14 Range-Minimum Queries

exam

10 February 2025

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CS566 (Wintersemester 2024/25)
Philipps-Universität Marburg
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Learning Outcomes

Unit 14: ***Range-Minimum Queries***

1. Know the *RMQ problem* and its *connection* to longest common extensions in strings.
2. Know and understand trivial RMQ solutions and *sparse tables*.
3. Know and understand the *Cartesian trees* data structure.
4. Know and understand the *exhaustive-tabulation technique* for RMQ with linear-time preprocessing.

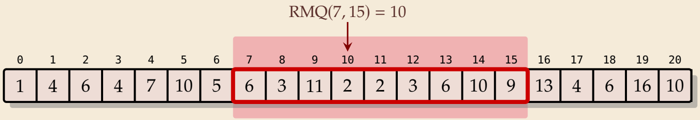
Outline

14 Range-Minimum Queries

14.1 Introduction

Range-minimum queries (RMQ)

- **Given:** Static array $A[0..n)$ of numbers (array/numbers don't change)
- **Goal:** Find minimum in a range;
  A known in advance and can be preprocessed



- **Nitpicks:**
  - Report *index* of minimum, not its value
  - Report *leftmost* position in case of ties

Clicker Question

Given the array from the slides, what is $\mathrm{RMQ}_A(1,6)$

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	4	6	4	7	10	5	6	3	11	2	2	3	6	10	9	13	4	6	16	10

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Rules of the Game

- comparison-based $\rightsquigarrow$ values don't matter, only relative order

- Two main quantities of interest:

  1. **Preprocessing time**: Running time $P(n)$ of the preprocessing step ($\rightsquigarrow$ space usage $\le P(n)$)

  2. **Query time**: Running time $Q(n)$ of one query (using precomputed data)

- Write $\langle P(n), Q(n) \rangle$ **time solution** for short

Clicker Question

What do you think, what running times can we achieve? For a $\langle P(n), Q(n) \rangle$ time solution, enter "<P(n),Q(n)>".

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14.2 RMQ, LCP, LCE, LCA — WTF?

Recall Unit 13

Application 4: Longest Common Extensions

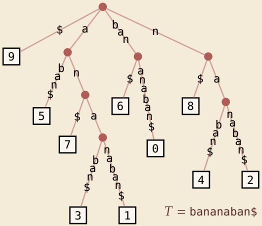
- We implicitly used a special case of a more general, versatile idea:

  The *longest common extension (LCE)* data structure:
  - **Given:** String $T[0..n)$
  - **Goal:** Answer LCE queries, i. e.,
    given positions i, j in T,
    how far can we read the same text from there?
    formally: $\mathrm{LCE}(i,j) = \max\{\ell : T[i..i+\ell) = T[j..j+\ell)\}$

⇝ use suffix tree of T!

(length of) longest common prefix of ith and jth suffix

- In $\mathcal{T}$: $\mathrm{LCE}(i,j) = \mathrm{LCP}(T_i, T_j)$ ⇝ same thing, different name!
  $=$ string depth of *lowest common ancester (LCA)* of leaves $\boxed{i}$ and $\boxed{j}$



- in short: $\mathrm{LCE}(i,j) = \mathrm{LCP}(T_i, T_j) = \mathrm{stringDepth}(\mathrm{LCA}(\boxed{i}, \boxed{j}))$

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Recall Unit 13

Efficient LCA

How to find lowest common ancestors?

- Could walk up the tree to find LCA $\rightsquigarrow$ $\Theta(n)$ worst case
- Could store all LCAs in big table $\rightsquigarrow$ $\Theta(n^2)$ space and preprocessing

Amazing result: Can compute data structure in $\Theta(n)$ time and space that finds any LCA in **constant(!) time**.

- a bit tricky to understand
- but a theoretical breakthrough
- and useful in practice

and suffix tree construction inside . . .

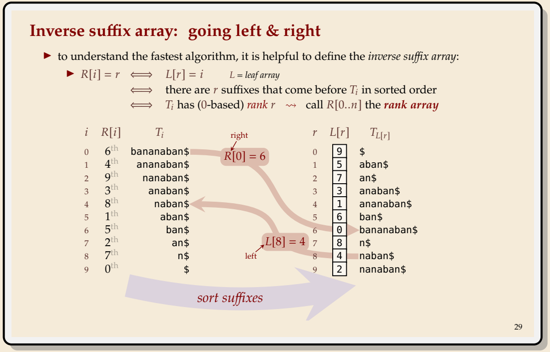
$\rightsquigarrow$ for now, use $O(1)$ LCA as black box.

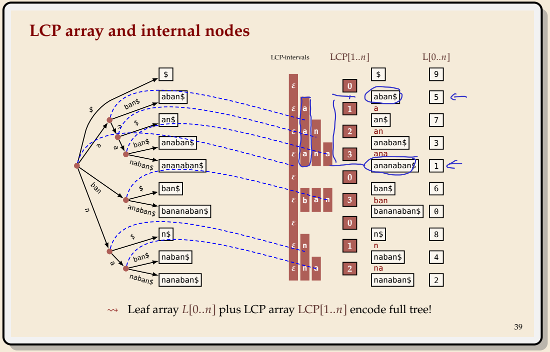
$\rightsquigarrow$ After linear preprocessing (time & space), we can find LCEs in $O(1)$ time.

