

# 9

## Graph Algorithms

9 December 2024

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# Learning Outcomes

## Unit 9: *Graph Algorithms*

1. Know basic terminology from graph theory, including types of graphs.
2. Know adjacency matrix and adjacency list representations and their performance characteristics.
3. Know graph-traversal based algorithm, including efficient implementations.
4. Be able to proof correctness of graph-traversal-based algorithms.
5. Know algorithms for maximum flows in networks.
6. Be able to model new algorithmic problems as graph problems.

# Outline

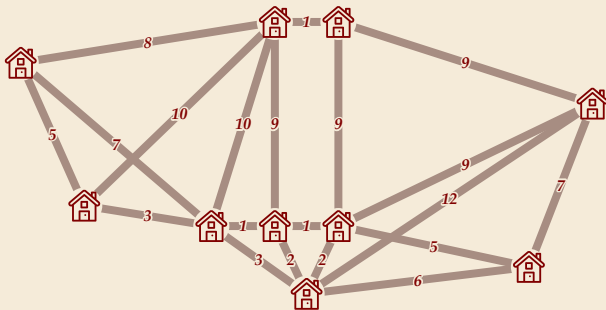
## 9 Graph Algorithms

- 9.1 Introduction & Definitions
- 9.2 Graph Representations
- 9.3 Graph Traversal
- 9.4 BFS and DFS
- 9.5 Advanced Uses of DFS
- 9.6 Network flows
- 9.7 The Ford-Fulkerson Method
- 9.8 The Edmonds-Karp Algorithm

## **9.1 Introduction & Definitions**

# Graphs in real life

- ▶ a graph is an abstraction of *entities* with their (pairwise) *relationships*
- ▶ abundant examples in real life (often called network there)
  - ▶ social networks: e. g. persons and their friendships, ... *Five/Six? degrees of separation*
  - ▶ physical networks: cities and highways, roads networks, power grids etc., the Internet, ...
  - ▶ content networks: world wide web, ontologies, ...
  - ▶ ...



Many More examples, e. g., in Sedgewick & Wayne's videos:

<https://www.coursera.org/learn/algorithms-part2>

# Flavors of Graphs

- ▶ Since graphs are used to model so many different entities and relations, they come in several variants

Property	Yes	No
edges are one-way	<i>directed</i> graph ( <i>digraph</i> )	<i>undirected</i> graph
$\leq 1$ edge between $u$ and $v$	<i>simple</i> graph	<i>multigraph</i> / with <i>parallel</i> edges
edges can lead from $v$ to $v$	with <i>loops</i>	(loop-free)
edges have weights	<i>(edge-) weighted</i> graph	<i>unweighted</i> graph

☺ any combination of the above can make sense . . .

- ▶ Synonyms:
  - ▶ **vertex** („Knoten“) = node = point = „Ecke“
  - ▶ **edge** („Kante“) = arc = line = relation = arrow = „Pfeil“
  - ▶ **graph** = network

# Graph Theory

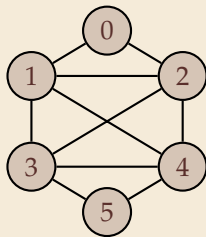
- ▶ default: unweighted, undirected, loop-free & simple graphs
- ▶ *Graph*  $G = (V, E)$  with
  - ▶  $V$  a finite of *vertices*
  - ▶  $E \subseteq [V]^2$  a set of *edges*, which are 2-subsets of  $V$ :  $[V]^2 = \{e : e \subseteq V \wedge |e| = 2\}$

## Example

$$V = \{0, 1, 2, 3, 4, 5\}$$

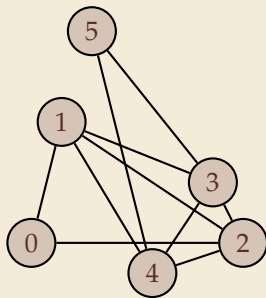
$$E = \{\{0, 1\}, \{1, 2\}, \{1, 4\}, \{1, 3\}, \{0, 2\}, \\ \{2, 4\}, \{2, 3\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\}.$$

## Graphical representation



like so ...

=



... or so

(same graph)

# Digraphs

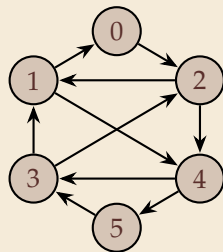
- ▶ default digraph: unweighted, loop-free & simple
- ▶ *Digraph (directed graph)*  $G = (V, E)$  with
  - ▶  $V$  a finite of *vertices*
  - ▶  $E \subseteq V^2 \setminus \{(v, v) : v \in V\}$  a set of (*directed*) *edges*,  
 $V^2 = V \times V = \{(x, y) : x \in V \wedge y \in V\}$  2-tuples / ordered pairs over  $V$

