of $v$ (iterate through (out-) neighbors of $v$)
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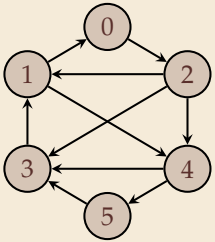
**Conventions:**

- (di)graph $G = (V, E)$ (omitted if clear from context)
- $n = |V|$, $m = |E|$
- in implementations assume $V = [0..n)$ (if needed, use symbol table to map complex objects to $V$)

# Adjacency Matrix Representation

- adjacency matrix $A \in \{0,1\}^{n \times n}$ of $G$: matrix with $A[u,v] = [uv \in E]$ $= \begin{cases} 1 & uv \in E \\ 0 & \text{sonst} \end{cases}$
  - works for both directed and undirected graphs (undirected $\rightsquigarrow A = A^T$ symmetric)
  - can use a weight $w(uv)$ or multiplicity in $A[u,v]$ instead of $0/1$
  - can represent loops via $A[v,v]$

**Example:**




$$A = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

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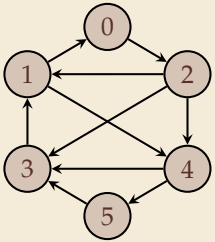


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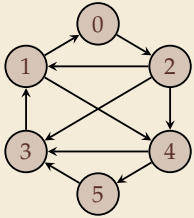
`isAdjacent` in $O(1)$ time

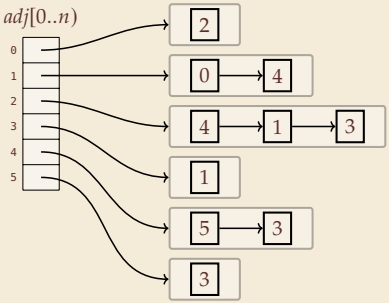
$O(n^2)$ (bits of) space wasteful for sparse graphs

`adj(`$v$`)` iteration takes $O(n)$ (independent of $d(v)$)

# Adjacency List Representation

- Store a linked list of neighbors for each vertex $v$:
  - $adj[0..n)$ bag of neighbors (as linked list)
  - undirected edge $\{u, v\}$ $\rightsquigarrow$ $v$ in $adj[u]$ and $u$ in $adj[v]$
  - weighted edge $uv$ $\rightsquigarrow$ store pair $(v, w(uv))$ in $adj[u]$
  - multiple edges and loops can be represented



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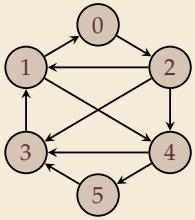
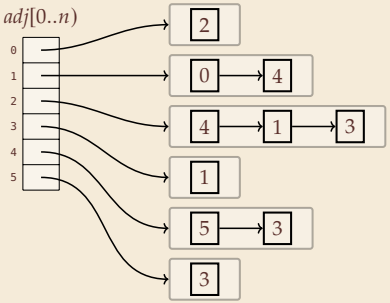



👎 `isAdjacent(`$u$`,`$v$`)` takes $\Theta(d(u))$ time (worst case)

👍 `adj(`$v$`)` iteration $O(1)$ per neighbor

👍 $\Theta(n + m)$ (words of) space for any graph ($\ll \Theta(n^2)$ bits for moderate $m$)

$\rightsquigarrow$ de-facto standard for graph algorithms

# Graph Types and Representations

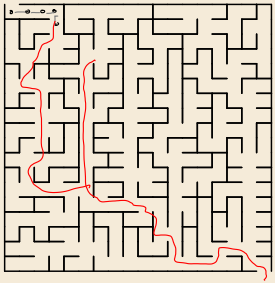
- Note that adj matrix and lists for undirected graphs effectively are representation of directed graph with directed edges both ways
  - conceptually still important to distinguish!
- multigraphs, loops, edge weights all naturally supported in adj lists
  - good if we allow and use them
  - but requires explicit checks to enforce simple / loopfree / bidirectional!
- we focus on **static graphs**
  dynamically changing graphs much harder to handle

# 9.3 Graph Traversal

# Generic Graph Traversal

- Plethora of graph algorithms can be expressed as a systematic exploration of a graph
  - depth-first search, breadth-first search
  - connected components
  - detecting cycles
  - topological sorting
  - Hierholzer's algorithm for Euler walks
  - strong components
  - testing bipartiteness
  - Dijkstra's algorithm
  - Prim's algorithm
  - Lex-BFS for perfect elimination orders of chordal graphs
  - ...

visiting all nodes & edges



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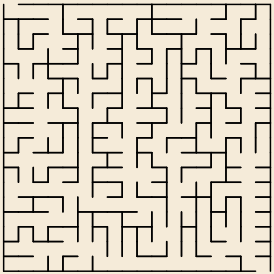


⇝ Formulate generic traversal algorithm

- first in abstract terms to argue about correctness
- then again for concrete instance with efficient data structures

# Tricolor Graph Traversal

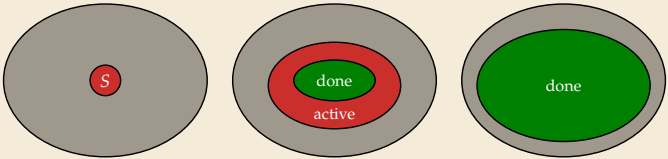
**Tricolor Graph Search:**

- maintain vertices in 3 (dynamic) sets
  - **Gray: unseen vertices**
    The traversal has not reached these vertices so far.
  - **Green: done vertices** (a.k.a. visited vertices)
    These vertices have been visited and all their edges have been explored already.
  - **Red: active vertices** (a.k.a. frontier („Rand") of traversal)
    All others, i. e., vertices that have been reached and some unexplored edges remain; initially some selected start vertices $S$.

**Invariant:**
No edges from ***done*** to ***unseen*** vertices

want to color green
⇝ to do that need neighbors here red

- (implicitly) maintain status of each edge
  - **not yet used**
  - **used edge**
- Vertices "want" to turn green.



# Generic Tricolor Graph Traversal – Code

```
1  procedure genericGraphTraversal(G, S)
2      // (di)graph G = (V, E) and start vertices S ⊆ V
3      C[0..n) := unseen // Color array, all cells initialized to unseen
4      for s ∈ S do C[s] := active end for
5      unusedEdges := E
6      while ∃v : C[v] == active
7          v := nextActiveVertex()  // Freedom 1: Which frontier vertex?
8          if ∄vw ∈ unusedEdges // no more edges from v ⇝ done with v
9              C[v] := done
10         else
11             w := nextUnusedEdge(v)  // Freedom 2: Which of its edges?
12             if C[w] == unseen
13                 C[w] := active
14             end if
15             unusedEdges.remove(vw)
16         end if
17     end while
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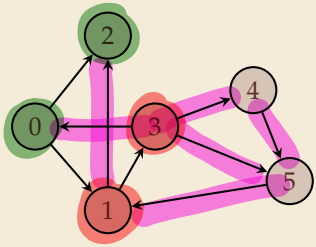


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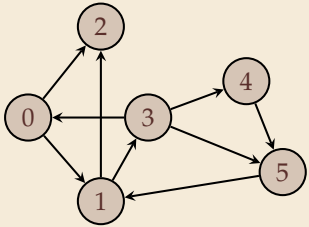


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- Implementations of nextActiveVertex() and nextUnusedEdge($v$) depends on (and defines!) specific traversal-based graph algorithms

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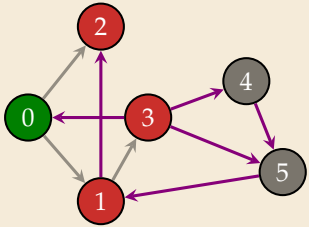


**Invariant:**
No edges from *done* to *unseen* vertices

- Implementations of nextActiveVertex() and nextUnusedEdge($v$) depends on (and defines!) specific traversal-based graph algorithms

# Generic Reachability

- Any choices nextActiveVertex() and nextUnusedEdge($v$) suffice to find exactly the vertices reachable from $S$ in ***done***

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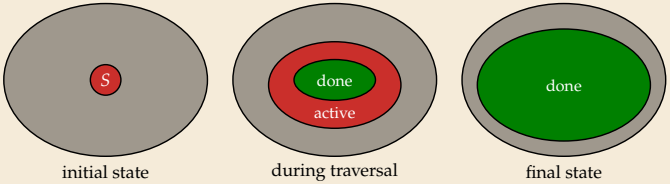
  2. For every ***done*** or ***active*** vertex $v$, there exists a path from $s \in S$ to $v$.




IB : (1) ✓ (2) ✓

IS : (i) line 9 color v grey
no edges to unseen from v
(1) ✓
(2) by IH ✓
(ii) line 11 color w red
(1) ✓
(2) by IH $s \rightsquigarrow v \to w$
hence $s \rightsquigarrow w$ ✓

# Generic Reachability

- Any choices nextActiveVertex() and nextUnusedEdge($v$) suffice to find exactly the vertices reachable from $S$ in ***done***

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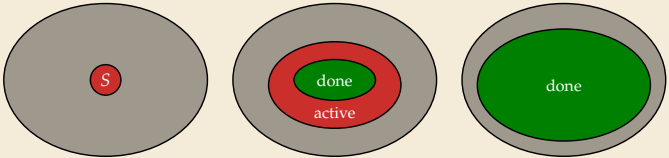


$\rightsquigarrow$ in final state:

- $v \in$ ***done*** $\rightsquigarrow$ path from $S$ $\rightsquigarrow$ reachable from $S$
- $v \in$ ***unseen*** $\rightsquigarrow$ not reachable from ***done*** $\supseteq S$ $\rightsquigarrow$ not reachable from $S$

(assume not. then $s \rightsquigarrow v$ first vertex on path that is unseen has done neighbor $\lightning$ (1))

# Data Structures for Frontier

- We need efficient support for
  - test $\exists v : C[v] =$ ***active***, nextActiveVertex()
  - test $\exists vw \in$ *unusedEdges*, nextUnusedEdge($v$)
  - *unusedEdges*.remove($vw$)

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# Data Structures for Frontier

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  - *unusedEdges* represented implicitly: edge used iff previously returned by $i$
    - $\rightsquigarrow$ don't need *unusedEdges*.remove($vw$)
  - Implement $\exists v : C[v] =$ ***active*** via *frontier*.isEmpty()
  - Implement $\exists vw \in$ *unusedEdges* via $i$.hasNext() assuming $(v, i) \in$ *frontier*
  - Implement nextUnusedEdge($v$) via $i$.next() assuming $(v, i) \in$ *frontier*

$\rightsquigarrow$ all operations apart from nextActiveVertex() in $O(1)$ time

$\rightsquigarrow$ *frontier* requires $O(n)$ extra space

# 9.4 BFS and DFS

# Breadth-First Search

- Maintain *frontier* in a **queue** (FIFO: first in, first out)

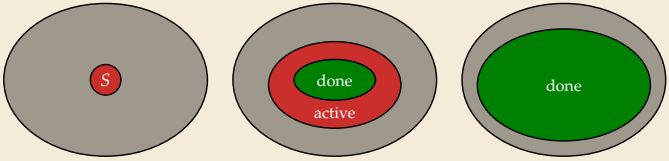
# Breadth-First Search

$\text{distance}_G(v, S) = \min_{s \in S} \text{distance}_G(v, s)$

"#edges" = length of shortest path from $s$ to $v$

- Maintain *frontier* in a **queue** (FIFO: first in, first out)

- **Invariant:**
  1. No edges from done to unseen vertices
  2. All ***done*** (& active) or ***active*** vertices are reached via a **shortest path** (fewest edges) from $S$
  3. Vertices enter and leave *frontier* in order of increasing distance from $S$




initial state | during traversal | final state

⇝ in final state, we reach all reachable vertices via shortest paths

Proof of BFS Invariant. (corrected version)

(1) follows from the generic proof for tricolor traversal

abbreviate $\delta(v) = \min_{s \in S} \text{distance}_G(s, v)$

IB: (2) only $s \in S$ are active or done.

we reach $s$ via path $[s]$ (length 0) ✓

(3) only $s \in S$ have entered frontier so far. these are minimal wrt. $\delta$

IH: Invariant is fulfilled up to now.

IS: consider next active vertex $v$.

(i) $v$ has no more unused edges. $\Rightarrow$ mark $v$ as done

(2) $v$ only changes color $\Rightarrow$ still holds by IH

(3) by IH, $v$ entered frontier at correct time & queue keeps order $\Rightarrow$ (3) still holds

(ii) $v$ has another edge $vw$. We further distinguish:

(ii-a) $w$ is active or done $\Rightarrow$ no changes

$\Rightarrow$ invariant still holds

(ii-b) $w$ is unseen $\Rightarrow$ enqueue $w$ and make it active

by IH, we reached $v$ by shortest path of $\delta(v)$ edges

and any node $u$ with $\delta(u) < \delta(v)$ is done (since before $v$)

$\Rightarrow \delta(w) = \delta(v) + 1$ and we reach $w$ via a shortest path $\Rightarrow$ (2) ✓

It remains to show that any $x$ with $\delta(x) = \delta(v)$ is active or done by now;

then adding $w$ to frontier now is according to sorted order by $\delta$.

Suppose the path to $x$ goes via $u : s \rightsquigarrow u \to x$ then $\delta(u) < \delta(v)$ and $u$ is done

$\Rightarrow$ all edges $u \to x$ have been used and thus $x$ is active or done

(We cannot have active or done $y$ with $\delta(y) > \delta(w)$ in frontier by IH;

their addition before $w$ would have violated the invariant at that time.)

$\Rightarrow$ (3) ✓

□.

# Breadth-First Search

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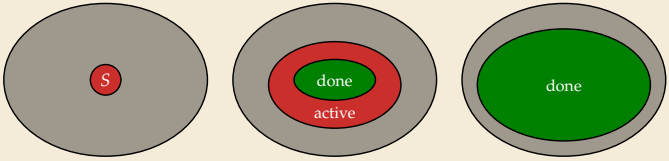


⇝ in final state, we reach all reachable vertices via shortest paths

- To preserve that knowledge, we collect extra information during traversal
  - $parent[v]$ stores predecessor on path from $S$ via which $v$ was reached
  - $distFromS[v]$ stores the length of this path

# Breadth-First Search – Code