Finally: Longest common extensions

- In Unit 13: Left question open how to compute LCA in suffix trees
- But: Enhanced Suffix Array makes life easier!

$$\mathrm{LCE}(i,j) \;=\; \mathrm{LCP}\Big[\mathrm{RMQ}_{\mathrm{LCP}}\big(\min\{R[i],R[j]\}+1,\ \max\{R[i],R[j]\}\big)\Big]$$





RMQ Implications for LCE

- Recall: Can compute (inverse) suffix array and LCP array in $O(n)$ time

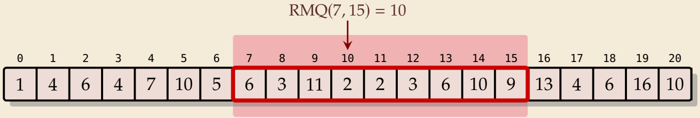
⇝ A $\langle P(n), Q(n) \rangle$ time RMQ data structure implies a $\langle P(n), Q(n) \rangle$ time solution for longest-common extensions

$+O(n)$

$\langle O(n), O(1) \rangle$ promised

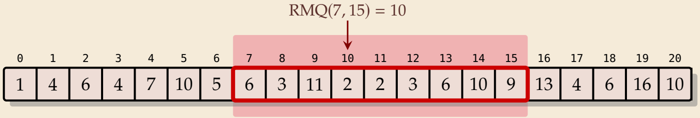
14.3 Trivial Solutions & Sparse Tables

Trivial Solutions



- Two easy solutions show extreme ends of scale:

Trivial Solutions



- Two easy solutions show extreme ends of scale:

1. **Scan on demand**

  - no preprocessing at all
  - answer $\text{RMQ}(i, j)$ by scanning through $A[i..j]$, keeping track of min

$\rightsquigarrow$ $\langle O(1), O(n) \rangle$

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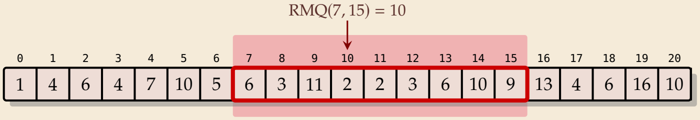

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2. **Precompute all**
   - Precompute all answers in a big 2D array $M[0..n)[0..n)$
   - queries simple: $\mathrm{RMQ}(i, j) = M[i][j]$

   $\rightsquigarrow \langle O(n^3), O(1) \rangle$

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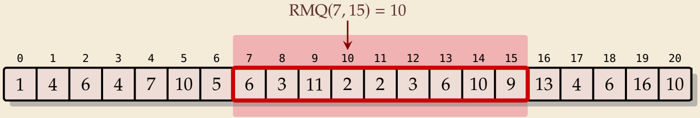

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   $\rightsquigarrow \langle O(n^3), O(1) \rangle$
   - Preprocessing can reuse partial results $\rightsquigarrow \langle O(n^2), O(1) \rangle$

Sparse Table

- **Idea:** Like “precompute-all”, but keep only some entries
- store $M[i][j]$ iff $\ell = j - i + 1$ is 2^k.
  - $\rightsquigarrow \le n \cdot \lg n$ entries
  - $\rightsquigarrow$ Can be stored as $M'[i][k]$

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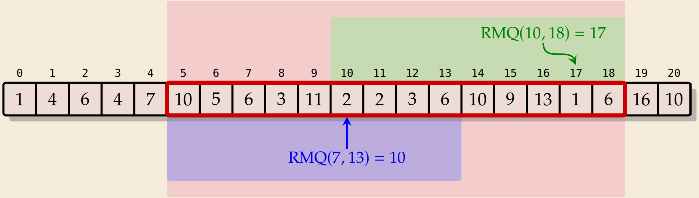

1. Find k with $\ell/2 \leq 2^k \leq \ell$
2. Cover range $[i..j]$ by
   2^k positions right from i and
   2^k positions left from j
3. $\text{RMQ}(i, j) = \arg\min\{A[rmq_1], A[rmq_2]\}$

   with $rmq_1 = \text{RMQ}(i, i + 2^k - 1)$
   $rmq_2 = \text{RMQ}(j - 2^k + 1, j)$

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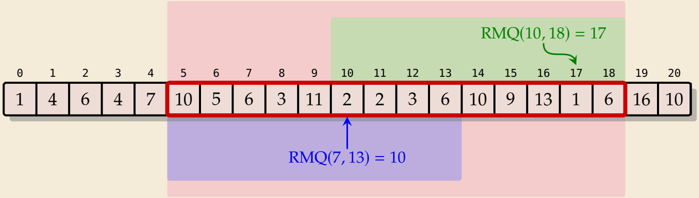

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   with $rmq_1 = \mathrm{RMQ}(i, i + 2^k - 1)$
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- Preprocessing can be done in $O(n \log n)$ times

$\rightsquigarrow \langle O(n \log n), O(1)\rangle$ time solution!