## Example

$$V = \{0, 1, 2, 3, 4, 5\}$$

$$E = \{(0, 2), (1, 0), (1, 4), (2, 1), (2, 4), \\ (3, 1), (3, 2), (4, 3), (4, 5), (5, 3)\}$$

## Graphical representation





# Graph Terminology

## Undirected Graphs

- ▶  $V(G)$  set of vertices,  $E(G)$  set of edges
- ▶ write  $uv$  (or  $vu$ ) for edge  $\{u, v\}$
- ▶ edges *incident* at vertex  $v$ :  $E(v)$
- ▶  $u$  and  $v$  are *adjacent* iff  $\{u, v\} \in E$ ,
- ▶ *neighborhood*  $N(v) = \{w \in V : w \text{ adjacent to } v\}$
- ▶ *degree*  $d(v) = |E(v)|$
- ▶ *walk*  $w$  of length  $n$ : sequence of vertices  $w[0..n]$  with  $\forall i \in [0..n) : w[i]w[i+1] \in E$
- ▶ *path*  $p$  is a (vertex-) simple walk: without duplicate vertices except possibly its endpoints
- ▶ *edge-simple* walk: no edge used twice
- ▶ *cycle*  $c$  is a closed path, i. e.,  $c[0] = c[n]$
- ▶  $G$  is *connected*  
iff for all  $u \neq v \in V$  there is a path from  $u$  to  $v$
- ▶  $G$  is *acyclic* iff  $\nexists$  cycle (of length  $n \geq 1$ ) in  $G$

## Directed Graphs (where different)

- ▶  $uv$  for  $(u, v)$
- ▶ iff  $(u, v) \in E \vee (v, u) \in E$
- ▶ in-/out-neighbors  $N_{\text{in}}(v), N_{\text{out}}(v)$
- ▶ in-/out-degree  $d_{\text{in}}(v), d_{\text{out}}(v)$
- ▶ *strongly connected* for digraphs  
(*weakly connected* = connected ignoring directions)

# Typical graph-processing problems

- ▶ **Path:** Is there a path between  $s$  and  $t$ ?  
**Shortest path:** What is the shortest path (distance) between  $s$  and  $t$ ?
- ▶ **Cycle:** Is there a cycle in the graph?  
**Euler tour:** Is there a cycle that uses each edge exactly once?  
**Hamilton(ian) cycle:** Is there a cycle that uses each vertex exactly once.
- ▶ **Connectivity:** Is there a way to connect all of the vertices?  
**MST:** What is the best way to connect all of the vertices?  
**Biconnectivity:** Is there a vertex whose removal disconnects the graph?
- ▶ **Planarity:** Can you draw the graph in the plane with no crossing edges?
- ▶ **Graph isomorphism:** Are two graphs the same up to renaming vertices?

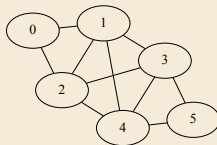
← can vary a lot, despite superficial similarity of problems

**Challenge:** Which of these problems  
can be computed in (near) linear time?  
in reasonable polynomial time?  
are intractable?

# Tools to work with graphs

- ▶ Convenient GUI to edit & draw graphs: *yEd live*  
[yworks.com/yed-live](http://yworks.com/yed-live)
- ▶ *graphviz* cmdline utility to draw graphs
  - ▶ Simple text format for graphs: DOT

```
graph G {  
    0 -- 2;    2 -- 4;  
    1 -- 0;    2 -- 3;  
    1 -- 4;    3 -- 4;  
    1 -- 3;    3 -- 5;  
    2 -- 1;    4 -- 5;  
}
```



```
dot -Tpdf graph.dot -Kfdp > graph.pdf
```

- ▶ graphs are typically not built into programming languages, but libraries exist
  - ▶ e. g. part of *Google Guava* for Java
  - ▶ they usually allow arbitrary objects as vertices
  - ▶ aimed at ease of use

## 9.2 Graph Representations

# Graphs in Computer Memory

- ▶ We defined graphs in set-theoretic terms. . .  
but computers can't directly deal with sets efficiently

↪ need to choose a *representation* for graphs.

- ▶ which is better depends on the required operations

## Key Operations:

- ▶  $\text{isAdjacent}(u, v)$   
Test whether  $uv \in E$
- ▶  $\text{adj}(v)$   
Adjacency list of  $v$  (iterate through (out-) neighbors of  $v$ )
- ▶ most others can be computed based on these

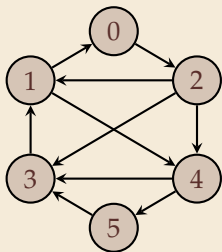
## Conventions:

- ▶ (di)graph  $G = (V, E)$  (omitted if clear from context)
- ▶  $n = |V|$ ,  $m = |E|$
- ▶ in implementations assume  $V = [0..n)$  (if needed, use symbol table to map complex objects to  $V$ )

# Adjacency Matrix Representation

- ▶ adjacency matrix  $A \in \{0, 1\}^{n \times n}$  of  $G$ : matrix with  $A[u, v] = [uv \in E]$ 
  - ▶ works for both directed and undirected graphs (undirected  $\rightsquigarrow A = A^T$  symmetric)
  - ▶ can use a weight  $w(uv)$  or multiplicity in  $A[u, v]$  instead of 0/1
  - ▶ can represent loops via  $A[v, v]$

Example:

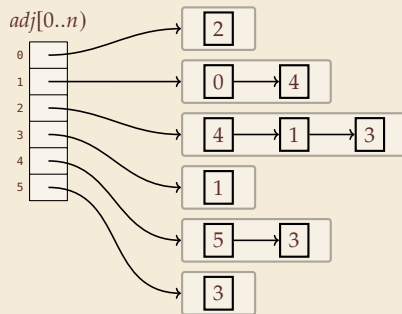
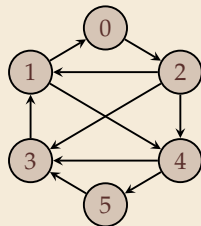


$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

- 👍 isAdjacent in  $O(1)$  time
- 👎  $O(n^2)$  (bits of) space wasteful for sparse graphs
- 👎 adj( $v$ ) iteration takes  $O(n)$  (independent of  $d(v)$ )