```
1  procedure bfs(G, S)
2      // (di)graph G = (V, E) and start vertices S ⊆ V
3      C[0..n) := unseen // New array initialized to all unseen
4      frontier := new Queue;
5      parent[0..n) := NOT_VISITED;  distFromS[0..n) := ∞
6      for s ∈ S
7          parent[s] := NONE;  distFromS[s] := 0
8          C[s] := active;  frontier.enqueue((s, G.adj[s].iterator()))
9      end for
10     while ¬frontier.isEmpty()
11         (v, i) := frontier.peek()
12         if ¬i.hasNext() // v has no unused edge
13             C[v] := done;  frontier.dequeue()
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- *parent* stores a *shortest-path tree/forest*
- can retrieve shortest path to $v$ from some vertex $s \in S$ (backwards) by following *parent*$[v]$ iteratively

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- *parent* stores a *shortest-path tree/forest*
- can retrieve shortest path to $v$ from some vertex $s \in S$ (backwards) by following *parent*$[v]$ iteratively
- running time $\Theta(n + m)$
- extra space $\Theta(n)$

# Depth-First Search

- Maintain *frontier* in a **stack** (LIFO: last in, first out)
  - only consider $S = \{s\}$
  - usual mode of operation: call `dfs`($v$) for all ***unseen*** $v$, for $v = 0, \ldots, n-1$

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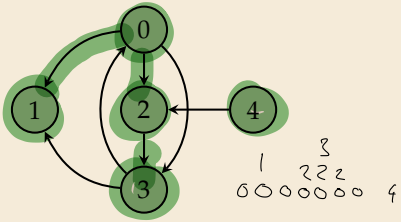


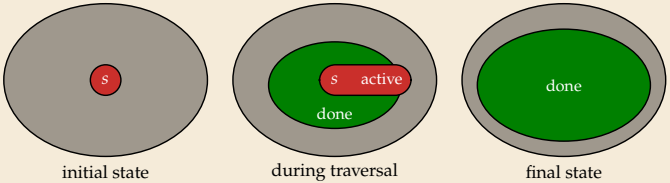


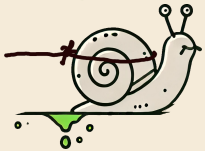

# Depth-First Search – Code

```
1  procedure dfsTraversal(G)
2      C[0..n) := unseen
3      for v := 0, ..., n − 1
4          if C[v] == unseen
5              dfs(G, v)
6
7  procedure dfs(G, s)
8      frontier := new Stack;
9      C[s] := active; frontier.push((s, G.adj[s].iterator()))
10     while ¬frontier.isEmpty()
11         (v, i) := frontier.top()
12         if ¬i.hasNext() // v has no unused edge
13             C[v] := done; frontier.pop(); postorderVisit(v)
14         else
15             w := i.next(); visitEdge(vw)
16             if C[w] == unseen
17                 preorderVisit(w)
18                 C[w] := active; frontier.push((w, G.adj[w].iterator()))
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20         end if
21     end while
```

- define *hooks* to implement further operations
  - preorder: visit $v$ when made ***active*** (start of $T(v)$)
  - postorder: visit $v$ when marked ***done*** (end of $T(v)$)
  - visitEdge: do something for every edge
- if needed, can store DFS forest via *parent* array

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# Simple DFS Application: Connected Components

- In an undirected graph, find all *connected components*.
  - **Given:** simple undirected $G = (V, E)$
  - **Goal:** assign component ids $CC[0..n)$, s.t. $CC[v] = CC[u]$ iff $\exists$ path from $v$ to $u$

