12

Dynamic Programming

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CS566 (Wintersemester 2024/25)
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Learning Outcomes

Unit 12: *Dynamic Programming*

1. Be able to apply the DP paradigm to solve new problems.

12 Dynamic Programming

12.1 Elements of Dynamic Programming

Introduction

applicable to many problems ↙

- *Dynamic Programming (DP)* is a powerful algorithm **design pattern** for exact solutions to **optimization** problems
- Some commonalities with Greedy Algorithms,
  but with an element of brute force added in

  DP = "careful brute force" (Erik Demaine)
- often yields polynomial time, but usually not linear time algorithms
- for many problems the *only* way we know to build efficient algorithms

- **Naming fun:** The term "dynamic programming", due to Richard Bellman from around 1953,
  does not refer to computer programming; rather to a program (= plan, schedule) changing with time.
  It seems to have been at least partly marketing babble devoid of technical meaning . . .

Plan of the Unit

1. Abstract steps of DP (briefly)
2. Details on a concrete example (*matrix chain multiplication*)
3. More examples!

The 6 Steps of Dynamic Programming

1. Define **subproblems** (and relate to original problem)
2. **Guess** (part of solution) ⇝ local brute force
3. Set up **DP recurrence** (for quality of solution)
4. Recursive implementation with **Memoization**
5. Bottom-up **table filling** (topological sort of subproblem dependency graph)
6. **Backtracing** to reconstruct optimal solution

- Steps 1–3 require insight / creativity / intuition;
  Steps 4–6 are mostly automatic / same each time

⇝ Correctness proof usually at level of DP recurrence

👍 running time too! worst case time = #subproblems · time to find single best guess

When does DP (not) help?

- *No Silver Bullet*
  DP is the most widely applicable design technique, but can't *always* be applied

1. Vitally important for DP to be correct:

Bellman's Optimality Criterion

For a *correctly guessed* fixed part of the solution,
***any* optimal solution to the corresponding subproblems**
must yield an *optimal solution* to the overall problem (once combined).

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> ***any* optimal solution to the corresponding subproblems**
> **must yield an *optimal solution* to the overall problem (once combined).**

at most polynomial in n

2. Also, the total **number of different subproblems** should be *"small"*
   (DP potentially still works correctly otherwise, but won't be *efficient*.)

12.2 DP & Matrix Chain Multiplication

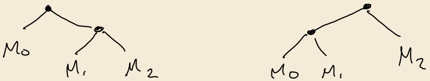
The Matrix-Chain Multiplication Problem

Consider the following exemplary problem

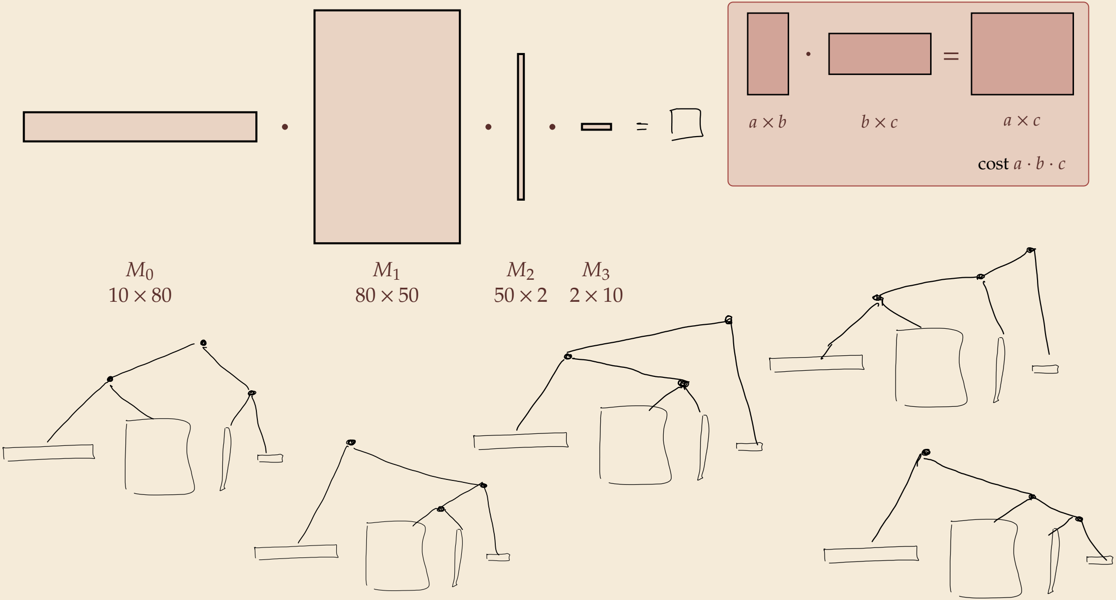
- We have a product $M_0 \cdot M_1 \cdot \cdots \cdot M_{n-1}$ of n matrices to compute
- Since (matrix) multiplication is associative, it can be evaluated in different orders.
- For non-square matrices of different sizes, different order can change costs dramatically
  - Assume elementary matrix multiplication algorithm:
  - ⇝ Multiplying $a \times b$-matrix with $b \times c$ matrix costs $a \cdot b \cdot c$ integer multiplications
- **Given:** Row and column counts $\overset{r}{r}[0..n)$ and $\overset{c}{c}[0..n)$ with $r[i+1] = c[i]$ for $i \in [0..n-1)$ (corresponding to matrices $M_0, \ldots, M_{n-1}$ with $M_i \in \mathbb{R}^{r[i] \times c[i]}$)
- **Goal:** parenthesization of the product chain with minimal cost

really a binary tree with n leaves!

$(M_0 \cdot (M_1 \cdot M_2))$ $((M_0 \cdot M_1) \cdot M_2)$



Matrix-Chain Multiplication – Example



M_0
10×80

M_1
80×50

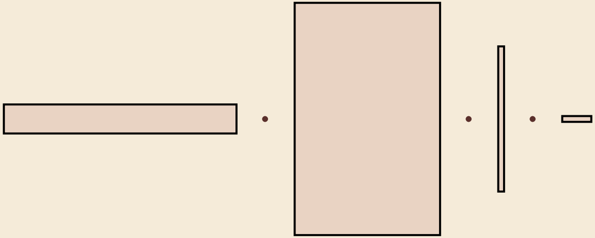
M_2
50×2

M_3
2×10

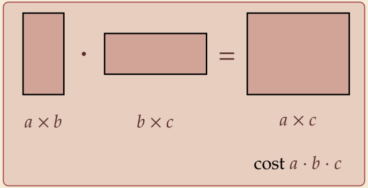
$a \times b$ · $b \times c$ = $a \times c$

cost $a \cdot b \cdot c$

Matrix-Chain Multiplication – Example



M_0 10×80 · M_1 80×50 · M_2 50×2 · M_3 2×10



Parenthesization	Cost (integer multiplications)		
$M_0 \cdot (M_1 \cdot (M_2 \cdot M_3))$	$1000 + 40\,000 + 8000$	=	49 000
$M_0 \cdot ((M_1 \cdot M_2) \cdot M_3)$	$8000 + 1600 + 8000$	=	17 600
$(M_0 \cdot M_1) \cdot (M_2 \cdot M_3)$	$40\,000 + 1000 + 5000$	=	46 000
$(M_0 \cdot (M_1 \cdot M_2)) \cdot M_3$	$8000 + 1600 + 200$	=	9 800
$((M_0 \cdot M_1) \cdot M_2) \cdot M_3$	$40\,000 + 1000 + 200$	=	41 200

first or last operation

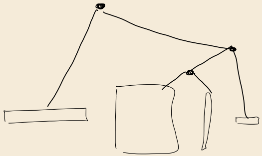
Greedy fails both ways!