14.4 Cartesian Trees

RMQ & LCA

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
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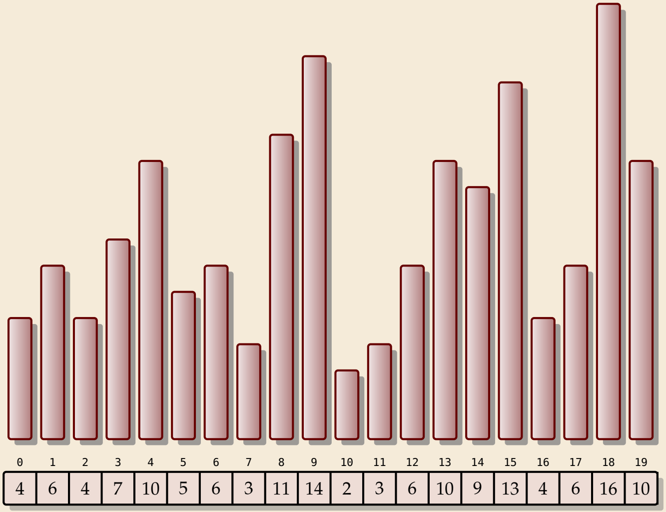
RMQ & LCA

(max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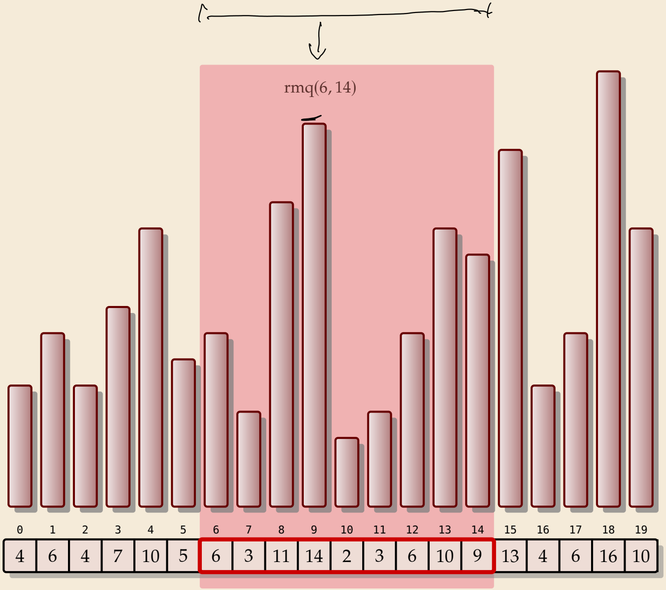
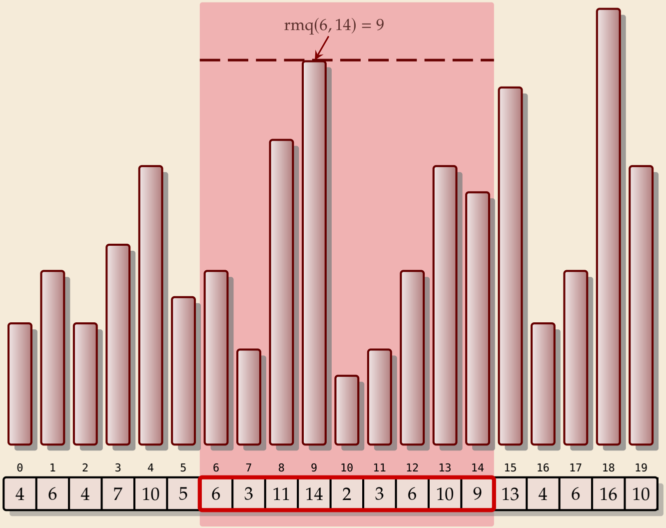
RMQ & LCA



- **Range-max queries** on array A:
  $$\mathrm{rmq}_A(i, j) = \arg\max_{i \le k \le j} A[k]$$
  $$= \textit{index} \text{ of max}$$

RMQ & LCA



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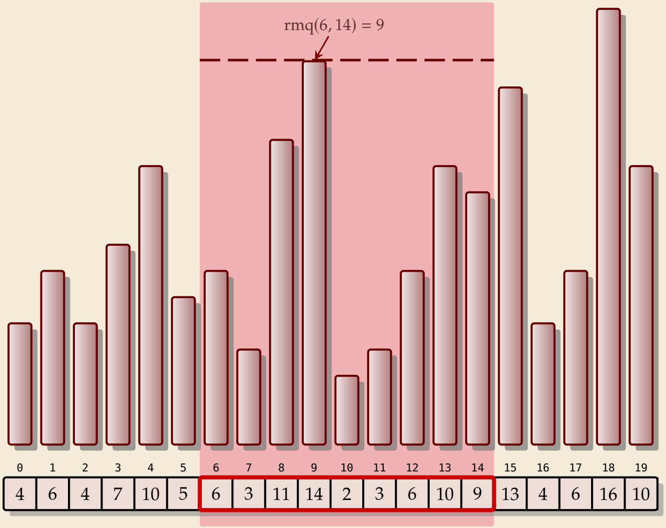

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- **Task:** Preprocess A,
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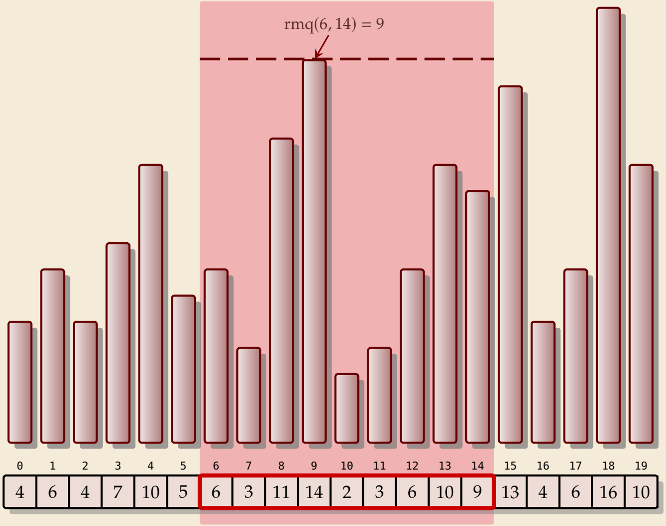
RMQ & LCA