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  - **Given:** simple undirected $G = (V, E)$
  - **Goal:** assign component ids $CC[0..n)$, s.t. $CC[v] = CC[u]$ iff $\exists$ path from $v$ to $u$

```
1  procedure connectedComponents(G):
2      // undirected graph G = (V, E) with V = [0..n)
3      C[0..n) := unseen
4      CC[0..n) := NONE
5      id := 0
6      for v := 0, ..., n − 1
7          if C[v] == unseen
8              dfs(G, v)
9              id := id + 1
10     return CC
11
12 procedure preorderVisit(v):
13     CC[v] := id
```

```
1  // same as before
2  procedure dfs(G, s)
3      frontier := new Stack;
4      C[s] := active; frontier.push((s, G.adj[s].iterator()))
5      while ¬frontier.isEmpty()
6          (v, i) := frontier.top()
7          if ¬i.hasNext() // v has no unused edge
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17         end if
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```

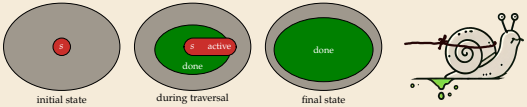
# Dijkstra's Algorithm & Prim's Algorithm

- On edge-weighted graphs, we can use tricolor traversal with a *priority queue* as *frontier*
- Dijkstra's Algorithm for shortest paths from $s$ in digraphs with weakly positive edge weights
  - priority of vertex $v$ = length of shortest path known so far from $s$ to $v$
- Prim's Algorithm for finding a minimum spanning tree
  - priority of vertex $v$ = weight of cheapest edge connecting $v$ to current tree

$\rightsquigarrow$ Detailed discussion in Unit 11

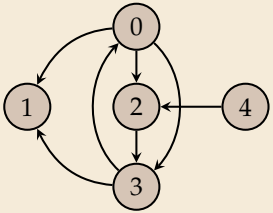
# 9.5 Advanced Uses of DFS

# Properties of DFS




- **Recall DFS Invariant 3:**
  The *active* vertices form a single **path** from $s$

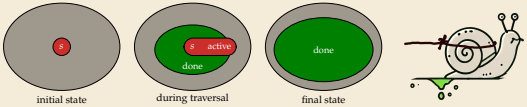
**input graph** $G$




**DFS forest**

**stack over time**

# Properties of DFS

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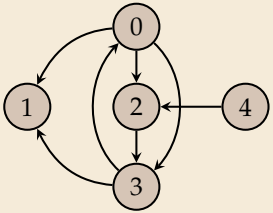


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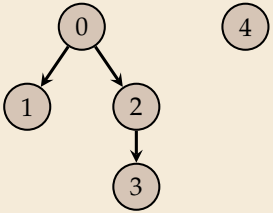
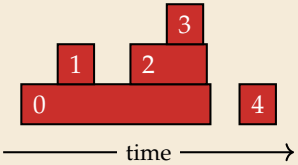



**stack over time**



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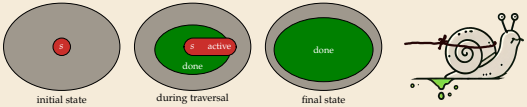


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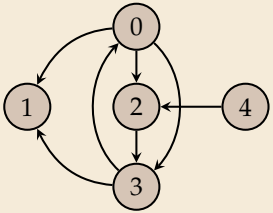
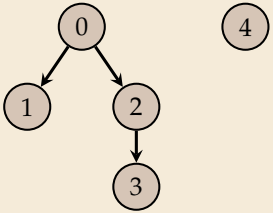
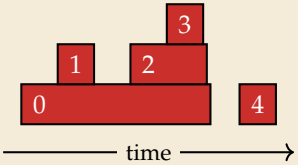



**DFS forest**

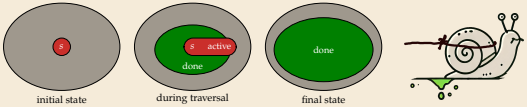



**stack over time**




$\rightsquigarrow$ Each vertex $v$ spends *time interval* $T(v)$ as ***active*** vertex

# Properties of DFS

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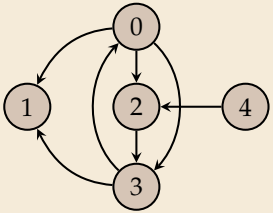


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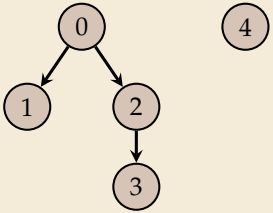


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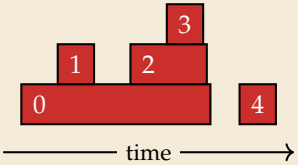
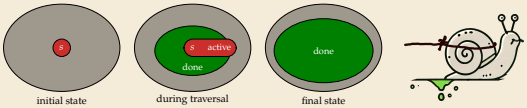



$\rightsquigarrow$ Each vertex $v$ spends *time interval* $T(v)$ as ***active*** vertex

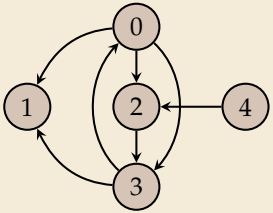
***1.*** *frontier* is stack $\rightsquigarrow$ $\{T(v) : v \in V\}$ forms *laminar set family*: ("disjoint or contained")
either $T(v) \cap T(w) = \emptyset$ or $T(v) \subseteq T(w)$ or $T(v) \supseteq T(w)$

# Properties of DFS

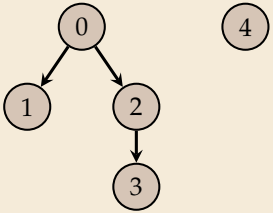



- **Recall DFS Invariant 3:**
  The ***active*** vertices form a single **path** from $s$

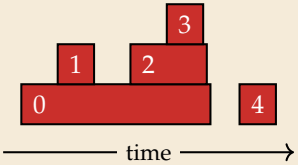
**input graph** $G$




**DFS forest**




**stack over time**




$\rightsquigarrow$ Each vertex $v$ spends *time interval* $T(v)$ as ***active*** vertex

***1.*** *frontier* is stack $\rightsquigarrow$ $\{T(v) : v \in V\}$ forms *laminar set family*: ("disjoint or contained")
either $T(v) \cap T(w) = \emptyset$ or $T(v) \subseteq T(w)$ or $T(v) \supseteq T(w)$

***2.*** **Parenthesis Theorem:** $T(v) \supseteq T(w)$ **iff** $v$ is ancestor of $w$ in DFS tree

'$\Rightarrow$' during $T(v)$, all discovered vertices become descendants of $v$

'$\Leftarrow$' $T(v)$ covers $v$'s entire subtree, which contains $w$'s subtree

# Properties of DFS – Unseen-Path Theorem

- **Unseen-Path Theorem:** In a DFS forest of a (di)graph $G$, $w$ is a descendant of $v$ **iff** at the time of `preorderVisit`($v$), there is a path from $v$ to $w$ using only ***unseen*** vertices.

at the time v was marked

# Properties of DFS – Unseen-Path Theorem

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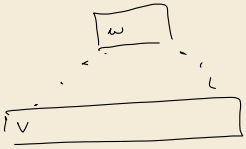
'$\Rightarrow$' If $w$ is a descendant of $v$, $T(w) \subseteq T(v)$ by the Parenthesis Theorem.
Hence the path from $v$ to $w$ in the DFS tree consists (at time of `preorderVisit`($v$)) of solely ***unseen*** vertices.

# Properties of DFS – Unseen-Path Theorem

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Hence the path from $v$ to $w$ in the DFS tree consists (at time of `preorderVisit`($v$)) of solely ***unseen*** vertices.

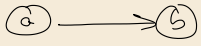
'$\Leftarrow$' Suppose towards a contradiction that there was a $w$ with an ***unseen*** path $p[0..\ell]$ with $p[0] = v$ and $p[\ell] = w$, but $w$ is not a descendant of $v$. W.l.o.g. let $w$ be a first such vertex, i. e., $p[0], \ldots, p[\ell - 1] = u$ are descendants of $v$.
So $T(u) \subset T(v)$ $(*)$.
Upon processing $u$, we will discover edge $uw$, so whether or not $w$ is already ***done*** at this point, $w$ will be marked ***done*** before $u$. Hence $\max T(w) \leq \max T(u)$.
With $(*)$, we obtain $\min T(v) \leq \min T(u) \leq \max T(w) \leq \max T(u)$, so by laminarity, $T(w) \subset T(u) \subset T(v)$ and $w$ is a descendant of $v$ ↯.

# Topological Sorting & Cycle Detection

- **Application:** Given a set of tasks with precedence constraints of the form "$a$ must be done before $b$", can we find a legal ordering for all tasks?

  ⇝ Model as directed graph!

  - tasks are the vertices $V$
  - add an edge $(a, b)$ when $a$ must be done before $b$



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- **Definition:** $R[0..n)$ is a *topological (order) ranking* of digraph $G = (V, E)$ if $\forall (u, v) \in E : R[u] < R[v]$

  Hinweis: $R[v]$ freie Stelle für $v$

- **Lemma DAG iff topo:**
  A directed graph $G$ has a topological ranking **iff** it does not contain a directed cycle.

# Topological Sorting & Cycle Detection

- **Application:** Given a set of tasks with precedence constraints of the form "$a$ must be done before $b$", can we find a legal ordering for all tasks?
  - $\rightsquigarrow$ Model as directed graph!
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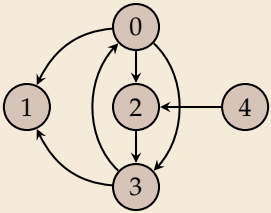
- **Lemma DAG iff topo:**
  A directed graph $G$ has a topological ranking **iff** it does not contain a directed cycle.
- **Topological Sorting**
  - **Given:** simple digraph $G = (V, E)$
  - **Goal:** Compute topological ranking of vertices $R[0..n)$ or output a directed cycle in $G$.
- Amazingly, can do all with one pass of DFS!

# DFS Edge Types

**input digraph** *G*



# DFS Edge Types

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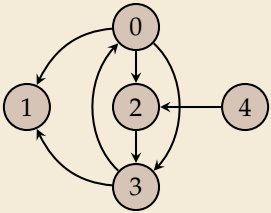
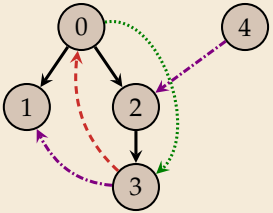
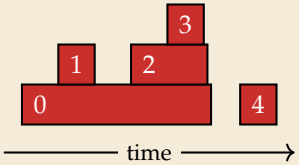



**DFS forest**




**stack over time**



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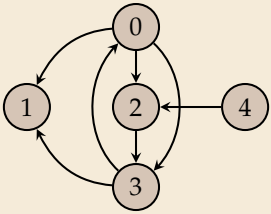


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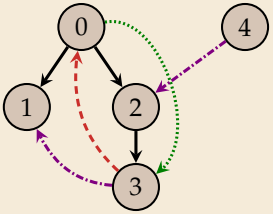


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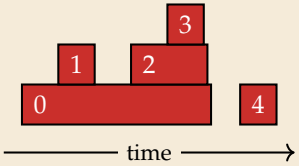


- During DFS traversal, an edge $vw$ has one of these 4 types: **example:**
  1. **tree edge:** ⟶ $w \in$ *unseen* ⇝ $vw$ part of DFS forest. $(0,1), (0,2), (2,3)$
  2. **back edges:** - - → $w \in$ *active*; ⇝ $w$ points to ancestor of $v$. $(3,0)$
  3. **forward edges*:** ····→ $w \in$ *done* $\wedge$ $w$ is descendant of $v$ in DFS tree. $(0,3)$
  4. **cross edges*:** -·→ $w \in$ *done* $\wedge$ $w$ is not descendant of $v$. $(3,0)$

  *only possible in *directed* graphs

# Cycle Detection

If $G$ contains a directed cycle, DFS will find a directed cycle:

- any back edge implies a cycle:
  - DFS visits an edge $(v, w)$ where $w \in$ ***active***, $w$ is already on the stack
  - $\rightsquigarrow$ DFS tree contains path $w \rightsquigarrow v$ and we have edge $v \to w$.

# Cycle Detection

If $G$ contains a directed cycle, DFS will find a directed cycle:

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  - $\rightsquigarrow$ DFS tree contains path $w \rightsquigarrow v$ and we have edge $v \rightarrow w$.

- conversely any cycle $C[0..k]$ once reached must have some back edge or cross edge (tree and forward edges go from smaller to larger preorder index)
  - cannot be a cross edge since cycle is strongly connected
    all cycle vertices must be descendants of first reached cycle vertex
  - $\rightsquigarrow$ cycle contributes a back edge

# DFS Postorder Implementation

```
1  procedure dfsPostorder(G):
2      C[0..n) := unseen
3      P[0..n) := NONE;  r := 0
4      parent[0..n) := NONE
5      cycle := NONE
6      for v := 0, ..., n − 1
7          if C[v] == unseen
8              dfs(G, v)
9      return (P, cycle)
10
11 procedure postorderVisit(v):
12     P[v] := r;  r := r + 1
13
14 procedure visitEdge(vw):
15     if C[w] == active
16         if cycle ≠ NONE return
17         while v ≠ w
18             cycle.append(v)
19             v := parent[v]
20         cycle.append(v)
```

```
1  // dfs is as in CC but with parent
2  procedure dfs(G, s)
3      frontier := new Stack;
4      parent[s] := NONE;
5      C[s] := active;  frontier.push((s, G.adj[s].iterator()))
6      while ¬frontier.isEmpty()
7          (v, i) := frontier.top()
8          if ¬i.hasNext() // v has no unused edge
9              C[v] := done;  frontier.pop()
10             postorderVisit(v)
11         else
12             w := i.next() // Advance i in adj[v]
13             visitEdge(vw)
14             if C[w] == unseen
15                 parent[w] := v;
16                 preorderVisit(w)
17                 C[w] := active;  frontier.push((w, G.adj[w].iterator()))
18             end if
19         end if
20     end while
```

# DFS Postorder & Topological Sort

- **DFS Postorder**: The DFS postorder numbers is a numbering $P[0..n)$ of $V$ such that $P[v] = r$ iff exactly $r$ vertices reached state ***done*** before $v$ in a DFS.

# DFS Postorder & Topological Sort

- **DFS Postorder**: The DFS postorder numbers is a numbering $P[0..n)$ of $V$ such that $P[v] = r$ iff exactly $r$ vertices reached state ***done*** before $v$ in a DFS.
- **Lemma rev postorder:** directed acyclic graph
  Let $G$ be a simple, connected DAG and $R[0..n)$ a *reverse DFS postorder* of $G$, i. e., $R[v] = n - 1 - P[v]$ for a DFS postorder $P[0..n)$. Then $R$ is a topological ranking of $G$.
- **Invariant:** If $v \in$ ***done*** and $(v, w) \in E$ then $w \in$ ***done*** and $R[v] < R[w]$.
  - initially true (***done*** $= \emptyset$)
  - upon `postorderVisit`($v$), all outgoing edges $vw$ lead to $w \in$ ***done*** (Parenthesis Theorem)

# Topological Sorting & Cycle Detection – Summary

- Putting everything together we obtain topological sorting
  - can produce either the *ranking* or the *sequence of vertices* in topological order, whatever is more convenient

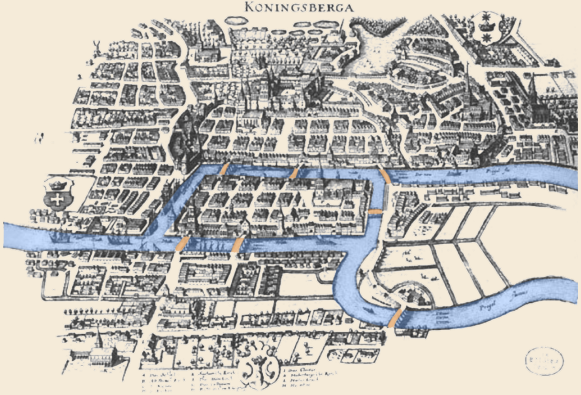
```
1 procedure topologicalRanking(P):
2     (P[0..n), cycle) := dfsPostorder(G)
3     if cycle ≠ NULL
4         return NOT_A_DAG
5     R[0..n) := NONE
6     for v := 0, ..., n − 1
7         R[v] = n − 1 − P[v]
8     return R
```