# Adjacency List Representation

- ▶ Store a linked list of neighbors for each vertex  $v$ :
  - ▶  $adj[0..n)$  bag of neighbors (as linked list)
  - ▶ undirected edge  $\{u, v\} \rightsquigarrow v \text{ in } adj[u] \text{ and } u \text{ in } adj[v]$
  - ▶ weighted edge  $uv \rightsquigarrow \text{store pair } (v, w(uv)) \text{ in } adj[u]$
  - ▶ multiple edges and loops can be represented



👎  $\text{isAdjacent}(u, v)$  takes  $\Theta(d(u))$  time (worst case)

👍  $\text{adj}(v)$  iteration  $O(1)$  per neighbor

👍  $\Theta(n + m)$  (words of) space for any graph ( $\ll \Theta(n^2)$  bits for moderate  $m$ )

$\rightsquigarrow$  de-facto standard for graph algorithms

# Graph Types and Representations

- ▶ Note that adj matrix and lists for undirected graphs effectively are representation of directed graph with directed edges both ways
  - ▶ conceptually still important to distinguish!
- ▶ multigraphs, loops, edge weights all naturally supported in adj lists
  - ▶ good if we allow and use them
  - ▶ but requires explicit checks to enforce simple / loopfree / bidirectional!
- ▶ we focus on **static graphs**  
dynamically changing graphs much harder to handle



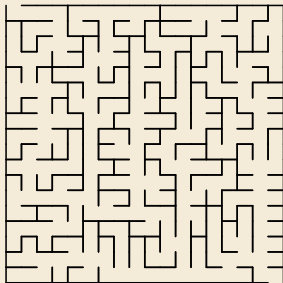
## 9.3 Graph Traversal

# Generic Graph Traversal

► Plethora of graph algorithms can be expressed as a systematic exploration of a graph

- depth-first search, breadth-first search
- connected components
- detecting cycles
- topological sorting
- Hierholzer's algorithm for Euler walks
- strong components
- testing bipartiteness
- Dijkstra's algorithm
- Prim's algorithm
- Lex-BFS for perfect elimination orders of chordal graphs
- ...

↑  
visiting all nodes & edges



~> Formulate generic traversal algorithm

- first in abstract terms to argue about correctness
- then again for concrete instance with efficient data structures

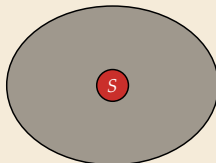
# Tricolor Graph Traversal

## Tricolor Graph Search:

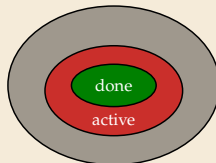
- ▶ maintain vertices in 3 (dynamic) sets
  - ▶ **Gray: unseen vertices**  
The traversal has not reached these vertices so far.
  - ▶ **Green: done vertices** (a.k.a. visited vertices)  
These vertices have been visited and all their edges have been explored already.
  - ▶ **Red: active vertices** (a.k.a. frontier („Rand“) of traversal)  
All others, i. e., vertices that have been reached and some unexplored edges remain; initially some selected start vertices  $S$ .
- ▶ (implicitly) maintain status of each edge
  - ▶ **not yet used**
  - ▶ **used edge**
- ▶ Vertices “want” to turn **green**.

### Invariant:

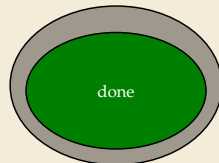
No edges from **done** to **unseen** vertices



initial state



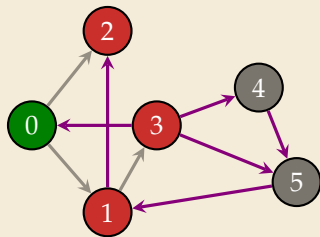
during traversal



final state

# Generic Tricolor Graph Traversal – Code

```
1 procedure genericGraphTraversal( $G, S$ )
2   // (di)graph  $G = (V, E)$  and start vertices  $S \subseteq V$ 
3    $C[0..n) := \text{unseen}$  // Color array, all cells initialized to unseen
4   for  $s \in S$  do  $C[s] := \text{active}$  end for
5    $\text{unusedEdges} := E$ 
6   while  $\exists v : C[v] == \text{active}$ 
7      $v := \text{nextActiveVertex}()$  // Freedom 1: Which frontier vertex?
8     if  $\nexists vw \in \text{unusedEdges}$  // no more edges from  $v \rightsquigarrow$  done with  $v$ 
9        $C[v] := \text{done}$ 
10    else
11       $w := \text{nextUnusedEdge}(v)$  // Freedom 2: Which of its edges?
12      if  $C[w] == \text{unseen}$ 
13         $C[w] := \text{active}$ 
14      end if
15       $\text{unusedEdges.remove}(vw)$ 
16    end if
17  end while
```



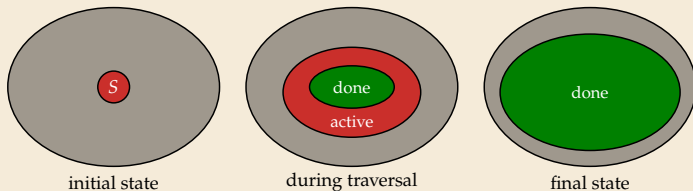
**Invariant:**

No edges from *done* to *unseen* vertices

- Implementations of `nextActiveVertex()` and `nextUnusedEdge(v)` depends on (and defines!) specific traversal-based graph algorithms

# Generic Reachability

- ▶ Any choices `nextActiveVertex()` and `nextUnusedEdge(v)` suffice to find exactly the vertices reachable from  $S$  in *done*
- ▶ **Invariant:**
  1. No edges from *done* to *unseen* vertices
  2. For every *done* or *active* vertex  $v$ , there exists a path from  $s \in S$  to  $v$ .