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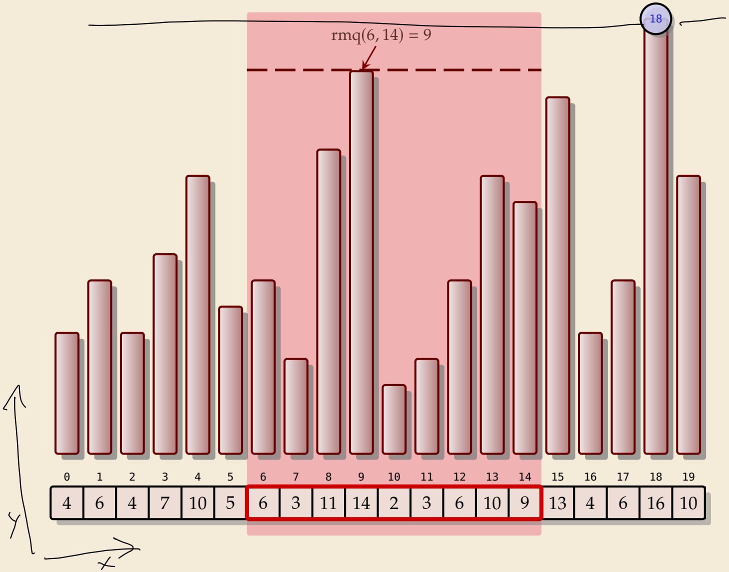

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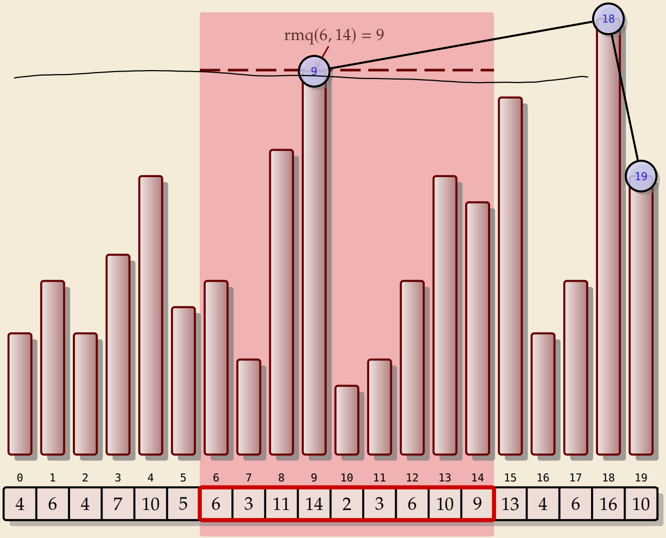
RMQ & LCA



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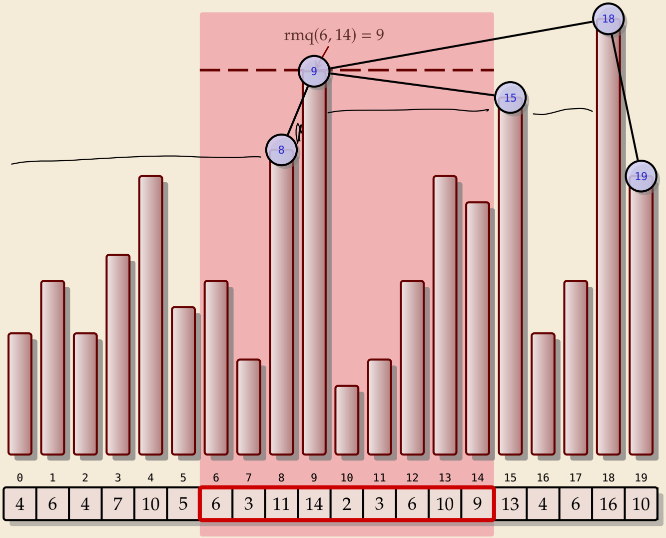

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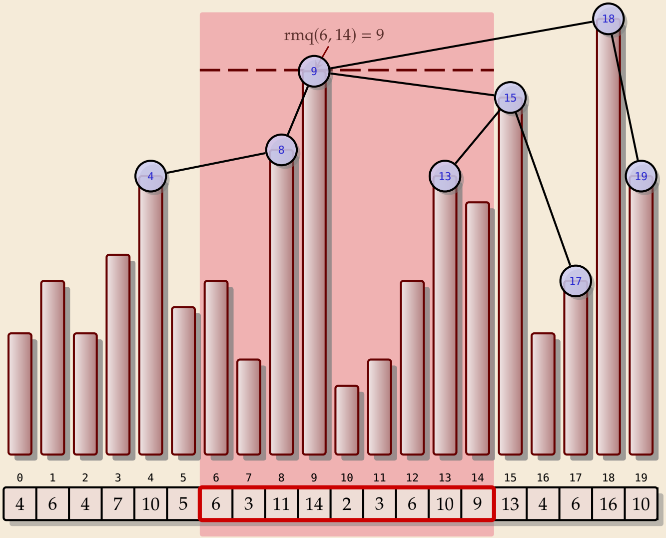