Matrix-Chain Multiplication – How about Brute Force?

If Greedy doesn't give optimal parenthesization, maybe just try all?

- parenthesizations for n matrices $=$ binary trees with n leaves (*evalution trees*)
  $=$ binary trees with $n-1$ (internal) nodes

- How many such trees are there?



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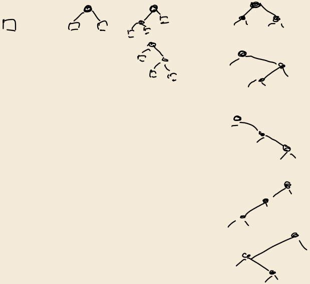
  $=$ binary trees with $\underbrace{n-1}_{m}$ (internal) nodes
- How many such trees are there?
  - Let's write $m = n - 1$;
  - $C_0 = 1, C_1 = 1, C_2 = 2, C_3 = 5$



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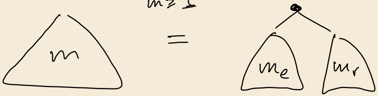
rank of root

$m \geq 1$



$m_\ell + m_r = m - 1$

$0 \leq m_\ell, m_r \leq m - 1$

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    generating functions / combinatorics / guess (OEIS!) & check . . .
  - Can show $C_n = \dfrac{1}{n+1}\dbinom{2n}{n} \sim \dfrac{1}{\sqrt{\pi}} \cdot \dfrac{4^n}{n^{3/2}}$

  ⇝ *exponentially many trees (almost 4^n)* $\qquad C_{20} = 6\,564\,120\,420, \;\; C_{30} = 3\,814\,986\,502\,092\,304$

⇝ A brute-force approach is utterly hopeless

⇝ *Dynamic programming to the rescue!*

Matrix-Chain Multiplication – Step 1: Subproblems

- Key ingredient for DP: Problem allows for recursive formulation
  Need to decide:
  1. What are the **subproblems** to consider?
  2. How can the **original problem** be expressed as subproblem(s)?

1. Subproblems
2. Guess!
3. DP Recurrence
4. Memoization
5. Table Filling
6. Backtrace

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Here:

1. **Subproblems** = Ranges of matrices $[i..j)$ $\quad 0 \le i \le j \le n$
   i. e., optimal parenthesization for each range $\underline{M_i, M_{i+1}, \ldots, M_{j-1}}$

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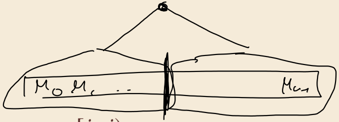

- **Intuition:**
  - Any subtree in binary multiplication tree covers some range $[i..j)$
    (matrix multiplication is not commutative $\rightsquigarrow$ left-right order has to stay)
  - left and right factors of a multiplication don't "see/influence" each other

Matrix-Chain Multiplication – Step 2: Guess

- Usually, any subproblem can be split into smaller subproblems in **several** ways
- Which way to decompose gives best solution not known *a priori*

⇝ What do we have to correctly *guess* to solve the problem?

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$\leadsto$ What do we have to correctly *guess* to solve the problem?

- Here: **Guess** last multiplication / root of binary tree

$\leadsto$ index $k \in [i+1..j)$ so that $[i..j)$ computed with **last** multiplication
$\underbrace{(M_i \cdot \cdots \cdot M_{k-1})} \cdot \underbrace{(M_k \cdot \cdots \cdot M_{j-1})}$

$\leadsto$ optimal parenthesization of $M_i, \ldots, M_{k-1}$ and $M_k, \ldots, M_{j-1}$ computed recursively (corresponds to subproblems $[i..k)$ and $[k..j)$)

try all k!

Matrix-Chain Multiplication – Step 3: DP Recurrence

- With subproblems and guessed part fixed,
  we try to express total **value/cost of solution** *recursively*

⇝ *We ignore the actual solution and just compute its cost!*

- Often good to prove correctness at level of recurrence

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subproblem $[i..j)$

- Here: **Recurrence** for $m(i,j)$ = total number of integer multiplications used in best parenthesization of $[i..j)$

⇝ Set up recurrence, including any base cases.

$$m(i,j) = \begin{cases} 0 & \text{if } j-i \le 1 \\ \underset{\text{best } k \text{ chosen by } \textit{local brute force}}{\min}\left\{ \boxed{\overset{\text{recursive cost}}{m(i,k)+m(k,j)} + \overset{\text{cost of last multiplication}}{r[i]\cdot r[k]\cdot c[j-1]}} : \overset{\text{guess}}{k} \in [i+1\,..\,j) \right\} & \text{otherwise} \end{cases}$$

Matrix-Chain Multiplication – Correctness

Claim: Let $\underline{m(i, j)}$ for $0 \le i \le j \le n$ be defined by the recurrence

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Then $m(i, j) =$ #integer multiplications in best parenthesization of $M_i \cdots M_{j-1}$.

Proof:

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Proof: By induction over $j - i$

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  In both cases, no multiplications are needed and $m(i, j) = 0$. ✓

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  - T's root must be a last product of left and right subterms $(M_i \cdots M_{k-1}) \cdot (M_k \cdots M_{j-1})$ for some $i < k < j$, with cost $r[i]r[k]c[j-1]$.

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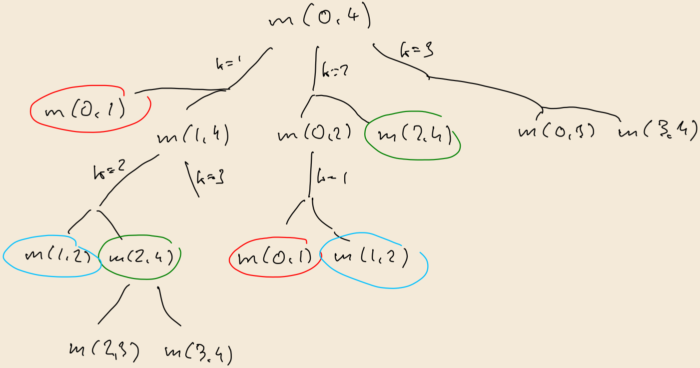
  $\rightsquigarrow$ By IH, the cost of T_ℓ and T_r are given by $m(i,k)$ and $m(k,j)$

  $\rightsquigarrow$ $m(i,j) = $ cost of T $\square$

$$m(i,j) = \begin{cases} 0 & \text{if } j-i \le 1 \\ \min\left\{ m(i,k) + m(k,j) + r[i] \cdot r[k] \cdot c[j-1] \;:\; k \in [i+1\,..\,j) \right\} & \text{otherwise} \end{cases}$$



Matrix-Chain Multiplication – Step 4: Memoization

- Write **recursive** function to compute recurrence
- But *memoize* all results! (symbol table: subproblem ↦ optimal cost)

⇝ First action of function: check if subproblem known

  - If so, return cached optimal cost
  - Otherwise, compute optimal cost and remember it!