↪ in final state:

- ▶  $v \in \text{done}$  ↪ path from  $S$  ↪ reachable from  $S$
- ▶  $v \in \text{unseen}$  ↪ not reachable from  $\text{done} \supseteq S$  ↪ not reachable from  $S$

# Data Structures for Frontier

- ▶ We need efficient support for
    - ▶ test  $\exists v : C[v] = \text{active}$ , `nextActiveVertex()`
    - ▶ test  $\exists vw \in \text{unusedEdges}$ , `nextUnusedEdge(v)`
    - ▶ `unusedEdges.remove(vw)`
  - ▶ Typical solution maintains **bag** “*frontier*” of *pairs*  $(v, i)$  where  $v \in V$  and  $i$  is an **iterator** in `adj[v]`
    - ▶ `unusedEdges` represented implicitly: edge used iff previously returned by  $i$ 
      - $\rightsquigarrow$  don't need `unusedEdges.remove(vw)`
    - ▶ Implement  $\exists v : C[v] = \text{active}$  via `frontier.isEmpty()`
    - ▶ Implement  $\exists vw \in \text{unusedEdges}$  via `i.hasNext()` assuming  $(v, i) \in \text{frontier}$
    - ▶ Implement `nextUnusedEdge(v)` via `i.next()` assuming  $(v, i) \in \text{frontier}$
- $\rightsquigarrow$  all operations apart from `nextActiveVertex()` in  $O(1)$  time
- $\rightsquigarrow$  *frontier* requires  $O(n)$  extra space

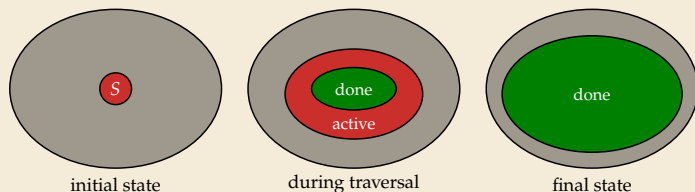
## 9.4 BFS and DFS

# Breadth-First Search

► Maintain *frontier* in a **queue** (FIFO: first in, first out)

► **Invariant:**

1. No edges from done to unseen vertices
2. All *done* or *active* vertices are reached via a **shortest path** from  $S$
3. Vertices enter and leave *frontier* in order of increasing distance from  $S$



⇒ in final state, we reach all reachable vertices via shortest paths

► To preserve that knowledge, we collect extra information during traversal

- *parent* $[v]$  stores predecessor on path from  $S$  via which  $v$  was reached
- *distFromS* $[v]$  stores the length of this path



# Breadth-First Search – Code

---

```
1 procedure bfs( $G, S$ )
2   // (di)graph  $G = (V, E)$  and start vertices  $S \subseteq V$ 
3    $C[0..n) := \text{unseen}$  // New array initialized to all unseen
4   frontier := new Queue;
5    $\text{parent}[0..n) := \text{NOT\_VISITED}$ ;  $\text{distFromS}[0..n) := \infty$ 
6   for  $s \in S$ 
7      $\text{parent}[s] := \text{NONE}$ ;  $\text{distFromS}[s] := 0$ 
8      $C[s] := \text{active}$ ; frontier.enqueue( $(s, G.\text{adj}[s].\text{iterator}())$ )
9   end for
10  while  $\neg \text{frontier.isEmpty}()$ 
11     $(v, i) := \text{frontier.peek}()$ 
12    if  $\neg i.\text{hasNext}()$  //  $v$  has no unused edge
13       $C[v] := \text{done}$ ; frontier.dequeue()
14    else
15       $w := i.\text{next}()$  // Advance  $i$  in  $\text{adj}[v]$ 
16      if  $C[w] == \text{unseen}$ 
17         $\text{parent}[w] := v$ ;  $\text{distFromS}[w] := \text{distFromS}[v] + 1$ 
18         $C[w] := \text{active}$ ; frontier.enqueue( $(w, G.\text{adj}[w].\text{iterator}())$ )
19      end if
20    end if
21  end while
```

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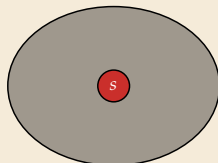
- ▶ *parent* stores a *shortest-path tree/forest*
- ▶ can retrieve shortest path to  $v$  from some vertex  $s \in S$  (backwards) by following *parent* $[v]$  iteratively
- ▶ running time  $\Theta(n + m)$
- ▶ extra space  $\Theta(n)$

# Depth-First Search

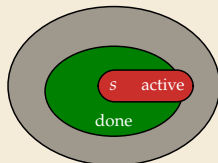
- ▶ Maintain *frontier* in a **stack** (LIFO: last in, first out)
  - ▶ only consider  $S = \{s\}$
  - ▶ usual mode of operation: call  $\text{dfs}(v)$  for all *unseen*  $v$ , for  $v = 0, \dots, n - 1$

▶ **Invariant:**

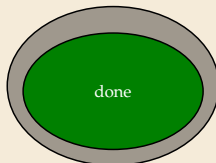
1. No edges from done to unseen vertices
2. All *done* or *active* vertices are reached via a path from  $s$
3. The *active* vertices form a single **path** from  $s$