```
1 procedure topologicalSort(P):
2     (P[0..n), cycle) := dfsPostorder(G)
3     if c ≠ NULL
4         return NOT_A_DAG
5     S[0..n) := NONE
6     for v := 0, ..., n − 1
7         S[n − 1 − P[v]] := v
8     return S
```

- $\Theta(n + m)$ time
- $\Theta(n)$ extra space

# Euler Cycles

**Euler Walk:** Walk using every edge in $G = (V, E)$ exactly once.

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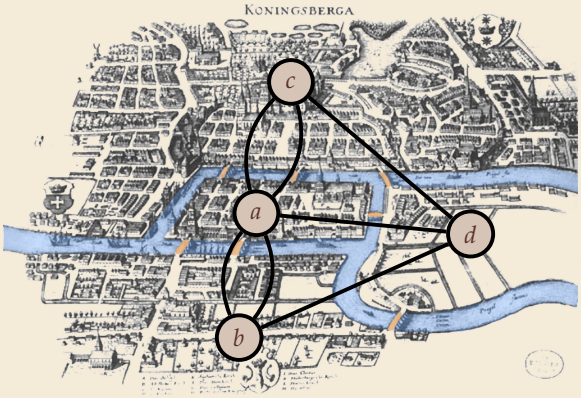


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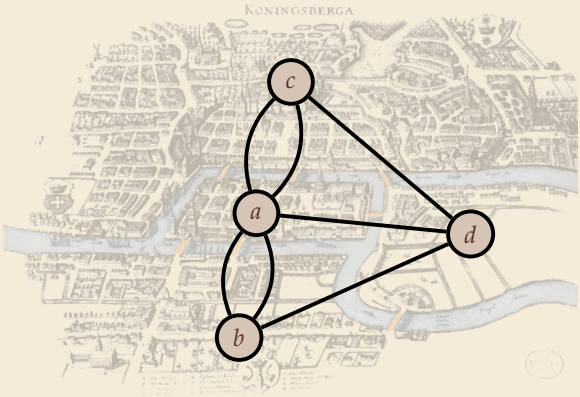
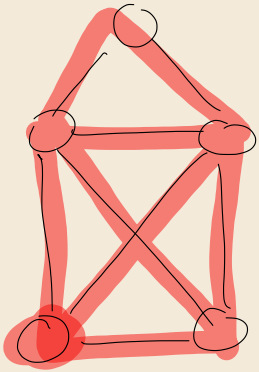


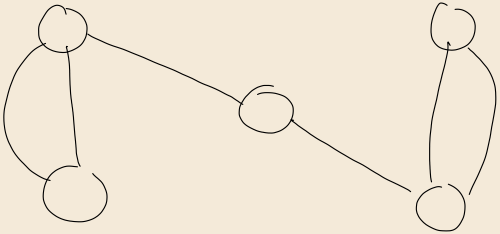


**Euler's Theorem:**
Euler walk exists iff $G$ connected and $0$ or $2$ vertices have odd degree.

'$\Rightarrow$' trivial (need to enter and exit intermediate vertices equally often)

'$\Leftarrow$' Following algorithm *constructs* Euler walk under this assumption





# Euler Cycles – Hierholzer's Algorithm

- use an *edge-centric DFS*
  - We mark *edges* (not vertices)
  - ⇝ stack = **edge-simple walk**
  - We remember iterator $i$ globally per $v$ to resume traversal

```
1  procedure eulerWalk(G):
2      // Assume G = (V, E) is connected (multi)graph
3      V_odd := {v ∈ V : d(v) odd}
4      if |V_odd| ∉ {0, 2} return NOT_EULERIAN
5      if V_odd = {x, y} then s := x else s := 0
6      euler[0..m) := NONE;  j := m − 1
7      visited[0..n, 0..n) := false // mark edges as visited
8      for v := 0, ..., n − 1
9          // globally remember next unexplored edge
10         nextEdge[v] := G.adj[w].iterator())
11     edgeDFS(s)
12     return euler
```

```
1  procedure edgeDFS(s):
2      frontier := new Stack;
3      frontier.push(s)
4      while ¬frontier.isEmpty()
5          v := frontier.top()
6          if ¬i.hasNext() // v has no unused edge
7              frontier.pop()
8              if ¬frontier.isEmpty()
9                  // assign edge leading here largest free index
10                 euler[j] := (frontier.top(), v);  j := j − 1
11             end if
12         else
13             w := i.next()
14             if ¬visited[v, w]
15                 visited[v, w] := true
16                 visited[w, v] := true
17                 frontier.push(w)
18             end if
19         end if
20     end while
```

# Clicker Question

Mark all correct statements about a `dfsTraversal` (Slide 21) of a **DAG** $G$:

A. Listing vertices in the order they are marked ***done*** is a topological sorting of $G$.

B. Listing vertices in the reverse order they are marked ***done*** is a topological sorting of $G$.

C. If $v$ is marked ***done*** before vertex $w$, there is a path $v \rightsquigarrow w$.

D. If $v$ is marked ***done*** before vertex $w$, there is a path $w \rightsquigarrow v$.

E. If $v$ is marked ***done*** before vertex $w$, there cannot be a path $v \rightsquigarrow w$.

F. If $v$ is marked ***done*** before vertex $w$, there cannot be a path $w \rightsquigarrow v$.

→ *sli.do/cs566*

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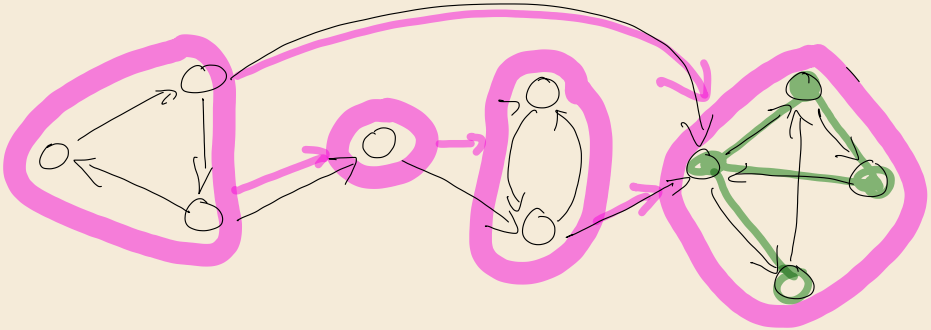
# Strong Components

- **Given:** digraph $G = (V, E)$
- **Goal:** component ids $SCC[0..n)$, s.t. $SCC[v] = SCC[u]$ iff $\exists$ directed path from $v$ to $u$



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  strongly connected component
- *Component DAG* $G^{SCC}$: contract SCCs intro single vertices
  $V(G^{SCC}) = \{C_1, \ldots, C_k\}$ with $C_1 \dot\cup \cdots \dot\cup C_k = V$;
  name by smallest vertex s.t. $i \leq j$ iff $\min C_i \leq \min C_j$
  - can't have cycles (↯ maximality of SCC)

  $\rightsquigarrow$ component DAG has a topological order $R^{SCC}[1..k]$

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  - can't have cycles (↯ maximality of SCC)
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If we call `dfs` on any $v$ in the **last** SCC $C$, it will discover all vertices in $C$, and only those!
(any edges between components lead *into* $C$ by topological order)

And we can iterate this backwards through any topological order to get all SCCs!

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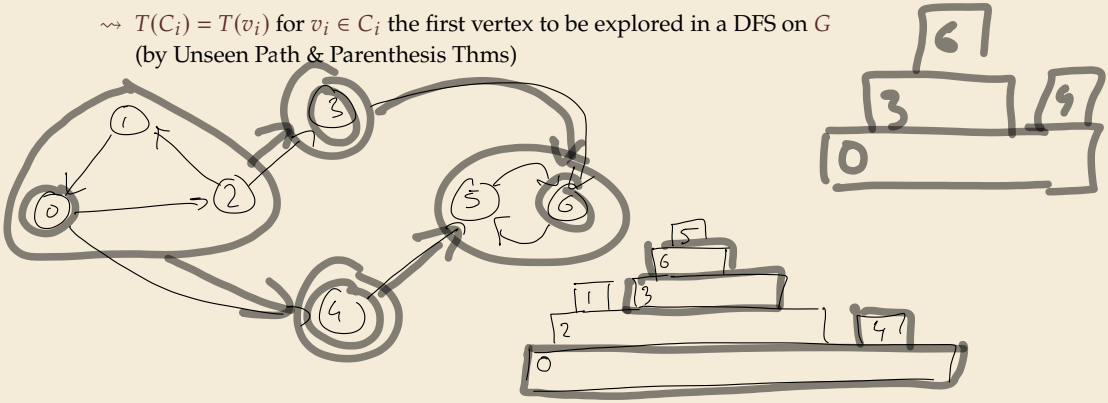
Can we efficiently find the topological order of $G^{SCC}$?
*Without knowing the components to start with??*

Amazingly, yes.

# Component Graph DFS

- Suppose we run `dfsTraversal` on $G$.
  - ⇝ We can extend time intervals to SCCs: $T(C_i) := \bigcup_{v \in C_i} T(v)$
  - ⇝ $T(C_i) = T(v_i)$ for $v_i \in C_i$ the first vertex to be explored in a DFS on $G$ (by Unseen Path & Parenthesis Thms)



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$\rightsquigarrow$ DFS on $G$ produces same $T(C_i)$ (up to time scaling) as DFS on $G^{\text{SCC}}$!

$\rightsquigarrow$ reverse DFS postorder on $G$ gives same relative order to $v_1, \ldots, v_k$ as reverse DFS postorder on $G^{\text{SCC}}$ gives as relative order to $C_1, \ldots, C_k$

# Component Graph DFS

- Suppose we run `dfsTraversal` on $G$.
  - $\rightsquigarrow$ We can extend time intervals to SCCs: $T(C_i) := \bigcup_{v \in C_i} T(v)$
  - $\rightsquigarrow$ $T(C_i) = T(v_i)$ for $v_i \in C_i$ the first vertex to be explored in a DFS on $G$ (by Unseen Path & Parenthesis Thms)

$\rightsquigarrow$ DFS on $G$ produces same $T(C_i)$ (up to time scaling) as DFS on $G^{\text{SCC}}$!

$\rightsquigarrow$ reverse DFS postorder on $G$ gives same relative order to $v_1, \ldots, v_k$ as reverse DFS postorder on $G^{\text{SCC}}$ gives as relative order to $C_1, \ldots, C_k$

We need **reverse** topological order on $G^{\text{SCC}}$, e. g., ***reversed*** *reverse DFS postorder*

- If we had the actual reverse DFS postorder on $G^{\text{SCC}}$, could just reverse again!
- But we only have reverse DFS postorder $S[0..n)$ on $G$!
- Reversing here would change $v_i$, i. e., which vertices of an SCC we see first

# Kosaraju-Sharir's Algorithm

- **Recall:** Want `reverse`(`topologicalRanking`($G^{\text{SCC}}$))

# Kosaraju-Sharir's Algorithm

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  Note: $A$ adj matrix of $G \rightsquigarrow A^T$ adj matrix of $G^T$
- For any DAG, we obtain a reverse topological order from reversing all edges: $\texttt{topologicalSort}(G^T)$

  If we reverse *iteration order* in dfsTraversal, we get $\texttt{reverse}(\texttt{topologicalSort}(G)) = \texttt{topologicalSort}(G^T)$

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$\rightsquigarrow$ Get that as induced ranking on $v_1, \ldots, v_k$ from reverse $\texttt{dfsPostorder}(G^T)$

# Kosaraju-Sharir's Algorithm – Code