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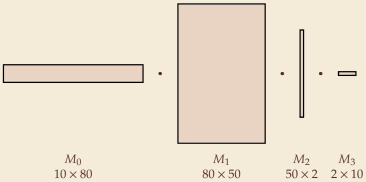
implements recurrence

```
procedure totalMults($r[i..j)$, $c[i..j)$):
    if $j - i \le 1$
        return 0
    else
        $best := +\infty$
        for $k := i+1, \ldots, j-1$
            $m_l := $ ~~cached~~TotalMults($r[i..k)$, $c[i..k)$)
            $m_r := $ ~~cached~~TotalMults($r[k..j)$, $c[k..j)$)
            $m := m_l + m_r + r[i] \cdot r[k] \cdot c[j-1]$
            $best := \min\{best, m\}$
        end for
        return $best$
```

$$m(i,j) \;=\; \begin{cases} 0 & \text{if } j-i \le 1 \\ \min\Big\{ m(i,k) + m(k,j) + r[i] \cdot r[k] \cdot c[j-1] \;:\; k \in [i+1..j) \Big\} & \text{otherwise} \end{cases}$$

```
procedure cachedTotalMults($r[i..j)$, $c[i..j)$):
    // $m[0..n)[0..n)$ initialized to NULL at start
    if $m[i][j]$ == NULL
        $m[i][j]$ := totalMults($r[i..j)$, $c[i..j)$)
    return $m[i,j]$
```

Matrix-Chain Multiplication – Example Memoization



$n = 4$

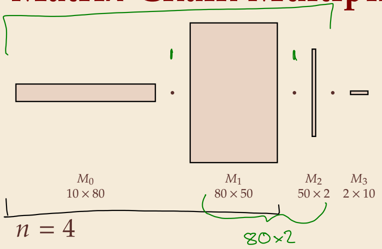
$r[0..n) = [10, 80, 50, 2]$

$c[0..n) = [80, 50, 2, 10]$

$m[i][j]$

i \ j	0	1	2	3	4
0	0	0			
1	—	0	0		
2	—	—	0	0	
3	—	—	—	0	0
4	—	—	—	—	0

Matrix-Chain Multiplication – Example Memoization



$n = 4$

$r[0..n) = [10, 80, 50, 2]$

$c[0..n) = [80, 50, 2, 10]$

m(0,2) = 10·80·50 = 40

40000

1600

m(0,3) = min{10·80·2 + m(1,3), m(0,2) + 10·50·2}

8000

traceback(0,4) k=3

(traceback(0,3)) · (traceback(3,4))

k=1 = M3

(traceback(0,1))(traceback(1,3))

= M0

(1,2) (2,3)

= M1 M2

= ((M0)·((M1)·(M2)))·(M3)

$m[i][j]$

i \ j	0	1	2	3	4
0	0	0	40000	9600	9800
1	—	0	0	8000	9600
2	—	—	0	0	1000
3	—	—	—	0	0
4	—	—	—	—	0

Matrix-Chain Multiplication – Runtime Analyses

```
procedure totalMults($r[i..j)$, $c[i..j)$):
    if $j - i \le 1$
        return 0
    else
        $best := +\infty$
        for $k := i + 1, \ldots, j - 1$
            $m_l := \text{cachedTotalMults}(r[i..k), c[i..k))$
            $m_r := \text{cachedTotalMults}(r[k..j), c[k..j))$
            $m := m_l + m_r + r[i] \cdot r[k] \cdot c[j-1]$
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        end for
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    return $m[i, j]$
```

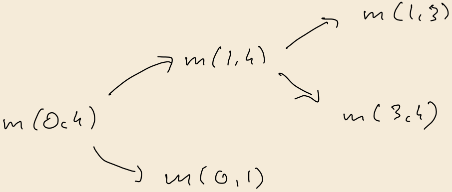
- With memoization, compute each subproblem at most once
- nonrecursive cost (totalMults): $O(j - i) = O(n)$
- Number of subproblems $[i..j)$ for $0 \le i \le j \le n$

$$\sum_{0 \le i \le j \le n} 1 \;=\; \sum_{i=0}^{n} \sum_{j=i}^{n} 1 \;=\; \Theta(n^2)$$

⇝ total running time $\overset{O}{\cancel{\Theta}}(n^3)$

Matrix-Chain Multiplication – Step 5: Table Filling

- Recurrence induces a DAG on subproblems (who calls whom)
  - Memoized recurrence traverses this DAG (DFS!)
  - We can slightly improve performance by systematically computing subproblems following a fixed topological order



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  - We can slightly improve performance by systematically computing subproblems following a fixed topological order
- **Topological order** here: by **increasing length** $\ell = j - i$, then by i

$$\underbrace{m(0,4)}_{\ell=4} > \underbrace{m(1,4) > m(0,3)}_{\ell=3} > \underbrace{m(2,4) > m(1,3) > m(0,2)}_{\ell=2} > \dots$$

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Matrix-Chain Multiplication – Step 5: Table Filling

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- **Topological order** here: by **increasing length** $\ell = j - i$, then by i

```
procedure totalMultsBottomUp($r[0..n)$, $c[0..n)$):
    $m[0..n)[0..n) := 0$ // initialize to 0
    for $\ell = 2, 3, \ldots, n$ // iterate over subproblems ...
        for $i = 0, 1 \ldots, n - \ell$ // ... in topological order
            $j := i + \ell$
            $m[i][j] := +\infty$
            for $k := i + 1, \ldots, j - 1$
                $q := m[i][k] + m[k][j] + r[i] \cdot r[k] \cdot c[j - 1]$
                $m[i][j] := \min\{m[i][j], q\}$
    return $m[0..n)[0..n)$
```

$m[i][j] = m(i,j)$

1. Subproblems
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Matrix-Chain Multiplication – Step 5: Table Filling

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    return $m[0..n)[0..n)$
```

- Same Θ-class as memoized recursive function
- In practice usually substantially faster
  - lower overhead
  - predictable memory accesses

Matrix-Chain Multiplication – Step 6: Backtracing

- So far, only determine the **cost** of an optimal solution
  - But we also want the solution itself
- By *retracing* our steps, we can determine/construct one!
- Here: output a parenthesized term recursively

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Matrix-Chain Multiplication – Step 6: Backtracing

- So far, only determine the **cost** of an optimal solution
  - But we also want the solution itself
- By *retracing* our steps, we can determine/construct one!
- Here: output a parenthesized term recursively

```
procedure matrixChainMult($r[0..n)$, $c[0..n)$):
    $m[0..n)[0..n) :=$ totalMultsBottomUp($r[0..n)$, $c[0..n)$)
    return traceback($[0..n)$)

procedure traceback($[i..j)$):
    if $j - i == 1$
        return $M_i$
    else
        for $k := i + 1, \ldots, j - 1$
            $q := m[i][k] + m[k][j] + r[i] \cdot r[k] \cdot c[j-1]$
            if $m[i][j] == q$
                return $\big($traceback($[i..k)$)$\big) \cdot \big($traceback($[k..j)$)$\big)$
        end for
    end if
```

1. Subproblems
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6. Backtrace

- follow recurrence a second time

Matrix-Chain Multiplication – Step 6: Backtracing

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            if $m[i][j] == q$
                return $($traceback($[i..k)$)$) \cdot ($traceback($[k..j)$)$)$
        end for
    end if
```

- follow recurrence a second time
- always have for running time: backtracing = O(computing M)

⇝ computing optimal cost and computing optimal solution have same complexity

- speedup possible by remembering correct guess k for each subproblem

Summary: The 6 Steps of Dynamic Programming

1. Subproblems
2. Guess!
3. DP Recurrence
4. Memoization
5. Table Filling
6. Backtrace

1. Define **subproblems** and how **original problem** is solved
2. What part of solution to **guess**?
3. Set up **DP recurrence** for quality/cost of solution
    - ⇝ Prove **correctness** here: induction over subproblems following recurrence
    - ⇝ Analyze running **time complexity** here: #subproblems · non-recursive time