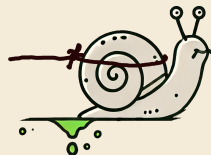
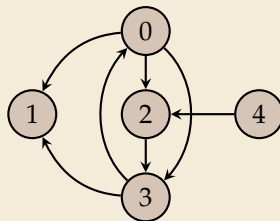
initial state



during traversal



final state



# Depth-First Search – Code

---

```
1 procedure dfsTraversal(G)
2   C[0..n] := unseen
3   for v := 0, ..., n - 1
4     if C[v] == unseen
5       dfs(G, v)
6
7 procedure dfs(G, s)
8   frontier := new Stack;
9   C[s] := active; frontier.push((s, G.adj[s].iterator()))
10  while ¬frontier.isEmpty()
11    (v, i) := frontier.top()
12    if ¬i.hasNext() // v has no unused edge
13      C[v] := done; frontier.pop(); postorderVisit(v)
14    else
15      w := i.next(); visitEdge(vw)
16      if C[w] == unseen
17        preorderVisit(w)
18        C[w] := active; frontier.push((w, G.adj[w].iterator()))
19      end if
20    end if
21  end while
```

- ▶ define *hooks* to implement further operations
  - ▶ preorder: visit  $v$  when made *active* (start of  $T(v)$ )
  - ▶ postorder: visit  $v$  when marked *done* (end of  $T(v)$ )
  - ▶ visitEdge: do something for every edge
- ▶ if needed, can store DFS forest via *parent* array
- ▶ running time  $\Theta(n + m)$
- ▶ extra space  $\Theta(n)$

# Simple DFS Application: Connected Components

- ▶ In an undirected graph, find all *connected components*.
  - ▶ **Given:** simple undirected  $G = (V, E)$
  - ▶ **Goal:** assign component ids  $CC[0..n]$ , s.t.  $CC[v] = CC[u]$  iff  $\exists$  path from  $v$  to  $u$

---

```
1 procedure connectedComponents( $G$ ):
2   // undirected graph  $G = (V, E)$  with  $V = [0..n)$ 
3    $C[0..n) := \text{unseen}$ 
4    $CC[0..n) := \text{NONE}$ 
5    $id := 0$ 
6   for  $v := 0, \dots, n - 1$ 
7     if  $C[v] == \text{unseen}$ 
8        $\text{dfs}(G, v)$ 
9        $id := id + 1$ 
10  return  $CC$ 
11
12 procedure preorderVisit( $v$ ):
13    $CC[v] := id$ 
```

---

---

```
1 // same as before
2 procedure dfs( $G, s$ )
3    $\text{frontier} := \text{new Stack}$ ;
4    $C[s] := \text{active}$ ;  $\text{frontier.push}((s, G.\text{adj}[s].\text{iterator}()))$ 
5   while  $\neg \text{frontier.isEmpty}()$ 
6      $(v, i) := \text{frontier.top}()$ 
7     if  $\neg i.\text{hasNext}()$  //  $v$  has no unused edge
8        $C[v] := \text{done}$ ;  $\text{frontier.pop}()$ 
9        $\text{postorderVisit}(v)$ 
10    else
11       $w := i.\text{next}()$ ;  $\text{visitEdge}(vw)$ 
12      if  $C[w] == \text{unseen}$ 
13         $\text{preorderVisit}(w)$ 
14         $C[w] := \text{active}$ 
15         $\text{frontier.push}((w, G.\text{adj}[w].\text{iterator}()))$ 
16      end if
17    end if
18  end while
```

---

# Dijkstra's Algorithm & Prim's Algorithm

- ▶ On edge-weighted graphs, we can use tricolor traversal with a *priority queue* as *frontier*
- ▶ Dijkstra's Algorithm for shortest paths from  $s$  in digraphs with weakly positive edge weights
  - ▶ priority of vertex  $v$  = length of shortest path known so far from  $s$  to  $v$
- ▶ Prim's Algorithm for finding a minimum spanning tree
  - ▶ priority of vertex  $v$  = weight of cheapest edge connecting  $v$  to current tree

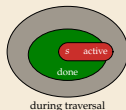
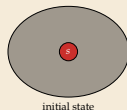
⇒ Detailed discussion in Unit 11

## 9.5 Advanced Uses of DFS

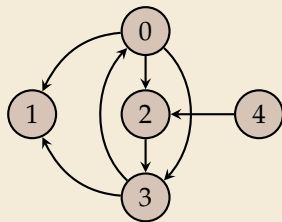
# Properties of DFS

## ► Recall DFS Invariant 3:

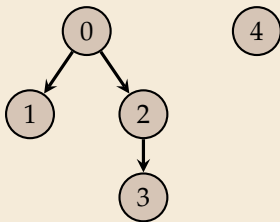
The **active** vertices form a single **path** from  $s$



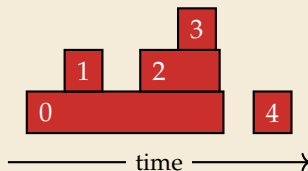
input graph  $G$



DFS forest



stack over time



$\rightsquigarrow$  Each vertex  $v$  spends *time interval*  $T(v)$  as **active** vertex

1. **frontier** is stack  $\rightsquigarrow \{T(v) : v \in V\}$  forms **laminar set family**: (“disjoint or contained”)  
either  $T(v) \cap T(w) = \emptyset$  or  $T(v) \subseteq T(w)$  or  $T(v) \supseteq T(w)$