```
1  procedure strongComponents(G):
2      // directed graph G = (V, E) with V = [0..n)
3      G^T = (V, {wv : vw ∈ E})
4      P[0..n) := dfsPostorder(G^T) // postorder numbers
5      for v ∈ V do S[P[v]] := v end for // postorder sequence
6      // Rest like connectedComponents (with permuted vertices)
7      C[0..n) := unseen
8      SCC[0..n) := NONE
9      id := 0
10     for j := n − 1, . . . , 0 // reverse postorder seq
11         v := S[j]
12         if C[v] == unseen
13             dfs(G, v)
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  - but derivation more natural this way?
- as all our traversals: $\Theta(n + m)$ time, $\Theta(n)$ extra space

# 9.6 Network flows

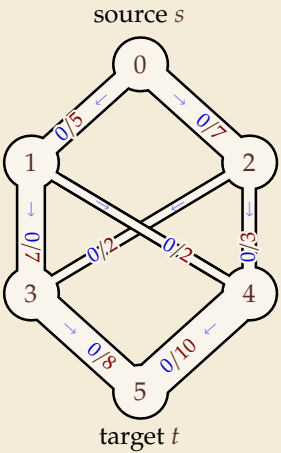
# Clicker Question

Prior knowledge from linear optimization; check all apply.

- A I've seen LPs in lectures before.
- B I could model an application problem as (I)LP.
- C I know algorithms for solving LPs.
- D I know what weak and strong duality in LPs are.
- E I could dualize an LP given to me.
- F I know about the complexity of LPs and ILPs.
- G LPs for me only mean music on vinyl.

→ *sli.do/cs566*

# Networks and Flows – Informal




Informally, imagine a network of water pipes.

- Water can flow through the pipes up to a flow capacity limit (up to $c(e)$ liters per second, say).
- There's infinite water pressing into the source $s$ and infinite drain capacity at the sink / target $t$
- At all other junctions, inflow = outflow (no leakage)

$\rightsquigarrow$ How much water can flow through the network?

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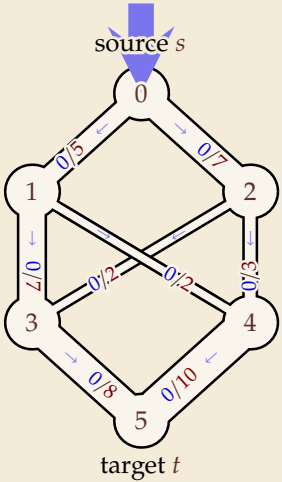


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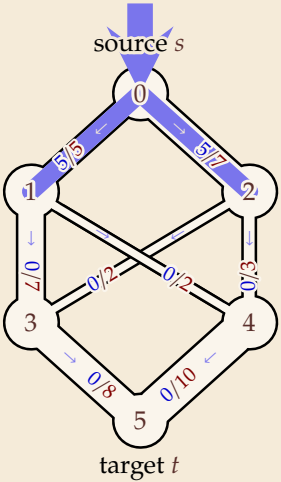


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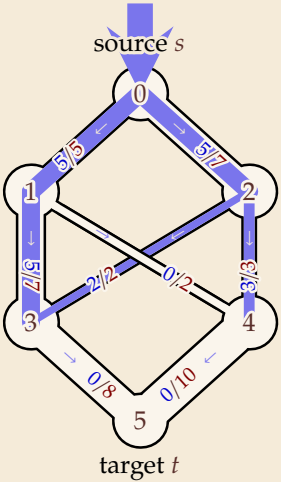


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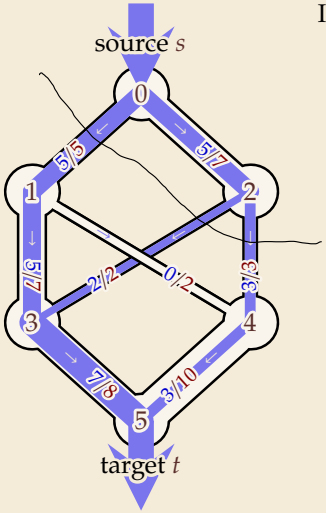


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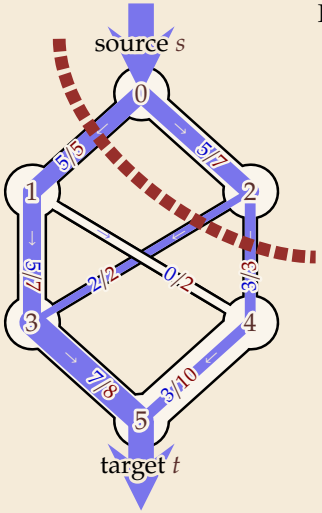


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In this example:

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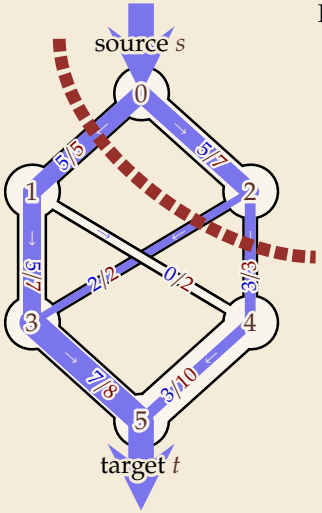


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- ⇝ shown flow is maximal

Remainder of this unit: general version of above (+ efficient algorithms)

# Networks and Flows – Definitions

- *s-t-(flow) network*:
  - **simple**, **directed**, **connected** graph $G = (V, E)$, no antiparallel edges ($vw \in E \rightsquigarrow wv \notin E$) — for notational convenience only
  - *edge capacities* $c : E \to \mathbb{R}_{\geq 0}$
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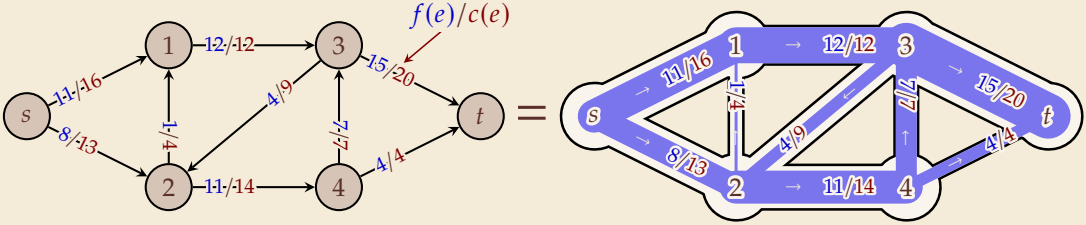

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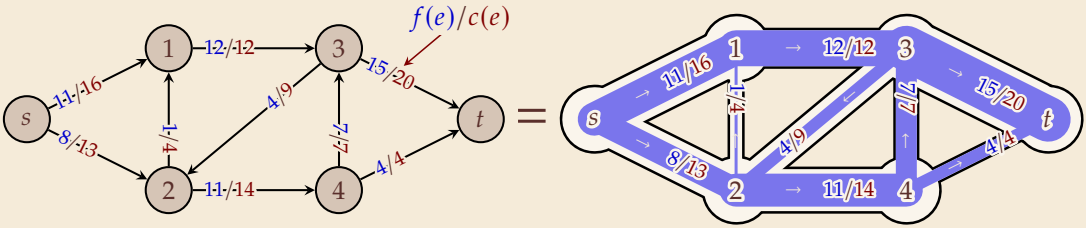

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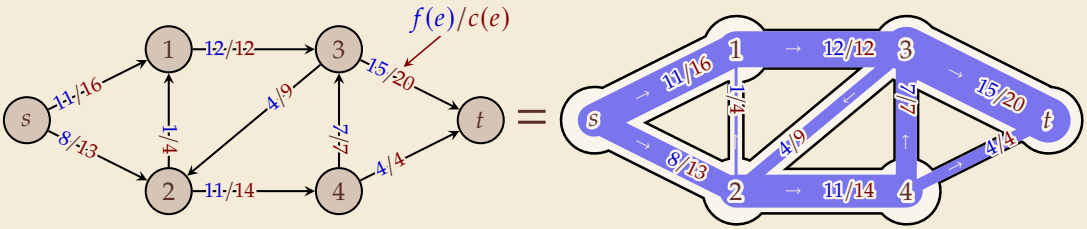
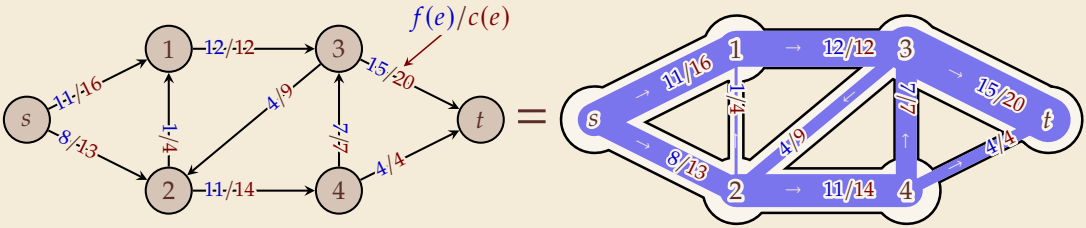


# Max-Flow Problem




- **Maximum-Flow Problem:**
  - **Given:** $s$-$t$-flow network
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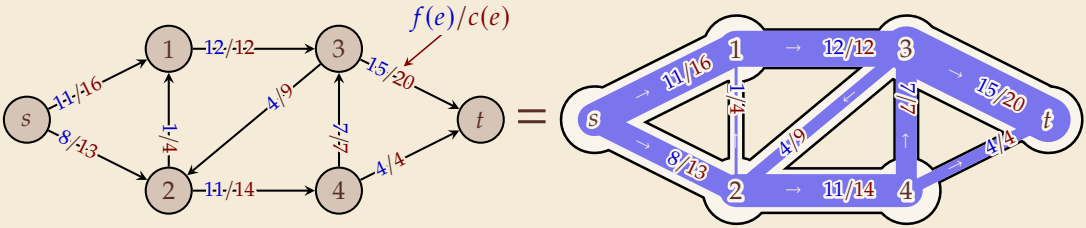


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- $\mathbb{N}$ vs $\mathbb{R}$
  - We focus on integral capacities here (as we will see) can restrict ourselves to integral flows
  - but: ideally want algorithms that work with arbitrary real numbers, too

# Multiple Sources & Sinks, Antiparallel Edges

- Some of the restrictions can be generalized easily.
- We forbid **loops** and **antiparallel** edges.