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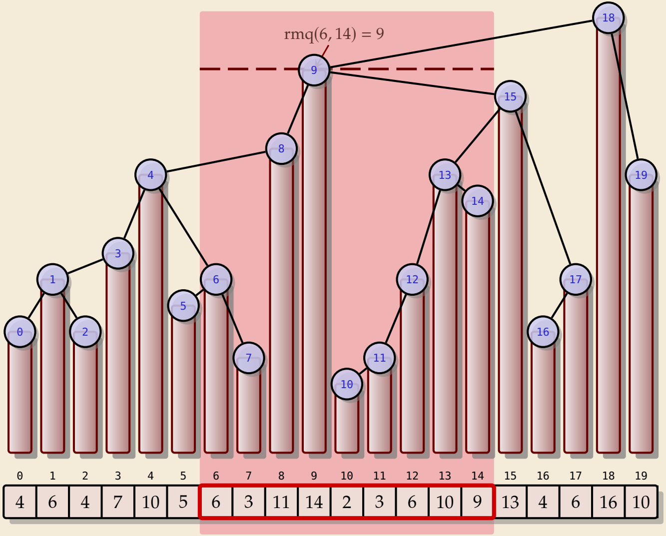

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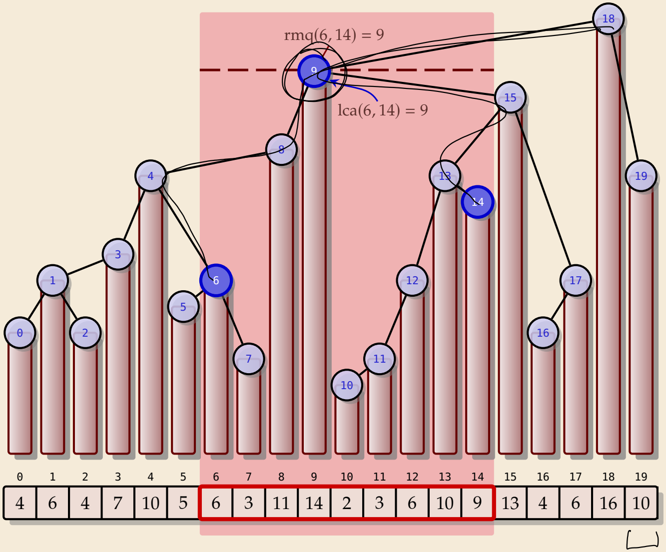

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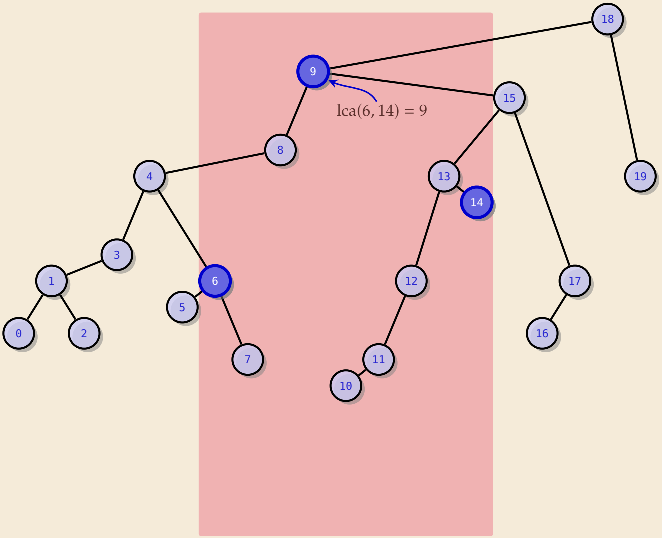
RMQ & LCA



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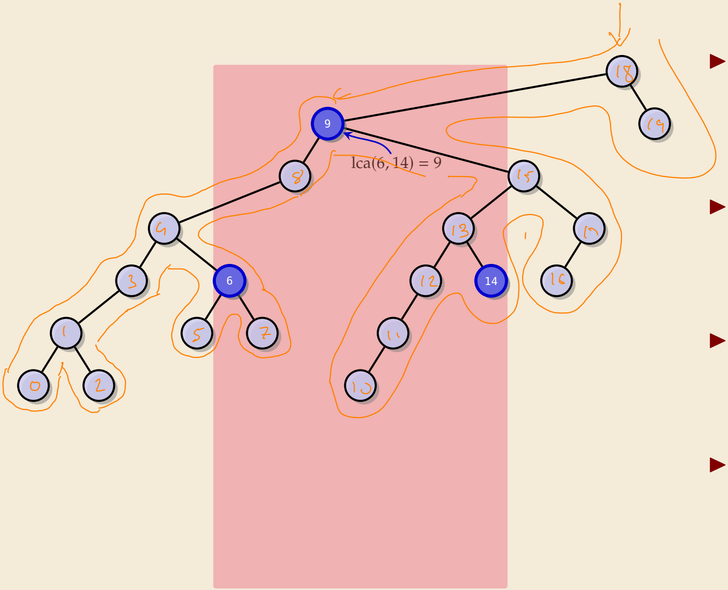

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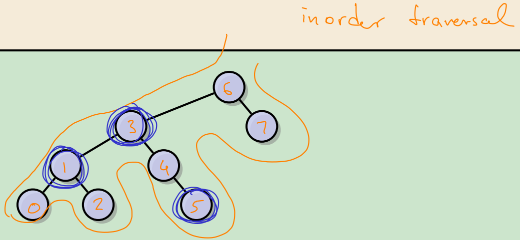
  of ith and jth node in inorder

Clicker Question



Given the (max-oriented) Cartesian tree for A on the left, what is $\mathrm{RMQ}_A(1, 5)$?