— (Basically) cookie-cutter approach from here on —

4. Recursive implementation with **Memoization**: mutually recursive functions with cache

or

5. Bottom-up **table filling**: define topological order of subproblem dependency graph
6. **Backtracing** to reconstruct optimal solution: Recursively retrace cost recurrence

12.3 Greedy as Special Case of DP

Dynamic Greedy

- Every Greedy Algorithm can also be seen as a DP algorithm **without guessing**

⇝ For new problems, it can help to first follow the DP roadmap and
then check if we can select the "correct" guess without local brute force

Dynamic Greedy

- ▶ Every Greedy Algorithm can also be seen as a DP algorithm **without guessing**

$\rightsquigarrow$ For new problems, it can help to first follow the DP roadmap and then check if we can select the "correct" guess without local brute force

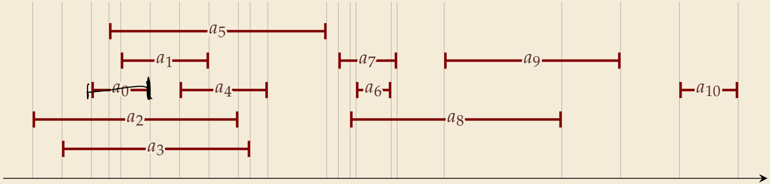
- ▶ If so, we then recurse on a single branch of subproblems

$\rightsquigarrow$ Greedy Algorithm doesn't need memoization or bottom-up table filling, but can do direct recursion instead

Recall Unit 11

The Activity selection problem

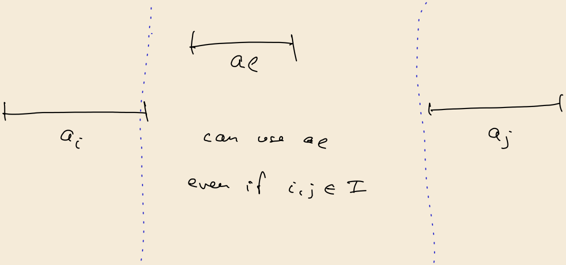
- **Activity Selection:** scheduling for *single* machine, jobs with *fixed* start and end times
  pick a *subset* of jobs without *conflicts*
  Formally:
  - **Given:** Activities $A = \{a_0, \ldots, a_{n-1}\}$, each with a start time s_i and finish time f_i $(0 \le s_i < f_i < \infty)$
  - **Goal:** Subset $I \subseteq [0..n)$ of tasks such that $i, j \in I \wedge i \neq j \implies [s_i, f_i) \cap [s_j, f_j) = \emptyset$ and $|I|$ is maximal among all such subsets
  - We further assume that jobs are sorted by finish time, i. e., $f_0 \le f_1 \le \cdots \le f_{n-1}$ (if not, easy to sort them in $O(n \log n)$ time)



DP Algorithm for Activity Selection

1. **Subproblems:** $A_{i,j} = \{a_\ell \in A : s_\ell \geq f_i \wedge f_\ell \leq s_j\}$
   (after a_i finishes and before a_j begins)

   Original problem: $A_{-1,n}$ with dummy tasks $f_{-1} = -\infty$, $s_n = +\infty$



can use a_ℓ

even if $i, j \in I$

1. Subproblems
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DP Algorithm for Activity Selection

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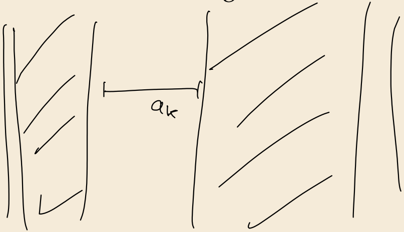
2. **Guess:** Task $k \in I^*$

3. **DP Recurrence:** Denote $c(i,j) = |I^*(A_{i,j})|$ = maximum #independent tasks in $A_{i,j}$

$$\rightsquigarrow \quad c(i,j) \quad = \quad \begin{cases} 0\,, & \text{if } A_{i,j} = \emptyset; \\ \max\{c(i,k) + c(k,j) + 1 : a_k \in A_{i,j}\} & \text{otherwise.} \end{cases}$$

4.–6. ***Omitted*** (could be done following the standard scheme)



DP Algorithm for Activity Selection

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4.–6. ***Omitted*** (could be done following the standard scheme)

- Problem-specific insight from Unit 11 ⇝ Can always use $\underline{k = \min\{k : a_k \in A_{ij}\}}$
  (earliest finish time)

  No guess needed!

12.4 The Bellman-Ford Algorithm

Recall Shortest Paths

- **Single Source Shortest Path Problem (SSSPP)**
  - **Given:** directed, edge-weighted, simple graph $G = (V, E, c)$ with edge costs $c : E \to \mathbb{R}$, a start vertex $s \in V$
  - **Goal:** a data structure that reports for every $v \in V$:
    $\delta_G(s, v)$: the shortest-path distance from s to v
    $\texttt{spath}(v)$: a shortest path from s to v (if it exists)
- $\delta_G(s, v) = \boxed{\inf\left(\{+\infty\} \cup \{c(w) : w \text{ an } s\text{-}v\text{-walk in } G\}\right)}$
  - Write δ instead of δ_G when graph clear from context

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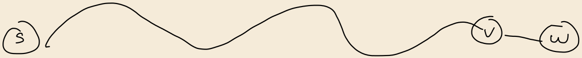
  - Write δ instead of δ_G when graph clear from context
- Here: Assume **negative-weight edges** are present (otherwise Dijkstra suffices)
  - but for now: assume there is **no negative cycle**

$\rightsquigarrow$ $\delta(s, v) > -\infty$ and can restrict to shortest **paths** (not walks)

Pfad Weg

Shortest Paths as DP – Last Edge Decomposition

- Idea: Every nontrivial shortest path has a **last edge**. *We don't know which; so guess!*



Shortest Paths as DP – Last Edge Decomposition

- Idea: Every nontrivial shortest path has a **last edge**. *We don't know which; so guess!*
  - ⇝ Subproblems: for $w \in V$, compute $\delta(s,w)$.
  - ⇝ Recurrence: $\delta(s,w) = \min\{\delta(s,v) + c(vw) : vw \in E\}$ $\delta(s,s) = 0$

Clicker Question

What is the problem with basing a DP algorithm on:
Subproblems: for $w \in V$, compute $\delta(s, w)$.
Recurrence: $\delta(s, w) = \min\left\{\delta(s, v) + c(vw) : vw \in E\right\}$

- A Bellman’s Optimality Criterion is not satisfied.
- B Does not yield to an efficient algorithm: too many subproblems.
- C Does not yield to an efficient algorithm: non-recursive cost too high.
- D Subproblem dependency graph is cyclic.
- E Subproblem dependency graph is not connected.
- F Does not always compute correct distances.

→ sli.do/cs566

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What is the problem with basing a DP algorithm on:
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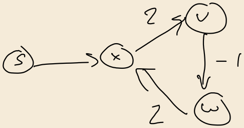
subproblem dependency graph is isomorphic to G^T! ⇝ doesn't work in general

- ⇝ Yields usable (terminating!) algorithm *iff* G is a DAG.

$$\begin{tikzcd}
 & \delta(s,v) \arrow[dd] \\
\delta(s,w) \arrow[ur] & \\
 & \delta(s,x) \arrow[ul]
\end{tikzcd}$$



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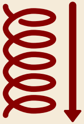

To break the cycles, let's turn them into a helix!

- Need to build "layers" in the subproblem dependency graph, so that edges can't go back up.