  - The presented algorithms actually work fine with both!
  - but proofs are cleaner to write without them
  - also: can always remove loops and (anti)parallel edges by adding a new vertex in the middle of the edge

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- We only allow a **single source** and a **single sink**
  - can add a **"supersource"** and **"supersink"** with capacity-$\infty$ edges to all sources resp. sinks

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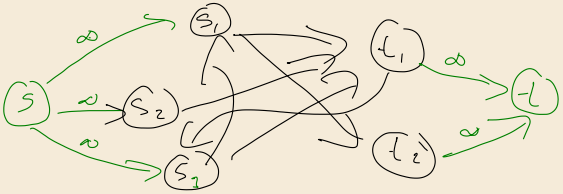

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1. **Disjoint Paths**
   - **Given:** Unweighted (di)graph $G = (V, E)$, vertices $s, t \in V$
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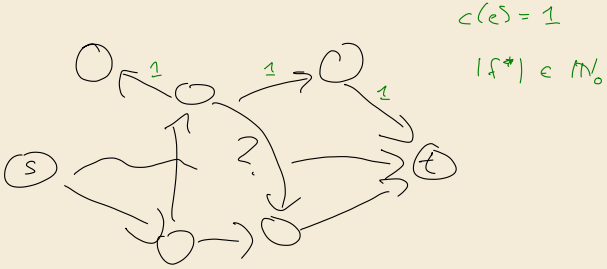

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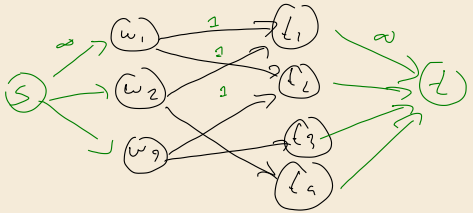


$f^*(e) \in \mathbb{N}_0$

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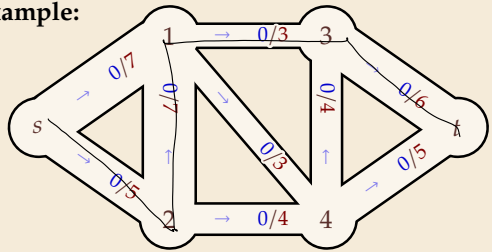
- Both problems can be solved by (in both cases, 1. and 3. are very efficient)
  1. constructing a specific flow network from their input data
  2. computing a maximum flow in that network
  3. "reading off" a solution for the original problem from the max flow

# 9.7 The Ford-Fulkerson Method

# Push Push Push!

- **Simple Idea:** Iteratively find a path from $s$ to $t$ that we can push more flow over.

**Example:**




***1.*** Push 3 units of flow over $s \to 2 \to 1 \to 3 \to t$

**Problem:** Cannot undo mistakes.
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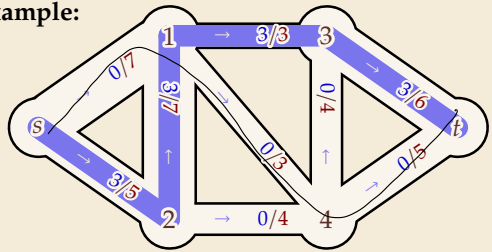


1. Push 3 units of flow over $s \to 2 \to 1 \to 3 \to t$
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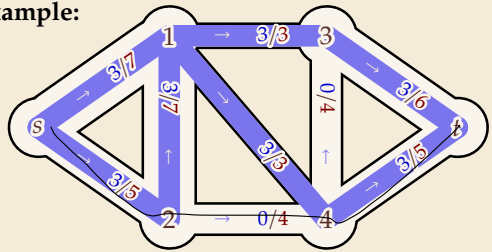


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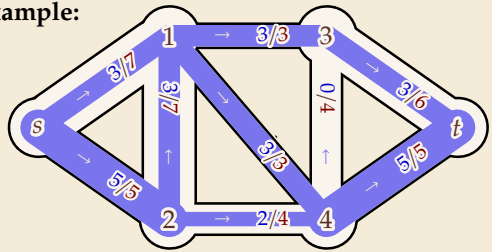


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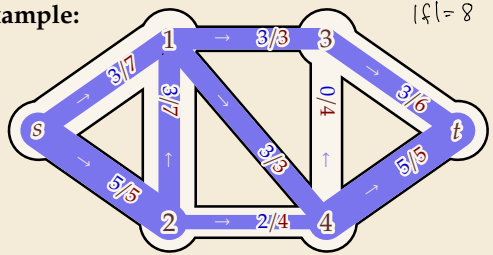


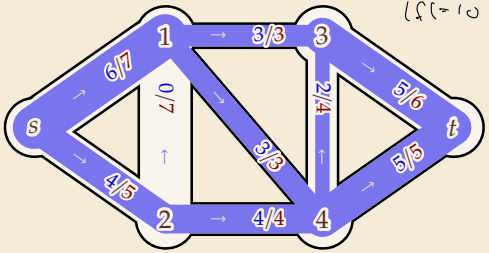


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⚠ **But:** resulting flow is **not** optimal!

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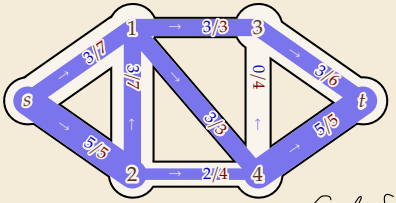


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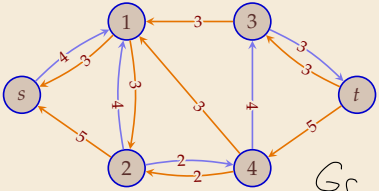


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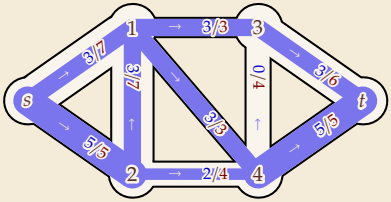


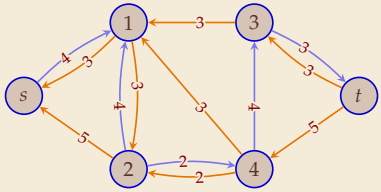


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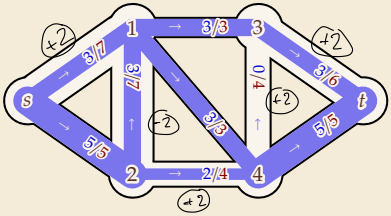


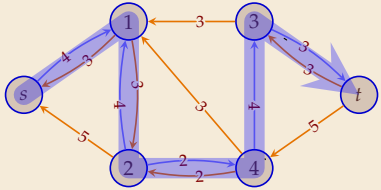


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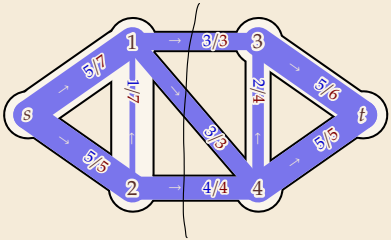


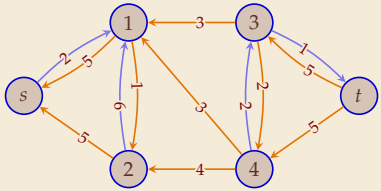


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# Cuts

- **Goal:** Certificate for maximum flows
- *s-t-cut* $(S, T)$: partition $S \dot\cup T = V$, $s \in S$, $t \in T$

  disjoint union

  - *net flow* across cut:
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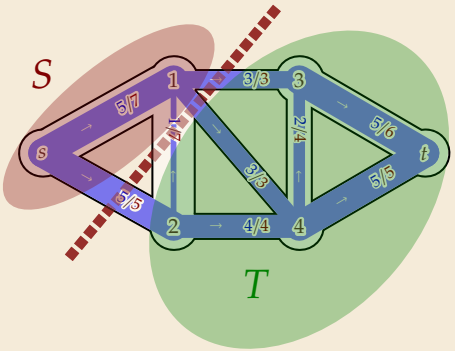

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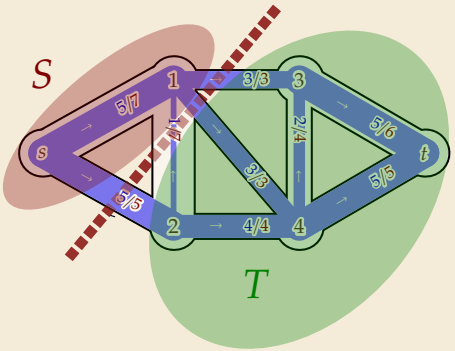


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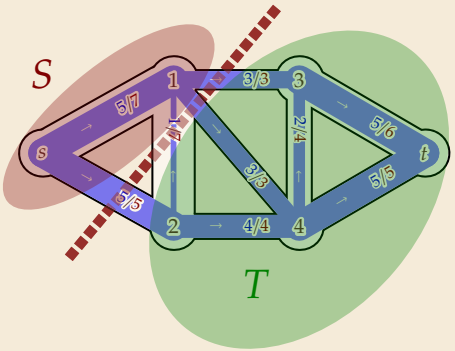
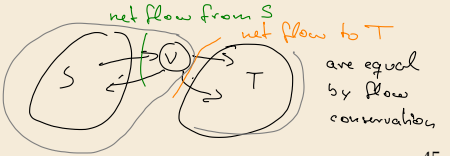



- $f(S,T) = 5+3+3-1 = 10$
- $c(S,T) = 5+3+3 = 11$

- **Lemma:** For any cut $(S,T)$, we have $f(S,T) = |f|$.
  (flow conservation!)