2. **Parenthesis Theorem**:  $T(v) \supseteq T(w)$  **iff**  $v$  is ancestor of  $w$  in DFS tree

‘ $\Rightarrow$ ’ during  $T(v)$ , all discovered vertices become descendants of  $v$

‘ $\Leftarrow$ ’  $T(v)$  covers  $v$ ’s entire subtree, which contains  $w$ ’s subtree

# Properties of DFS – Unseen-Path Theorem

- **Unseen-Path Theorem:** In a DFS forest of a (di)graph  $G$ ,  $w$  is a descendant of  $v$  iff at the time of  $\text{preorderVisit}(v)$ , there is a path from  $v$  to  $w$  using only *unseen* vertices.

‘ $\Rightarrow$ ’ If  $w$  is a descendant of  $v$ ,  $T(w) \subseteq T(v)$  by the Parenthesis Theorem.

Hence the path from  $v$  to  $w$  in the DFS tree consists (at time of  $\text{preorderVisit}(v)$ ) of solely *unseen* vertices.

‘ $\Leftarrow$ ’ Suppose towards a contradiction that there was a  $w$  with an *unseen* path  $p[0..\ell]$  with  $p[0] = v$  and  $p[\ell] = w$ , but  $w$  is not a descendant of  $v$ . W.l.o.g. let  $w$  be a first such vertex, i. e.,  $p[0], \dots, p[\ell - 1] = u$  are descendants of  $v$ .

So  $T(u) \subset T(v)$  (\*).

Upon processing  $u$ , we will discover edge  $uw$ , so whether or not  $w$  is already *done* at this point,  $w$  will be marked *done* before  $u$ . Hence  $\max T(w) \leq \max T(u)$ .

With (\*), we obtain  $\min T(v) \leq \min T(u) \leq \max T(w) \leq \max T(u)$ , so by laminarity,  $T(w) \subset T(u) \subset T(v)$  and  $w$  is a descendant of  $v$  ⚡.



# Topological Sorting & Cycle Detection

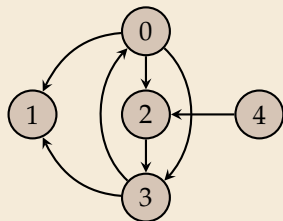
- ▶ **Application:** Given a set of tasks with precedence constraints of the form “ $a$  must be done before  $b$ ”, can we find a legal ordering for all tasks?

↪ Model as directed graph!

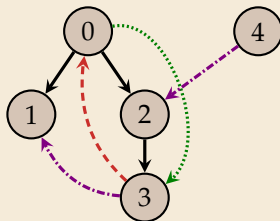
- ▶ tasks are the vertices  $V$
- ▶ add an edge  $(a, b)$  when  $a$  must be done before  $b$
- ▶ **Definition:**  $R[0..n]$  is a *topological (order) ranking* of digraph  $G = (V, E)$  if  $\forall (u, v) \in E : R[u] < R[v]$
- ▶ **Lemma DAG iff topo:**  
A directed graph  $G$  has a topological ranking **iff** it does not contain a directed cycle.
- ▶ **Topological Sorting**
  - ▶ **Given:** simple digraph  $G = (V, E)$
  - ▶ **Goal:** Compute topological ranking of vertices  $R[0..n]$  or output a directed cycle in  $G$ .
- ▶ Amazingly, can do all with one pass of DFS!

# DFS Edge Types

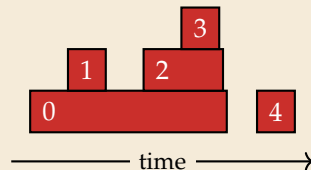
input digraph  $G$



DFS forest



stack over time



► During DFS traversal, an edge  $vw$  has one of these 4 types:

1. **tree edge:**  $\longrightarrow w \in \text{unseen} \rightsquigarrow vw$  part of DFS forest.
2. **back edges:**  $--\longrightarrow w \in \text{active}; \rightsquigarrow w$  points to ancestor of  $v$ .
3. **forward edges\*:**  $\cdots\cdots\longrightarrow w \in \text{done} \wedge w$  is descendant of  $v$  in DFS tree.
4. **cross edges\*:**  $-\cdots\longrightarrow w \in \text{done} \wedge w$  is not descendant of  $v$ .

\*only possible in *directed* graphs

**example:**

$(0, 1), (0, 2), (2, 3)$   
 $(3, 0)$   
 $(0, 3)$   
 $(3, 0)$

# Cycle Detection

If  $G$  contains a directed cycle, DFS will find a directed cycle:

- ▶ any back edge implies a cycle:
  - ▶ DFS visits an edge  $(v, w)$  where  $w \in \text{active}$ ,  $w$  is already on the stack
  - $\rightsquigarrow$  DFS tree contains path  $w \rightsquigarrow v$  and we have edge  $v \rightarrow w$ .
- ▶ conversely any cycle  $C[0..k]$  once reached must have some back edge or cross edge (tree and forward edges go from smaller to larger preorder index)
  - ▶ cannot be a cross edge since cycle is strongly connected  
all cycle vertices must be descendants of first reached cycle vertex
  - $\rightsquigarrow$  cycle contributes a back edge