Shortest Paths as DP – Last Edge Decomposition

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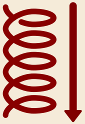

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- Need to build "layers" in the subproblem dependency graph, so that edges can't go back up.
- **Subproblems:** (w,ℓ) for $w \in V$, $\ell \in [0..n)$, compute $\delta_{\le \ell}(s,w)$
  where $\underline{\delta_{\le \ell}(s,v)} = \min\left(\{+\infty\} \cup \{c(w) : w \text{ an } s\text{-}v\text{-walk with } \le \boldsymbol{\ell} \text{ edges}\}\right)$
- **Original problems:** $\ell = n-1$ (without negative cycles, paths suffice)

Shortest Paths as DP – Last Edge Decomposition

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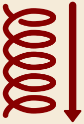

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- **Subproblems:** (w, ℓ) for $w \in V$, $\ell \in [0..n)$, compute $\delta_{\leq \ell}(s, w)$
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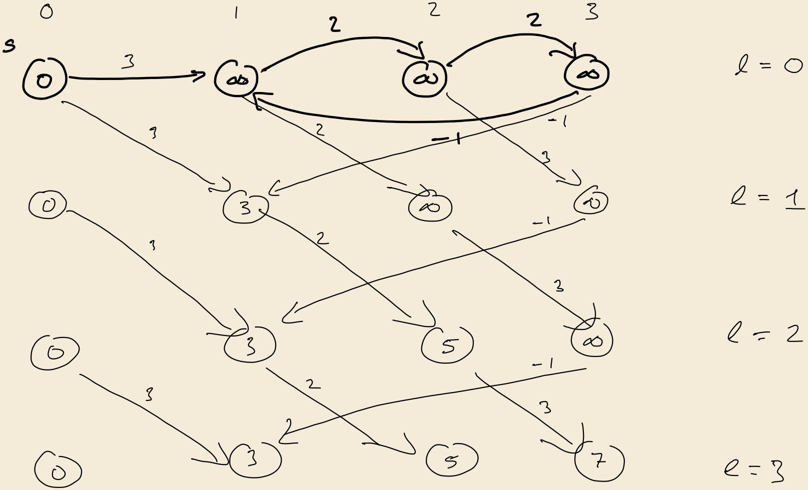
- **Original problems:** $\ell = n - 1$ (without negative cycles, paths suffice)
- **Recurrence:** $\underline{\delta_{\leq \ell}(s, w)} = \begin{cases} \infty & \text{if } \ell = 0 \text{ and } s \neq w \\ 0 & \text{if } \ell = 0 \text{ and } s = w \\ \min\{\delta_{\leq \ell - 1}(s, v) + c(vw) : vw \in E\} & \text{otherwise} \end{cases}$

Shortest Paths as DP – Length Layers



Hold On – What about negative cycles?

- The recurrence for $\delta_{\leq \ell}$ seems to work fine with *negative* edges

  But G could contain a **negative-weight cycle** C . . .

$$\delta_{\leq \ell}(s,w) = \begin{cases} \infty & \text{if } \ell = 0 \text{ and } s \neq w \\ 0 & \text{if } \ell = 0 \text{ and } s = w \\ \min\{\delta_{\leq \ell-1}(s,v) + c(vw) : vw \in E\} & \text{otherwise} \end{cases}$$

Isn't that a contradiction to the non-existence of shortest paths?

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Isn't that a contradiction to the non-existence of shortest paths?

- No. If we restrict the length, shortest walks always exist.
- But: If there is a negative cycle $C[0..k]$ with paths $s \rightsquigarrow C$ and $C \rightsquigarrow w$,

  then $\delta_{\leq \ell}(s,w) > \delta_{\leq \ell+k}(s,w) > \delta_{\leq \ell+2k}(s,w) > \cdots$ (and $\delta(s,w) = -\infty$)

Hold On – What about negative cycles?

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  But G could contain a **negative-weight cycle** C ...

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$\rightsquigarrow$ We can *detect* if any negative cycle is reachable from s by including more layers $\ell \geq n$ and check if some vertex still improves.

  - *How many further layers do we need / when is it safe to stop?*

Detecting negative cycles

We can detect reachable negative cycles by including just the *single* extra layer $\ell = n$!

Lemma: $\exists w : \delta_{\le n}(s,w) < \delta_{\le n-1}(s,w)$ *iff* negative cycle reachable from s

“⇒”

- ▶ If some vertex w improves further, i. e., $\delta_{\le n}(s,w) < \delta_{\le n-1}(s,w)$
  a walk $W[0..n]$ with $c(W) = \delta_{\le n}(s,w)$ was the **shortest** way to reach w
- ⇝ W is a non-simple walk, i. e., it contains a cycle
- ▶ Let $P[0..k]$ be the path resulting from W by shortcutting all cycles ⇝ $k \le n - 1$
- ⇝ $c(P) \ \ge \ \delta_{\le n-1}(s,w) \ > \ \delta_{\le n}(s,w) \ = \ c(W)$
- ⇝ $\exists$ negative cycle reachable from s

Detecting negative cycles

We can detect reachable negative cycles by including just the *single* extra layer $\ell = n$!

Lemma: $\exists w : \delta_{\leq n}(s,w) < \delta_{\leq n-1}(s,w)$ *iff* negative cycle reachable from s $\rightarrow$ subproblems (ℓ, v) $\ell \in [0..n]$

"$\Rightarrow$"

- If some vertex w improves further, i. e., $\delta_{\leq n}(s,w) < \delta_{\leq n-1}(s,w)$ a walk $W[0..n]$ with $c(W) = \delta_{\leq n}(s,w)$ was the **shortest** way to reach w

$\rightsquigarrow$ W is a non-simple walk, i. e., it contains a cycle

- Let $P[0..k]$ be the path resulting from W by shortcutting all cycles $\rightsquigarrow$ $k \leq n-1$

$\rightsquigarrow$ $c(P) \geq \delta_{\leq n-1}(s,w) > \delta_{\leq n}(s,w) = c(W)$

$\rightsquigarrow$ $\exists$ negative cycle reachable from s

"$\Leftarrow$"

- Conversely, let negative cycle $C[0..k]$ be reachable from s

$\rightsquigarrow$ $c(C) = \sum_{i=0}^{k-1} c(C[i]C[i+1]) < 0$

- Assume towards a contradiction that $\forall w : \delta_{\leq n}(s,w) = \delta_{\leq n-1}(s,w)$

$\rightsquigarrow$ $\forall vw \in E : \delta_{\leq n-1}(s,w) \leq \delta_{\leq n-1}(s,v) + c(vw)$ (no update in layer $\ell = n$)

- summing this inequality over $C[0..k]$ yields (abbreviating $\delta(w) := \delta_{\leq n-1}(s,w)$)

$$\sum_{i=1}^{k} \delta(C[i]) \leq \sum_{i=1}^{k} \Big(\delta(C[i-1]) + c(C[i]C[i+1])\Big) = \sum_{i=0}^{k-1} \delta(C[i]) + \underbrace{\sum_{i=1}^{k} c(C[i]C[i+1])}_{= c(C) < 0}$$