Def. $|S| = f(\{s\}, V\setminus\{s\})$ induction on $|S|$

is: add $v$ to $S$



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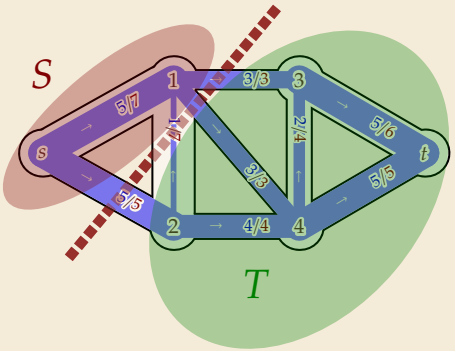


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- **Lemma:** For any cut $(S,T)$, we have $f(S,T) = |f|$.
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- **Corollary:** $|f| \le c(S,T)$ for any $s$-$t$-cut $(S,T)$

# The Max-Flow Min-Cut Theorem

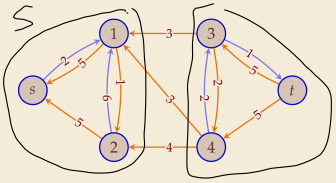
- **Max-Flow Min-Cut Theorem:**
  Let $f$ be a feasible flow in $s$-$t$-network $G = (V, E)$. Then the following conditions are equivalent:

  1. $|f| = c(S, T)$ for some cut $(S, T)$ of $G$.
  2. $f$ is a maximum flow in $G$
  3. The residual network $G_f$ has no augmenting path.

Proofs: "(1) => (2)" Corollary implies: every other flow $f'$ $|f'| \leq c(S,T) = |f|$

"(2) => (3)" contraposition if $G_f$ has $p$ from $s$ to $t$ => can increase $|f|$
=> $|f|$ not maximal

"(3) => (1)" $S = \{v : s \rightsquigarrow v\}$ $T = V \setminus S$ => $(S,T)$ is a cut




edges across $(S,T)$ are saturated forward edges or empty backward edges

$$f(S,T) = \sum_{v \in S} \sum_{w \in T} c(v,w) = c(S,T)$$

□

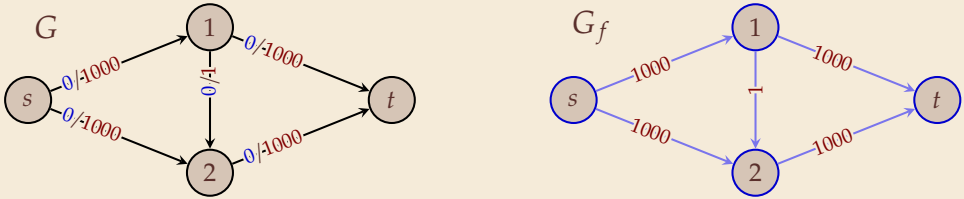
# Generic Ford-Fulkerson Method

```
1  procedure genericFordFulkerson(G = (V, E), s, t, c):
2      // G is a flow network with source s ∈ V, sink t ∈ V and capacities c : E → ℝ≥0
3      for vw ∈ E do f(vw) := 0 end for
4      while ∃ path p from s to t in G_f // Freedom: Which augmenting path?
5          Δ := min{c_f(e) : e ∈ p} // bottleneck capacity
6          for e ∈ p
7              if e ∈ E // forward edge
8                  f(e) := f(e) + Δ
9              else // backward edge
10                 f(e) := f(e) − Δ
11     return f
```

- Returned flow is a maximum flow $f^*$ (Max-Flow Min-Cut Theorem)
- If $c : E \to \mathbb{N}_0$, also $f : E \to \mathbb{N}_0$: For all $v, w \in V$ holds:
  - initially $f(vw) = 0 \in \mathbb{N}_0$
  - $c_f(vw)$ is difference of $c(vw) \in \mathbb{N}_0$ and $f(vw) \in \mathbb{N}_0$
  - $\Delta$ equal to some $c_f(v'w') \in \mathbb{N}_{\geq 1}$ ($E_f$ contains only non-zero capacity edges!)
  - $\rightsquigarrow$ new flow $f(vw) \pm \Delta \in \mathbb{N}_0$
- $\rightsquigarrow$ For integral capacities, always terminate after $\leq |f^*|$ iterations

# Bad Example

- Unfortunately, we might also take $|f^*|$ iterations!

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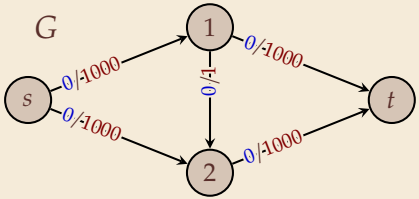


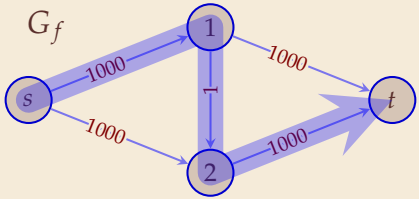


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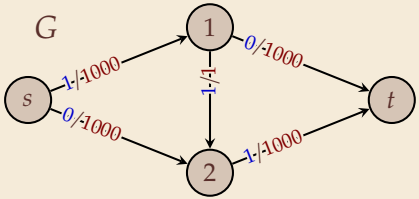


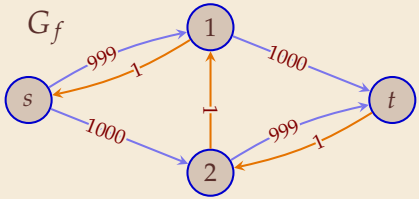


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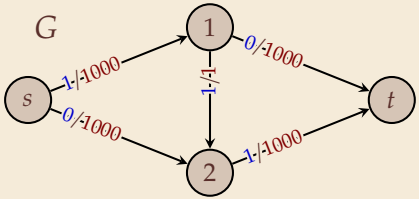


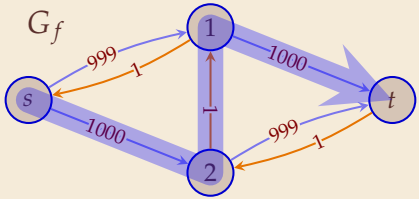


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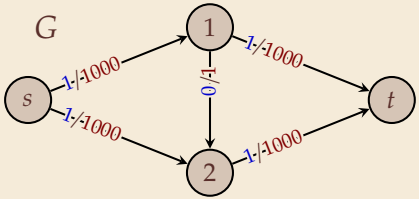


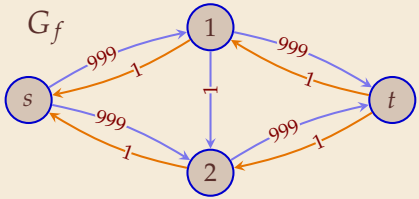


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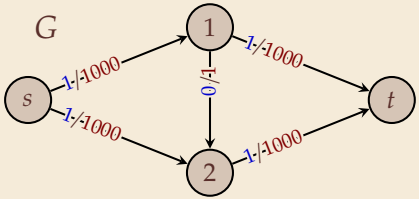


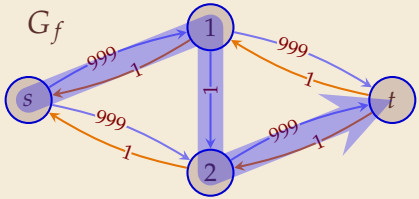


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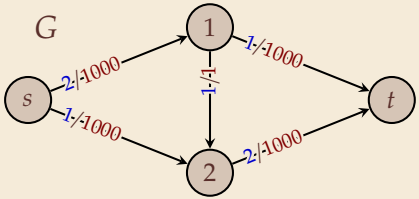


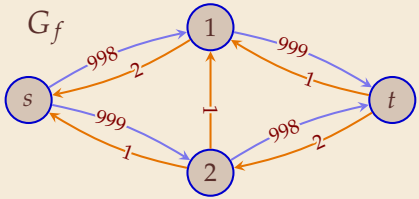


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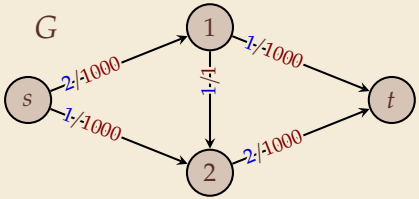


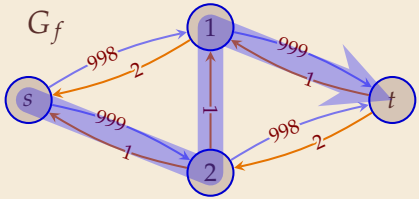


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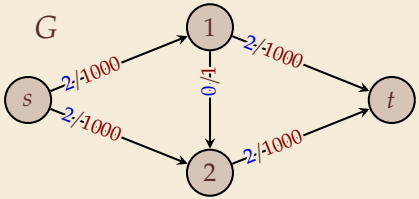


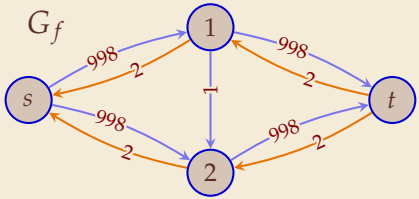


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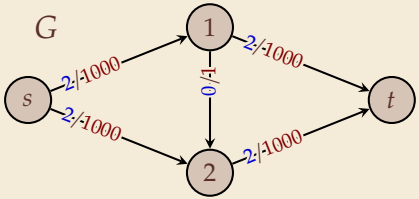


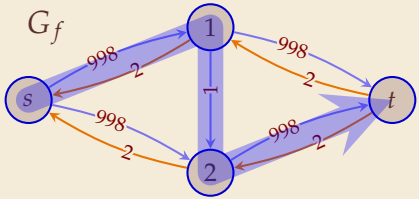


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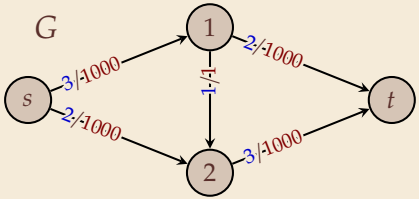


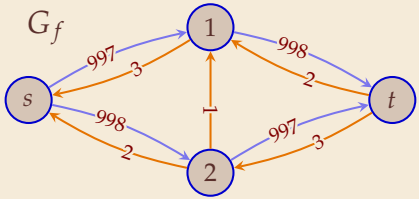


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# A Very Bad Example

- for irrational flows, might not even terminate
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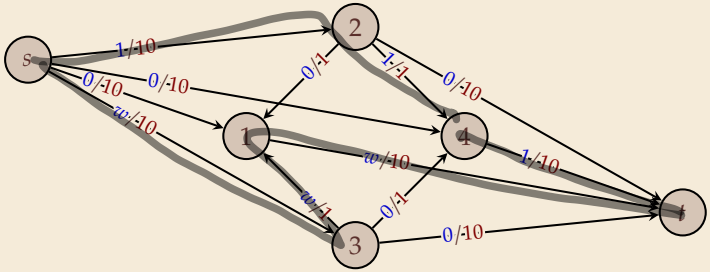
- $w = \varphi - 1 = (\sqrt{5} - 1)/2 \approx 0.618 \quad \rightsquigarrow \quad 1 - w = w^2 \approx 0.382$