# DFS Postorder Implementation

---

```
1 procedure dfsPostorder(G):
2   C[0..n) := unseen
3   P[0..n) := NONE; r := 0
4   parent[0..n) := NONE
5   cycle := NONE
6   for v := 0, ..., n - 1
7     if C[v] == unseen
8       dfs(G, v)
9   return (P, cycle)
10
11 procedure postorderVisit(v):
12   P[v] := r; r := r + 1
13
14 procedure visitEdge(vw):
15   if C[w] == active
16     if cycle ≠ NONE return
17     while v ≠ w
18       cycle.append(v)
19       v := parent[v]
20   cycle.append(v)
```

---

---

```
1 // dfs is as in CC but with parent
2 procedure dfs(G, s)
3   frontier := new Stack;
4   parent[s] := NONE;
5   C[s] := active; frontier.push((s, G.adj[s].iterator()))
6   while ¬frontier.isEmpty()
7     (v, i) := frontier.top()
8     if ¬i.hasNext() // v has no unused edge
9       C[v] := done; frontier.pop()
10      postorderVisit(v)
11   else
12     w := i.next() // Advance i in adj[v]
13     visitEdge(vw)
14     if C[w] == unseen
15       parent[w] := v;
16       preorderVisit(w)
17       C[w] := active; frontier.push((w, G.adj[w].iterator()))
18   end if
19   end if
20   end while
```

---

# DFS Postorder & Topological Sort

- ▶ **DFS Postorder:** The DFS postorder numbers is a numbering  $P[0..n)$  of  $V$  such that  $P[v] = r$  iff exactly  $r$  vertices reached state *done* before  $v$  in a DFS.
- ▶ **Lemma rev postorder:** directed acyclic graph  
Let  $G$  be a simple, connected DAG and  $R[0..n)$  a *reverse DFS postorder* of  $G$ , i. e.,  $R[v] = n - 1 - P[v]$  for a DFS postorder  $P[0..n)$ . Then  $R$  is a topological ranking of  $G$ .
- ▶ **Invariant:** If  $v \in \text{done}$  and  $(v, w) \in E$  then  $w \in \text{done}$  and  $R[v] < R[w]$ .
  - ▶ initially true ( $\text{done} = \emptyset$ )
  - ▶ upon `postorderVisit(v)`, all outgoing edges  $vw$  lead to  $w \in \text{done}$  (Parenthesis Theorem)

# Topological Sorting & Cycle Detection – Summary

- ▶ Putting everything together we obtain topological sorting
  - ▶ can produce either the *ranking* or the *sequence of vertices* in topological order, whatever is more convenient

---

```
1 procedure topologicalRanking( $P$ ):
2   ( $P[0..n], cycle$ ) := dfsPostorder( $G$ )
3   if  $cycle \neq \text{NULL}$ 
4     return NOT_A_DAG
5    $R[0..n] := \text{NONE}$ 
6   for  $v := 0, \dots, n - 1$ 
7      $R[v] = n - 1 - P[v]$ 
8   return  $R$ 
```

---

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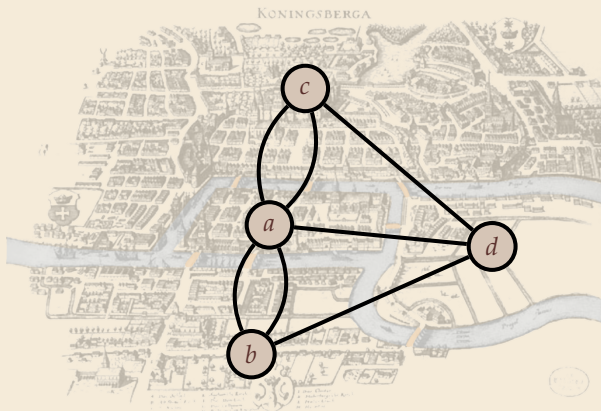
```
1 procedure topologicalSort( $P$ ):
2   ( $P[0..n], cycle$ ) := dfsPostorder( $G$ )
3   if  $c \neq \text{NULL}$ 
4     return NOT_A_DAG
5    $S[0..n] := \text{NONE}$ 
6   for  $v := 0, \dots, n - 1$ 
7      $S[n - 1 - P[v]] := v$ 
8   return  $S$ 
```

---

- ▶  $\Theta(n + m)$  time
- ▶  $\Theta(n)$  extra space

# Euler Cycles

**Euler Walk:** Walk using every edge in  $G = (V, E)$  exactly once.



## Euler's Theorem:

Euler walk exists iff  $G$  connected and 0 or 2 vertices have odd degree.

' $\Rightarrow$ ' trivial (need to enter and exit intermediate vertices equally often)

' $\Leftarrow$ ' Following algorithm *constructs* Euler walk under this assumption



# Euler Cycles – Hierholzer's Algorithm

- ▶ use an *edge-centric DFS*
  - ▶ We mark *edges* (not vertices)
- ↪ stack = **edge-simple walk**
- ▶ We remember iterator  $i$  globally per  $v$  to resume traversal

---

```
1 procedure eulerWalk(G):
2   // Assume  $G = (V, E)$  is connected (multi)graph
3    $V_{\text{odd}} := \{v \in V : d(v) \text{ odd}\}$ 
4   if  $|V_{\text{odd}}| \notin \{0, 2\}$  return NOT_EULERIAN
5   if  $V_{\text{odd}} = \{x, y\}$  then  $s := x$  else  $s := 0$ 
6    $euler[0..m] := \text{NONE}$ ;  $j := m - 1$ 
7    $visited[0..n, 0..n] := \text{false}$  // mark edges as visited
8   for  $v := 0, \dots, n - 1$ 
9     // globally remember next unexplored edge
10     $nextEdge[v] := G.adj[w].iterator()$ 
11  edgeDFS(s)
12  return euler
```