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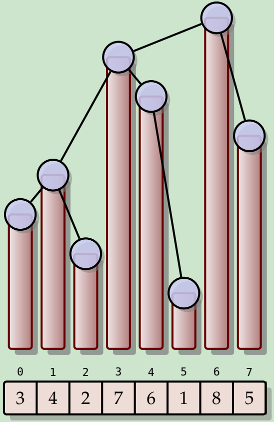
Clicker Question



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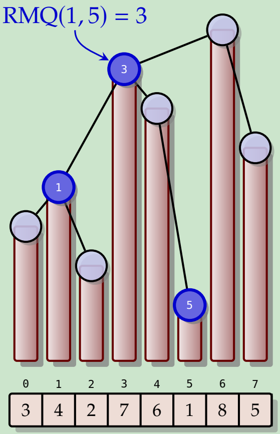

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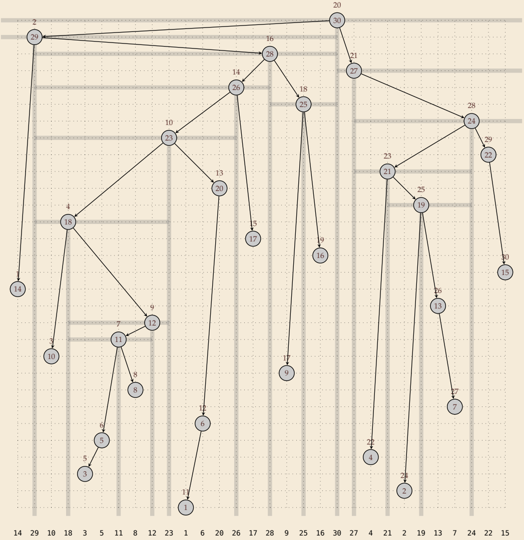
Cartesian Tree – Larger Example



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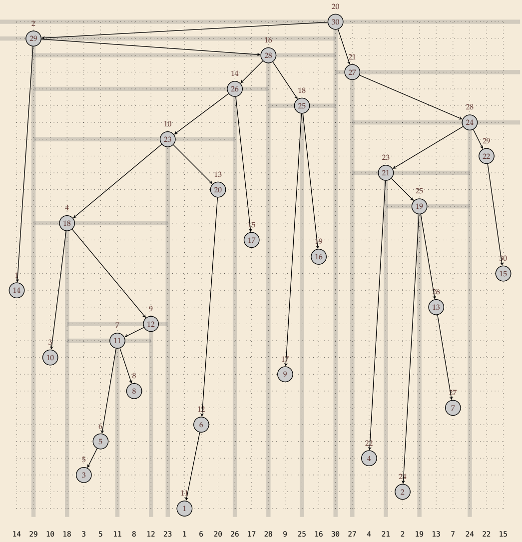

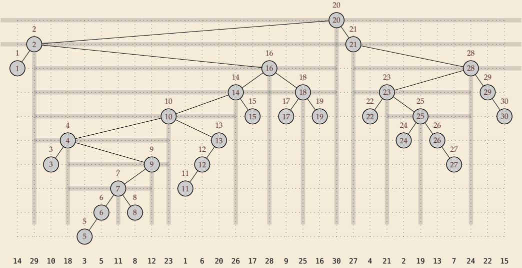

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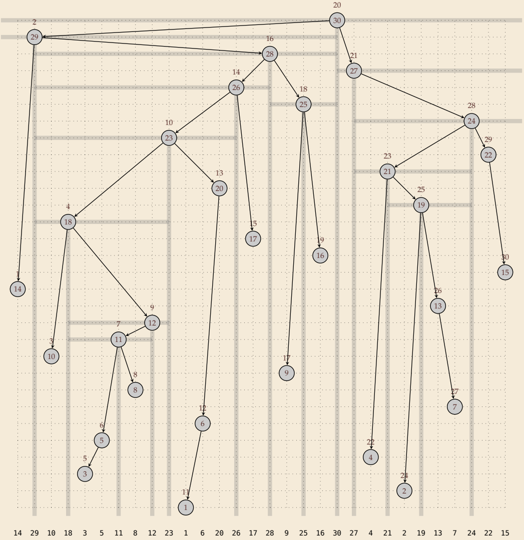
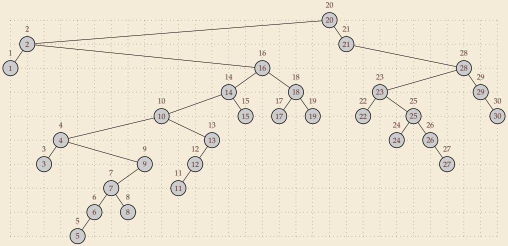
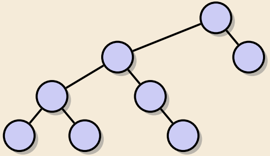




Counting binary trees



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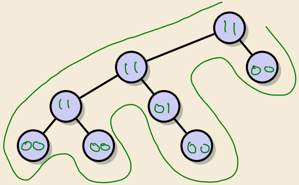

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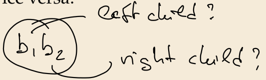
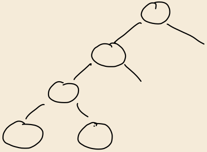


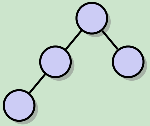


- How many different Cartesian trees are there for arrays of length n?
  - known result: *Catalan numbers* $\frac{1}{n+1}\binom{2n}{n}$
  - easy to see: $\leq 2^{2n}$

⇝ many arrays will give rise to the same Cartesian tree

Can we exploit that?