$=$

$\rightsquigarrow$ $0 \leq c(C) < 0$ ↯

$\square$

Shortest Paths as DP – Template Algorithm

1. Subproblems
2. Guess!
3. DP Recurrence
4. Memoization
5. Table Filling
6. Backtrace

- Strictly following the template works …
  - Subproblem order: by increasing $\ell \in [0..n]$ and $v \in V$
  - Bottom-up table filling:

```
procedure shortestPathsDP($G$, $s$):
    // Base case $\ell = 0$:
    $\delta[0..n][0..n) := +\infty$ // $\delta[\ell][v]$ will store $\delta_{\leq \ell}(s, v)$
    $\delta[0][s] := 0$
    for $\ell := 1, \ldots, n$ // layer
        for $w := 0, \ldots, n-1$ // vertex
            for $vw \in E$
                $\delta[\ell][w] := \min\{\delta[\ell][w], \underbrace{\delta[\ell-1][v]}_{\text{already computed}} + c(vw)\}$
    return $\delta$
```

$$\delta_{\leq \ell}(s, w) = \begin{cases} \infty & \text{if } \ell = 0 \text{ and } s \neq w \\ 0 & \text{if } \ell = 0 \text{ and } s = w \\ \min\{\delta_{\leq \ell-1}(s, v) + c(vw) : vw \in E\} & \text{otherwise} \end{cases}$$

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- . . . but some improvements are possible!
  - Iterating over *incoming* edges is not convenient
    - ⇝ order of updates within layer ℓ doesn't matter ⇝ iterate forwards!
  - only use final distances in the end; we waste space by keeping 2D array around
    - ⇝ can actually just do updates in place, using a single array δ
    - ⇝ Don't strictly solve subproblems (ℓ, v) any more! (but final result correct)

The Bellman-Ford Algorithm

```
procedure bellmanFord($G$, $s$):
    $dist[0..n) := +\infty$;  $pred[0..n) :=$ null
    $dist[s] := 0$
    for $\ell := 1, \ldots, \underline{n-1}$
        for $v := 0, \ldots, n-1$
            for $(w, c) \in G$.adj[$v$]
                if $dist[w] > dist[v] + c$        $s \rightsquigarrow v \to w$
relax(vw)           $dist[w] := dist[v] + c$
                    $pred[w] := v$ // remember for backtrace
    for $v := 0, \ldots, n-1$        // layer n
        for $(w, c) \in G$.adj[$v$]
            if $dist[w] > dist[v] + c$
                return HAS_NEGATIVE_CYCLE
    return $(dist, pred)$
```

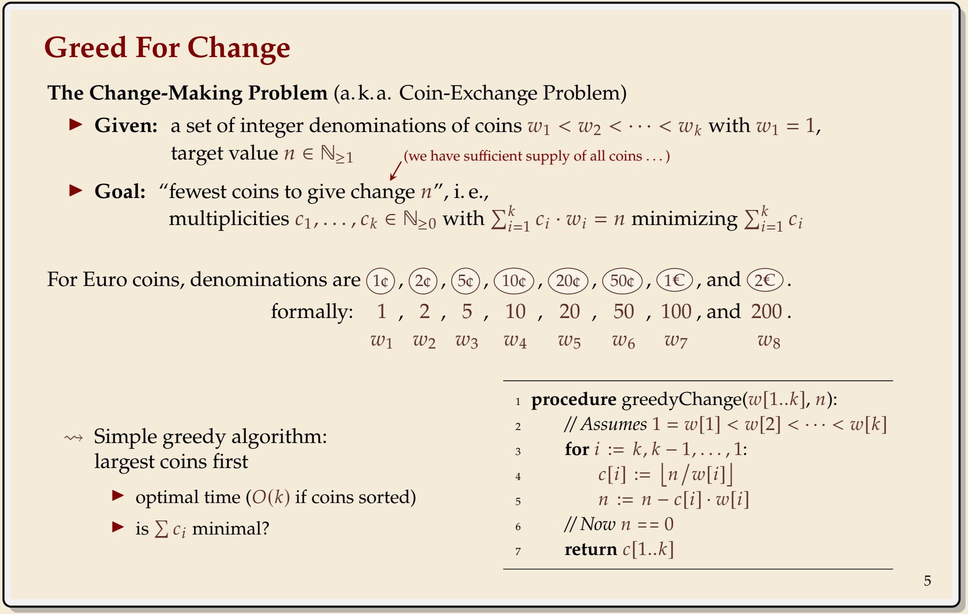
- Final algorithm (including shortest path tree via *pred*)
- **Correctness:**
  - by induction over loop iteration show $dist[w] \le \delta_{\le \ell}(s, w)$ and if finite, $dist[w]$ is $c(P)$ for some s-w-path
  - negative cycle detection from Lemma
- **Space:** $\Theta(n)$ $\sum_{v \in V} d_{out}(v) = m$
- **Running time:** $O(n(n + m))$

Extensions:

- Can be implemented in $O(nm)$ time by removing unreachable vertices from consideration
- Instead of only detecting a negative cycle, we can return one; we can also explicitly find all vertices with $\delta(s, w) = -\infty$ (needs another traversal).
- Can terminate with smaller ℓ if no distance changed ⇝ faster for some graphs

12.5 Making Change in Pre-1971 UK

Recall Unit 11



Greed For Change

The Change-Making Problem (a. k. a. Coin-Exchange Problem)

- **Given:** a set of integer denominations of coins $w_1 < w_2 < \cdots < w_k$ with $w_1 = 1$,
  target value $n \in \mathbb{N}_{\geq 1}$
  (we have sufficient supply of all coins . . .)
- **Goal:** "fewest coins to give change n", i. e.,
  multiplicities $c_1, \ldots, c_k \in \mathbb{N}_{\geq 0}$ with $\sum_{i=1}^{k} c_i \cdot w_i = n$ minimizing $\sum_{i=1}^{k} c_i$

For Euro coins, denominations are 1¢, 2¢, 5¢, 10¢, 20¢, 50¢, 1€, and 2€.

formally: $\underset{w_1}{1}$, $\underset{w_2}{2}$, $\underset{w_3}{5}$, $\underset{w_4}{10}$, $\underset{w_5}{20}$, $\underset{w_6}{50}$, $\underset{w_7}{100}$, and $\underset{w_8}{200}$.

⇝ Simple greedy algorithm:
largest coins first

- optimal time ($O(k)$ if coins sorted)
- is $\sum c_i$ minimal?

```
procedure greedyChange($w[1..k]$, $n$):
    // Assumes $1 = w[1] < w[2] < \cdots < w[k]$
    for $i := k, k-1, \ldots, 1$:
        $c[i] := \lfloor n / w[i] \rfloor$
        $n := n - c[i] \cdot w[i]$
    // Now $n == 0$
    return $c[1..k]$
```

Pre-Decimal English Coins

We discussed that for some (unwise) choices of denominations, Greedy cannot give optimal change.

Pre-Decimal English Coins

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Welcome to Britain until 1971!

British Pre-Decimal Coins:

- $\frac{1}{2}$ penny,
- 1 penny,
- 3 pence,
- 6 pence,
- shilling = 12 pence,
- florin = 24 pence
- half-crown = 30 pence
- crown = 60 pence
- pound = 240 pence
- guinea = $21 \cdot 12 = 252$ pence
  (obsolete as coin since 1816)

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⇝ Greedy would give 48 pence as 30p + 12p + 6p

- obviously, 2 florins are more efficient

⇝ How to solve exactly?

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⇝ How to solve exactly?

As the old saying goes . . .

Where Greedy fails, DP prevails.

(but mind details, and how it scales)

Making Change by DP

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Idea: Every solution must pick a first coin. Which one? Unclear, so guess!