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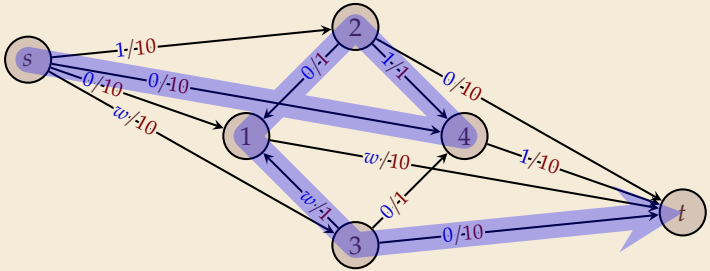

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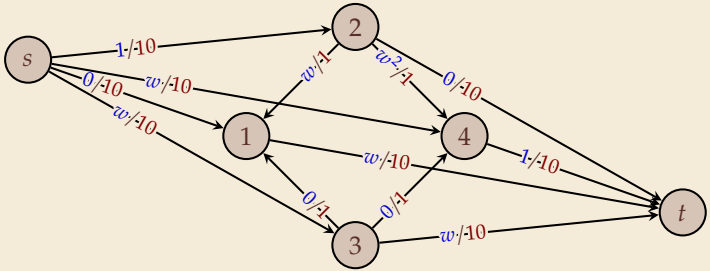

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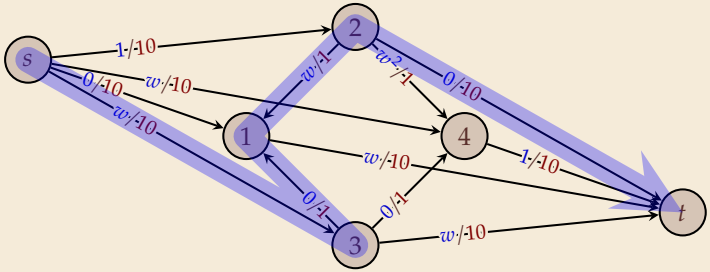

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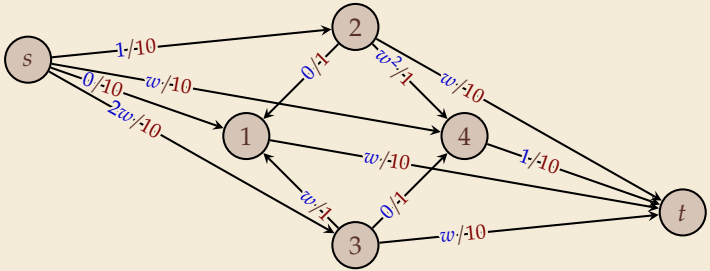

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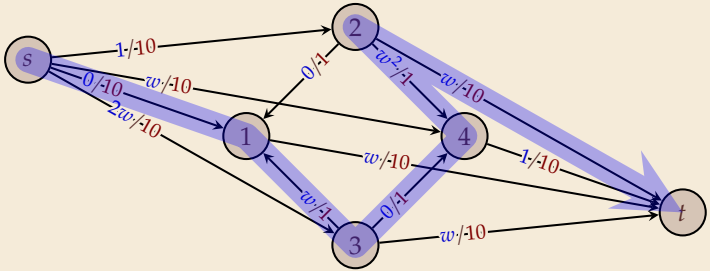

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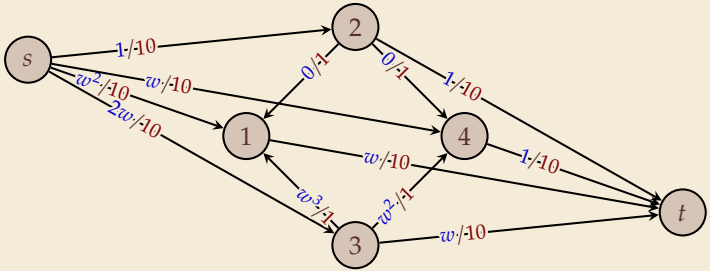

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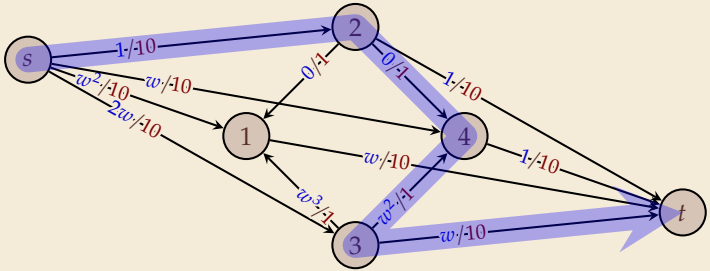

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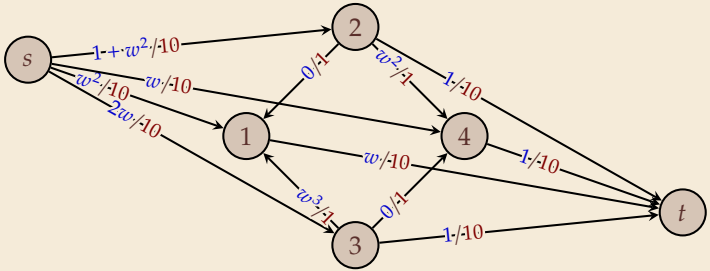

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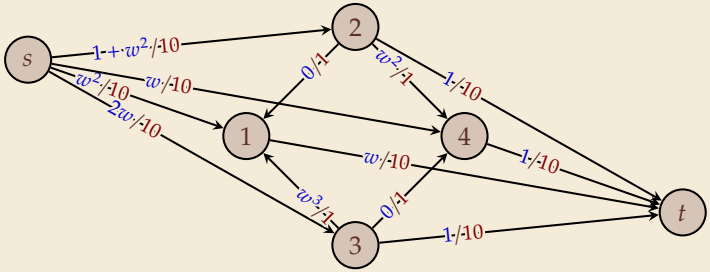


- after 2 paths, situation in 1-2-3-4 restored (rotated), but flows multiplied by $w$

$\rightsquigarrow$ augmenting paths have capacities $w$, $w$, $w^2$, $w^2$, $w^3$, $w^3$ ...

$\rightsquigarrow$ never terminate, never exceed $|f| \geq 5$

# 9.8 The Edmonds-Karp Algorithm

# Edmonds-Karp

- It turns out, many ways to choose augmenting paths systematically work fine
- Edmonds & Karp: take a shortest path (in #edges)

Java implementation
Sedgewick & Wayne

```
1  procedure EdmondsKarp(G = (V, E), s, t, c):
2      // G is a flow network with source s ∈ V, sink t ∈ V and capacities c : E → ℝ≥0
3      for vw ∈ E do f(vw) := 0 end for
4      while true
5          bfs(G_f, {s})
6          if distFrom[t] == ∞ return f
7          else p := pathTo(t)
8          Δ := min{c_f(e) : e ∈ p} // bottleneck capacity
9          for e ∈ p
10             if e ∈ E // forward edge
11                 f(e) := f(e) + Δ
12             else // backward edge
13                 f(e) := f(e) − Δ
14     end while
```

# Edmonds-Karp – Analysis

- **Theorem:** The Edmonds-Karp algorithm terminates after $O(nm)$ iterations with a maximum flow. The total running time is in $O(nm^2)$.

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- *Proof Plan:*
  - every augmenting path has a *critical* edge $vw$ contributing the bottleneck capacity
  - we will show:
    (1) distances of vertices from $s$ in $G_f$ weakly increase over time
    (2) before $vw$ can be a *critical* edge *again*, $v$'s distance increases by at least 2
  - $\rightsquigarrow$ each edge $vw$ is critical for at most $n/2$ augmenting paths ($v$'s distance $\in [1..n-2]$)
  - $\rightsquigarrow$ $O(nm)$ augmenting paths
  - each iteration runs one BFS, which costs $O(n+m) = O(m)$ times since $G$ is connected.

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- **Notation:**
  - Write $f_0, f_1, \ldots$ for values of $f$ during iterations of while loop
  - $\rightsquigarrow$ $G_{f_i}$ residual network after $i$th augmentation
  - Write $\delta_i(v)$ for shortest-path distance from $s$ to $v$ in $G_{f_i}$

# Edmonds-Karp – Analysis [2]

(1) ▶ **EK Monotonicity Lemma:** For all $i$ and $v \in V$, we have $\delta_{i+1}(v) \geq \delta_i(v)$.

- $f_i$: flow after $i$th augmentation
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*Proof:*

- by induction over $k$, the value of $\delta_i(v)$
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  - $\rightsquigarrow$ $\exists$ shortest path $p[0..k+1]$ in $G_{f_{i+1}}$ with $p[0] = s$ and $p[k+1] = v$.
  - $\rightsquigarrow$ For $w = p[k]$, $p[0..k]$ is a shortest path from $s$ to $w$ $\rightsquigarrow$ $k = \delta_{i+1}(w) \underset{\text{IH}}{\geq} \delta_i(w)$

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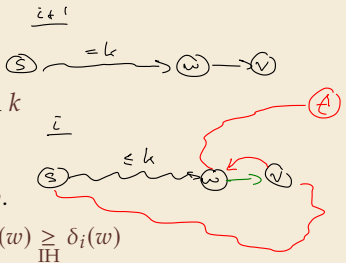


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  - Case 1: $wv \in E_{f_i}$ $\rightsquigarrow$ $\delta_i(v) \leq \delta_i(w) + 1$
  - Case 2: $wv \notin E_{f_i}$ $\rightsquigarrow$ reverse edge $vw$ in $i$th augmenting path, a shortest $s$-$t$-path
    $\rightsquigarrow$ $\delta_i(v) = \delta_i(w) - 1 \leq \delta_i(w) + 1$
  - in both cases: $\delta_{i+1}(v) = \delta_{i+1}(w) + 1 \geq \delta_i(w) + 1 \geq \delta_i(v)$

# Edmonds-Karp – Analysis [3]

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$\rightsquigarrow \delta_i(w) = \delta_i(v) + 1$ in $G_{f_i}$ vw not present

- before $vw$ reappears in $G_f$, need to have had $wv$ in augmenting path; say this first happens in iteration $j > i$ $\rightsquigarrow$ $\delta_j(v) = \delta_j(w) + 1$

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- by EK Monotonicity Lemma:
  $\delta_j(v) = \delta_j(w) + 1 \geq \delta_i(w) + 1 = \delta_i(v) + 2$

This concludes the proof of the theorem.

# Maximum Flow – Discussion

Edmonds-Karp is a robust choice

easy to implement (see Sedgewick Wayne for an elegant Java version!)

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- max-flow min-cut theorem is a special case of LP duality
- can also solve generalization of min-cost flows
  - each edge $vw$ has a cost $a(vw)$
  - cost of a flow $f$: $\sum a(vw) \cdot f(vw)$
  - demand $d$ at sink becomes part of constraints: $|f| \geq d$