Clicker Question



What binary string corresponds to the tree shown on the left?

(using the encoding just discussed)

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14.5 Exhaustive Tabulation

Four Russians?

The exhaustive-tabulation technique to follow is often called "Four Russians trick" . . .

- The algorithmic technique was published 1970 by
  V. L. Arlazarov, E. A. Dinitz, M. A. Kronrod, and I. A. Faradžev

- all worked in Moscow at that time . . . but not even clear if all are Russians!

  (Arlazarov and Kronrod are Russian)

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The exhaustive-tabulation technique to follow is often called "Four Russians trick" . . .

- The algorithmic technique was published 1970 by
  V. L. Arlazarov, E. A. Dinitz, M. A. Kronrod, and I. A. Faradžev
- all worked in Moscow at that time . . . but not even clear if all are Russians!

  (Arlazarov and Kronrod are Russian)

- American authors coined the othering term "Method of Four Russians"
  . . . name in widespread use

Bootstrapping

- We know a $\langle O(n \log n), O(1) \rangle$ time solution
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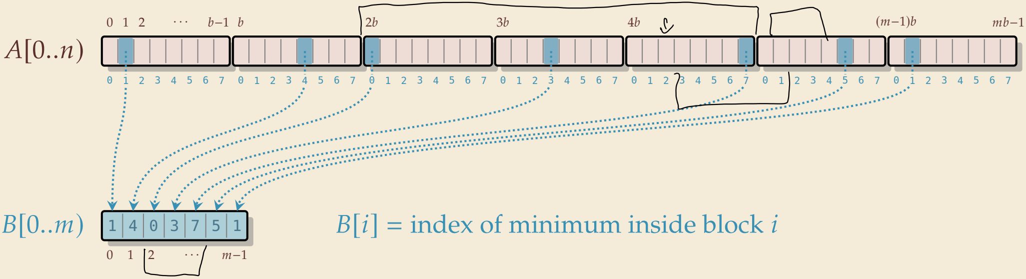
- Create array of block minima $B[0..m)$ for $m = \lceil n/b \rceil = O(n/\log n)$



$B[i]$ = index of minimum inside block i

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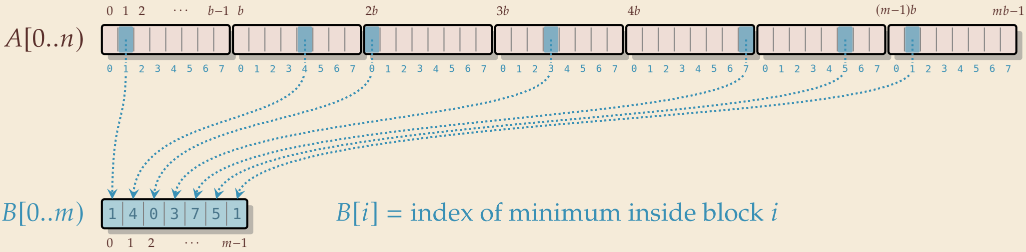

⇝ Use sparse tables for B.

⇝ Can solve RMQs in $B[0..m)$ in $\langle O(n), O(1) \rangle$ time

Query decomposition

- Query $\mathrm{RMQ}_A(i, j)$ covers
  - suffix of block $\ell = \lfloor i/m \rfloor$
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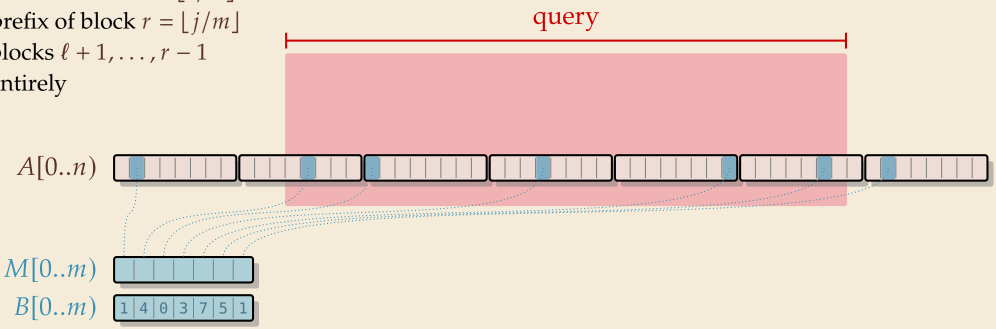

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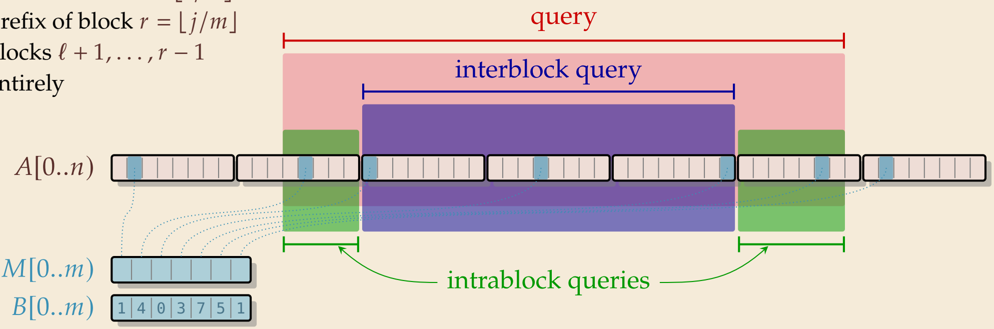