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$$C(m) = \begin{cases} 0 & \text{if } m = 0 \\ 1 + \min\{C(m - w_i) : i \in [1..k] \wedge \underset{\substack{\uparrow \\ \text{guess}}}{w_i} \leq m\} & \text{otherwise} \end{cases}$$

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- **Bottom-up implementation & Backtrace**

```
procedure dpChange($w[1..k]$, $n$):
    $C[0..n] := +\infty$
    $C[0] := 0$
    for $m := 1, \ldots, n$
        for $i := 1, \ldots, k$
            if $w[i] \overset{\leq}{\cancel{\geq}} m$
                $q := 1 + C[m - w[i]]$
                $C[m] := \min\{C[m], q\}$
    return $C$[illegible] [0..n]
```

```
procedure tracebackChange($w[1..k]$, $n$):
    $C[0..n] :=$ dpChange($w[1..k]$, $n$)
    $c[1..k] := 0$ // coin multiplicities
    $m := n$
    while $m > 0$
        for $i := 1, \ldots, k$
            if $w[i] \overset{\leq}{\cancel{\geq}} m \wedge C[m] == 1 + C[m - w[i]]$
                $c[i] := c[i] + 1$;   $m := m - w[i]$
    return $c[1..k]$
```

Clicker Question

What is the running time of dpChange($w[1..k]$, n)?

A Dunno.

B $O(m)$

C $O(n)$

D $O(k)$

E $O(k \log n)$

F $O(nk)$

G $O(n^2k)$

H $O(nk^2)$

I $O(n^2k^2)$

J $O(n^3k^2)$

→ `sli.do/cs566`

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Making Change by DP – Analysis

- **Input:** denominations of coins $w_1 < w_2 < \cdots < w_k$ with $w_1 = 1$, target value $n \in \mathbb{N}_{\geq 1}$

- **Space:** $\Theta(n)$

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How good is this running time?

- A linear function in both input numbers seems decent, right? (If k and n small, certainly Yes.)
  - Running time is also certainly a *polynomial* in n and k

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  - Reason: We give the input **number** n in **binary**, so n is exponential in its ***input size***.
  - ⚠ Must distinguish: *value* of a number (in the input) vs. *size* of the (encoding of the) input
  - $\rightsquigarrow$ dpChange is a ***pseudo-polynomial time*** algorithm

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  - ⇝ dpChange is a ***pseudo-polynomial time*** algorithm

$\neq$ quasi-polynomial

- Actually, the general making-change problem is NP-complete (!)

Knapsack

Let's look at slightly more interesting problem: *Knapsack* (*„Rucksack“*).

a.k.a. the burglar's problem

The 0/1-Knapsack Problem

- **Given:** k items with weights $w_1 \, \dots, w_k \in \mathbb{N}_{\geq 1}$ and values $v_1, \dots, v_k \in \mathbb{R}_{\geq 0}$;
  a weight budget $W \in \mathbb{N}$
- **Goal:** Subset $I \subseteq [1..k]$ such that $\sum_{i \in I} w_i \leq W$ with maximum $\sum_{i \in I} v_i$.

  Variant closer to Making change: Can use each item several times

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  - $\rightsquigarrow$ 2^k possible "states" to be in (items already used) (***0/1***-Knapsack)

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ϟϟ need a table of size $W \cdot 2^k$ ... *might as well do brute force then!*

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Knapsack by DP

- ⇝ *Force order to consider items in!*

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- Let's refine the guessing part to
  Guess: Whether or not to include the *last* item (k)
  - ⇝ For subproblem, restrict to items $1, \ldots, k-1$ (in either case)

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- ⇝ **Subproblems:** (ℓ, B) for $\ell \in [1..k]$ and $B \in [0..W]$

$$V(\ell, B) = \max_{I} \sum_{i \in I} v_i \text{ over sets of items } I \subset [1..\boldsymbol{\ell}] \text{ with } \textstyle\sum_{i \in I} w_i \le \boldsymbol{B}$$

Original problem corresponds to $V(k, W)$

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- **Recurrence:**

$$V(\ell, B) = \begin{cases} 0 & \text{if } \ell = 1 \wedge w_1 > B \\ v_1 & \text{if } \ell = 1 \wedge w_1 \le B \\ \max\{\overset{\text{take item } \ell}{\overbrace{v_\ell + V(\ell - 1, B - w_{\underset{\ell}{\cancel{k}}})}}, \overset{\text{don't take } \ell}{\overbrace{V(\ell - 1, B)}}\} & \text{otherwise} \end{cases}$$

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- **Recurrence:** $$V(\ell, B) = \begin{cases} 0 & \text{if } \ell = 1 \wedge w_1 > B \\ v_1 & \text{if } \ell = 1 \wedge w_1 \le B \\ \max\Big\{ \overset{\text{take item } \ell}{v_\ell + V(\ell - 1, B - w_k)},\ \overset{\text{don't take } \ell}{V(\ell - 1, B)} \Big\} & \text{otherwise} \end{cases}$$

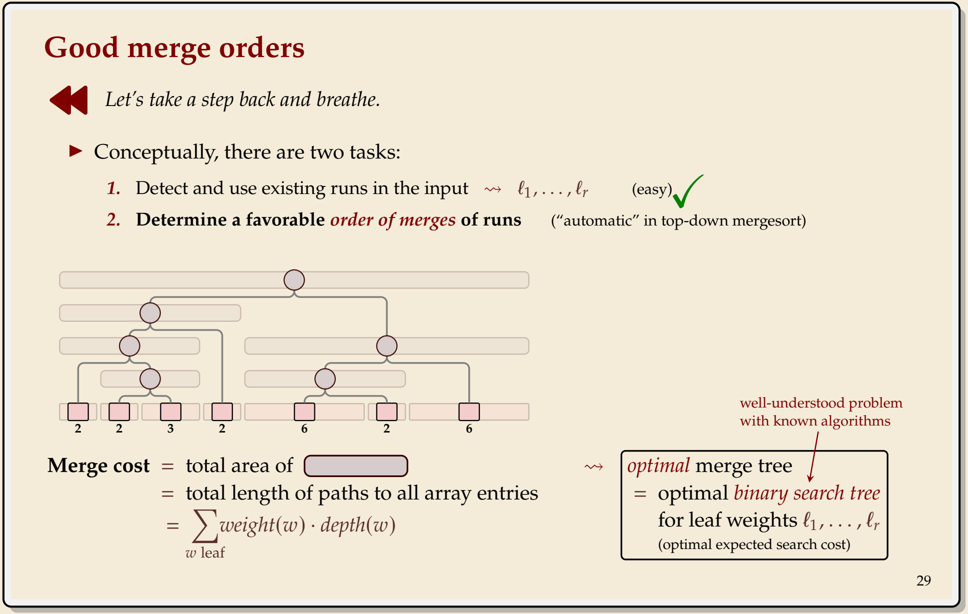
Cookie-Cutter Steps 4. – 6. ***Omitted***

- $V(\ell, \cdot)$ only needs $V(\ell - 1, \cdot)$ ⇝ two arrays $V[0..W]$ and $V_{\text{prev}}[0..W]$ suffice

⇝ $\Theta(W)$ **space**, $\Theta(W \cdot k)$ **time** (pseudo-polynomial algorithm)

12.6 Optimal Merge Trees & Optimal BSTs

Recall Unit 4



Good merge orders

Let's take a step back and breathe.