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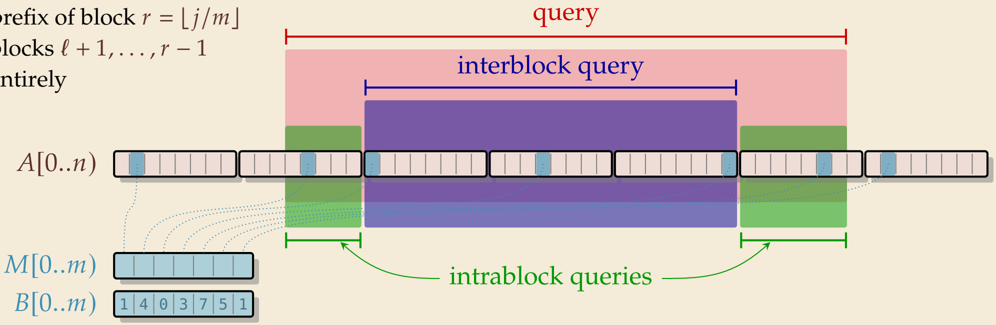

- $\mathrm{RMQ}_A(i,j) = \arg\min_{k\in K} A[k]$ with $K = \left\{ \begin{array}{l} \mathrm{RMQ}_{\text{block }\ell}\big(i-\ell b,\ (\ell+1)b-1\big), \\ b\cdot \mathrm{RMQ}_M\big(\ell+1, r-1\big) + \\ \qquad B\big[\mathrm{RMQ}_M\big(\ell+1,r-1\big)\big], \\ \mathrm{RMQ}_{\text{block }r}\big(rb,\ j-rb\big) \end{array} \right\}$

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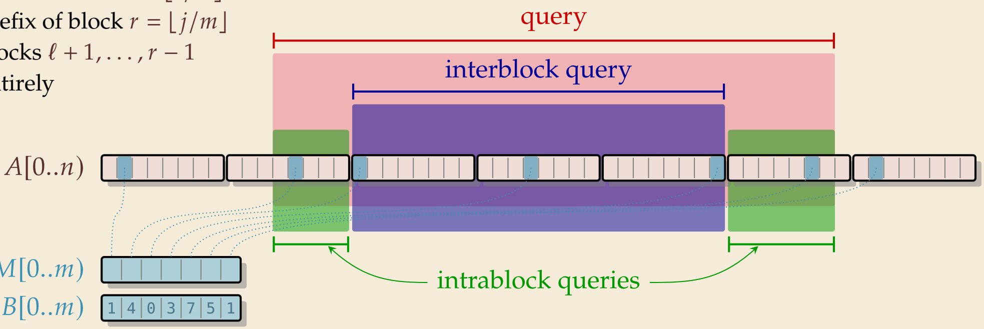

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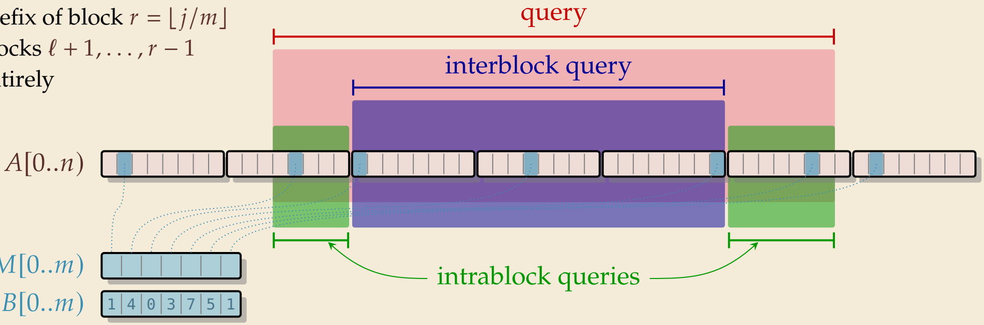

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Intrablock queries [1]

⇝ It remains to solve the intrablock queries!

- Want $\langle O(n), O(1) \rangle$ time overall

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⇝ ***Exhaustive Tabulation Technique:***

1. represent each subproblem by storing its *type* (here: encoding of Cartesian tree)
2. *enumerate* all possible subproblem types and their solutions
3. use type as index in a large *lookup table*

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Block type	i	j	$\mathrm{RMQ}(i, j)$
⋮			
01101001	0	1	0
"	0	2	0
"	0	3	3
"	1	2	1
	1	3	
		⋮	
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- $\le \sqrt{n}$ block types
- $\le b^2$ combinations for i and j

$\rightsquigarrow$ $\Theta\left(\sqrt{n} \cdot \log^2 n\right)$ rows

- each row can be computed in $O(\log n)$ time

$\rightsquigarrow$ overall preprocessing: $O(n)$ time!

Discussion

- $\langle O(n), O(1) \rangle$ time solution for RMQ ✓

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Research questions:

- Reduce the space usage
- Avoid access to A at query time