- Conceptually, there are two tasks:
  1. Detect and use existing runs in the input $\rightsquigarrow$ $\ell_1, \ldots, \ell_r$ (easy) ✓
  2. **Determine a favorable *order of merges* of runs** ("automatic" in top-down mergesort)

Merge cost = total area of ▭

= total length of paths to all array entries

$$= \sum_{w \text{ leaf}} weight(w) \cdot depth(w)$$

$\rightsquigarrow$ *optimal* merge tree

= optimal *binary search tree*

for leaf weights $\ell_1, \ldots, \ell_r$

(optimal expected search cost)

well-understood problem
with known algorithms

Optimal Alphabetic Trees

"well-understood problem with known algorithms" . . . *let's make it so* 😉

- **Given:** Leaf weights $\ell_0, \ldots, \ell_n$ normalized to $\ell_0 + \cdots + \ell_n = 1$
- **Goal:** Binary search tree T with $n + 1$ null pointers $L_0, \ldots, L_n$, such that

$$c(T) \;:=\; \sum_{i=1}^{n} \ell_i \cdot \mathrm{depth}_T(L_i) \text{ is minimized}$$

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- **Equivalent interpretations:**

  1. *Optimal Static BST* with keys $1, 2, \ldots, n$

     $\Pr[i+0.5) = \ell_i$

     #comparisons

     $\rightsquigarrow$ leaf L_i reached when searching for $i + 0.5$ $\rightsquigarrow$ $c(T)$ *expected* cost of *unsuccessful search*

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     $\leadsto$ leaf L_i reached when searching for $i + 0.5$ $\leadsto$ $c(T)$ *expected* cost of *unsuccessful search*
  2. *Alphabetic code* for $\sigma = n + 1$ symbols; like Huffman code, but *codewords must retain order* (if $i < j$ then the codeword for i lexicographically smaller than codeword for j)

     $\leadsto$ $c(T)$ expected codeword length

     - Inherit lower bound from Huffman codes: $c(T) \;\ge\; \mathcal{H}$ with $\mathcal{H} \;=\; \sum_{i=0}^{n} \ell_i \cdot \log_2\!\left(\frac{1}{\ell_i}\right)$

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3. *Merge tree* for adaptive sorting; $c(T) =$ merge cost per element.
   - Via Peeksort or Powersort know methods to achieve $c(T) \le \mathcal{H} + 2$
   - But neither are in general optimal

Optimal Alphabetic Trees by DP

- **Guess:** (Key in) root $r \in [1..n]$ of BST T (= #leaves in left subtree)

1. Subproblems
2. Guess!
3. DP Recurrence
4. Memoization
5. Table Filling
6. Backtrace

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  Original problem: $C(0, n+1)$

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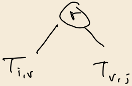
- **Recurrence:**

all leaves in subtree pay 1 at root...

$$C(i,j) \;=\; \begin{cases} 0 & \text{if } j-i=1 \\ \ell_i + \cdots + \ell_{j-1} + \min\{C(i,r)+C(r,j) : r \in [i+1..j-1]\} & \text{otherwise} \end{cases}$$

... plus cost to continue in left/right subtree



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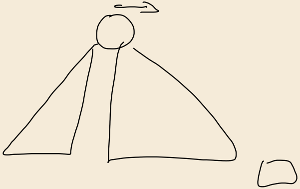
... plus cost to continue in left/right subtree

$\rightsquigarrow$ Obtain a $O(n^3)$ time and $O(n^2)$ space algorithm

Optimal Binary Search Trees

- Algorithm can be generalized to Optimal BSTs when also internal nodes have weights
  - Same DP subproblems
- Running time can be reduced to $\underline{O(n^2)}$ using *quadrangle inequality*
  - Intuitively: When adding more weight in right subtree, optimal root cannot move left.
  - Requires to remember r for each subproblem



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- For original alphabetic tree problem, can actually find optimal tree in $O(n \log n)$ time with a much more intricate algorithm

12.7 Edit Distance

Clicker Question

What does `diff A.txt B.txt` do?

→ *sli.do/cs566*

Edit Distance

Our last DP application here: (algorithmic foundation of) `diff`!

- `diff` is a classic Unix tool to compare two text files
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  - We've seen *Hamming distance* . . .
    But how to deal with strings of different lengths?
  - how to match common parts that are far apart?
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Edit Distance Problem

- **Given:** String $A[0..m)$ and $B[0..n)$ over alphabet $\Sigma = [0..\sigma)$.
- **Goal:** $d_{\text{edit}}(A, B) =$ minimal # symbol operations to transform A into B
  operations can be insertion/deletion/substitution of single character

Edit Distance Example

Example: edit distance $d_{\text{edit}}(\texttt{algorithm}, \texttt{logarithm})$?

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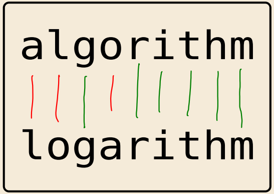


algorithm

logarithm

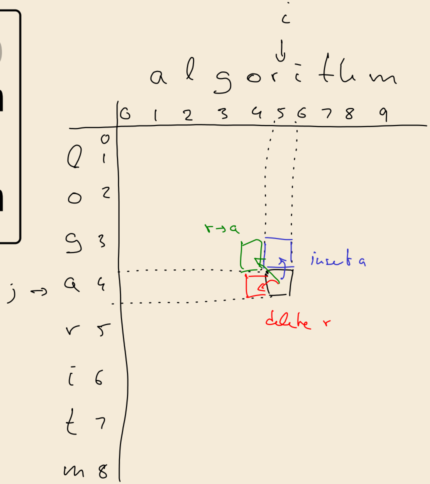
Edit Distance Example

Example: edit distance $d_{\text{edit}}(\texttt{algorithm}, \texttt{logarithm})$?

```
algorithm

logarithm
```

```
0123456789
al·gorithm
-|+|X|||||
·logarithm
```



Edit Distance by DP

1. **Subproblems:** (i, j) for $0 \le i \le m$, $0 \le j \le m$ compute $d_{\text{edit}}(A[0..i), B[0..j))$
2. **Guess:** What to do with last positions? (insert/delete/(mis)match)

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2. **Guess:** What to do with last positions? (insert/delete/(mis)match)
3. **Recurrence:** $D(i, j) = d_{\text{edit}}(A[0..i), B[0..j))$

$$D(i,j) \;=\; \begin{cases} i & A[0..i) \to \varepsilon \qquad \text{if } j = 0 \\ j & \varepsilon \to B(0..j) \qquad \text{if } i = 0 \\ \min \begin{cases} D(i-1, j) + 1, \\ D(i, j-1) + 1, \\ D(i-1, j-1) + \underbrace{\left[A[i-1] \ne B[j-1]\right]}_{} \end{cases} & \text{otherwise} \end{cases}$$

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$$D(i,j) = \begin{cases} i & \text{if } j = 0 \\ j & \text{if } i = 0 \\ \min \begin{cases} D(i-1,j)+1, \\ D(i,j-1)+1, \\ D(i-1,j-1) + \big[A[i-1] \neq B[j-1]\big] \end{cases} & \text{otherwise} \end{cases}$$

⇝ $O(nm)$ space and time
space can be improved to $O(\min\{n,m\})$ by remembering only 2 rows or columns

▶ An optimal *edit script* can be constructed by a backtrace (assuming we store entire D(i,j))

Generalized Edit Distances

- The variant we discussed is also called *Levenshtein distance*
  - all operation have cost 1

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  - inserting an $a \in \Sigma$
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  - deleting an occurrence of $a \in \Sigma$
  - inserting an $a \in \Sigma$
  - substituting $a \in \Sigma$ for $b \in \Sigma$
- Extensions of the algorithm can support: $\notin$ exam
  - **free** insert/delete at beginning/end of a string
  - *affine gap costs*, i. e., inserting/deleting k **consecutive** chars costs $c \cdot k + d$ for constants c and d
- extensions widely used to find approximate matches, e. g., in DNA sequences

Dynamic Programming – Summary

1. Subproblems
2. Guess!
3. DP Recurrence
4. Memoization
5. Table Filling
6. Backtrace

👍 Versatile and powerful algorithm design paradigm

👍 Once key idea (recurrence) clear, implementation rather straight-forward