---

---

```
1 procedure edgeDFS(s):
2   frontier := new Stack;
3   frontier.push(s)
4   while  $\neg \text{frontier.isEmpty}()$ 
5      $v := \text{frontier.top}()$ 
6     if  $\neg i.hasNext()$  //  $v$  has no unused edge
7       frontier.pop()
8       if  $\neg \text{frontier.isEmpty}()$ 
9         // assign edge leading here largest free index
10         $euler[j] := (\text{frontier.top}(), v)$ ;  $j := j - 1$ 
11      end if
12    else
13       $w := i.next()$ 
14      if  $\neg visited[v, w]$ 
15         $visited[v, w] := \text{true}$ 
16         $visited[w, v] := \text{true}$ 
17        frontier.push(w)
18      end if
19    end if
20  end while
```

---



# Strong Components

- ▶ **Given:** digraph  $G = (V, E)$
  - ▶ **Goal:** component ids  $SCC[0..n)$ , s.t.  $SCC[v] = SCC[u]$  iff  $\exists$  directed path from  $v$  to  $u$   
strongly connected component
  - ▶ **Component DAG**  $G^{SCC}$ : contract SCCs into single vertices  
 $V(G^{SCC}) = \{C_1, \dots, C_k\}$  with  $C_1 \dot{\cup} \dots \dot{\cup} C_k = V$ ;  
name by smallest vertex s.t.  $i \leq j$  iff  $\min C_i \leq \min C_j$ 
    - ▶ can't have cycles (⚡ maximality of SCC)
- $\rightsquigarrow$  component DAG has a topological order  $R^{SCC}[1..k]$



If we call dfs on any  $v$  in the **last** SCC  $C$ , it will discover all vertices in  $C$ , and only those!  
(any edges between components lead *into*  $C$  by topological order)

And we can iterate this backwards through any topological order to get all SCCs!



Can we efficiently find the topological order of  $G^{SCC}$ ?  
*Without knowing the components to start with??*

Amazingly, yes.

# Component Graph DFS

- ▶ Suppose we run `dfsTraversal` on  $G$ .

↪ We can extend time intervals to SCCs:  $T(C_i) := \bigcup_{v \in C_i} T(v)$

↪  $T(C_i) = T(v_i)$  for  $v_i \in C_i$  the first vertex to be explored in a DFS on  $G$   
(by Unseen Path & Parenthesis Thms)

↪ DFS on  $G$  produces same  $T(C_i)$  (up to time scaling) as DFS on  $G^{\text{SCC}}$ !


↪ reverse DFS postorder on  $G$  gives same relative order to  $v_1, \dots, v_k$  as  
reverse DFS postorder on  $G^{\text{SCC}}$  gives as relative order to  $C_1, \dots, C_k$



We need **reverse** topological order on  $G^{\text{SCC}}$ , e. g., *reversed reverse DFS postorder*

- ▶ If we had the actual reverse DFS postorder on  $G^{\text{SCC}}$ , could just reverse again!
- ▶ But we only have reverse DFS postorder  $S[0..n)$  on  $G$ !
- ⚡ Reversing here would change  $v_i$ , i. e., which vertices of an SCC we see first

# Kosaraju-Sharir's Algorithm

- ▶ **Recall:** Want  $\text{reverse}(\text{topologicalRanking}(G^{\text{SCC}}))$
- ▶ **Transpose/Reverse Graph of  $G = (V, E)$ :**  $G^T = (V, E^T)$  where  $E^T = \{wv : vw \in E\}$   
Note:  $A$  adj matrix of  $G \rightsquigarrow A^T$  adj matrix of  $G^T$
- ▶ For any DAG, we obtain a reverse topological order from reversing all edges:  
 $\text{topologicalSort}(G^T)$   If we reverse iteration order in  $\text{dfsTraversal}$ , we get  $\text{reverse}(\text{topologicalSort}(G)) = \text{topologicalSort}(G^T)$
- ▶ **Observation:**  $(G^T)^{\text{SCC}} = (G^{\text{SCC}})^T$ 
  - ▶ strong components not affected by edge reversals
- ▶ **Want:**  $\text{reverse}(\text{topologicalRanking}(G^{\text{SCC}}))$  (any ranking works, need not be reverse DFS postorder)
- $\rightsquigarrow$  Get it from:  $\text{topologicalRanking}((G^{\text{SCC}})^T) = \text{topologicalRanking}((G^T)^{\text{SCC}})$
- $\rightsquigarrow$  Get that as induced ranking on  $v_1, \dots, v_k$  from  $\text{reverse dfsPostorder}(G^T)$

# Kosaraju-Sharir's Algorithm – Code

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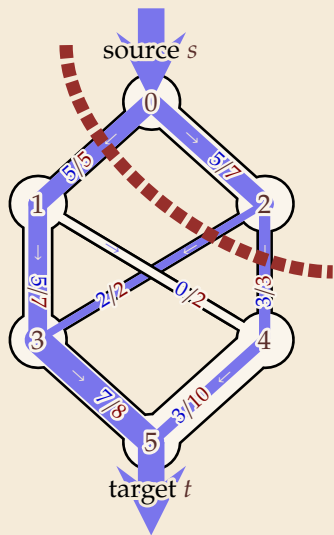
```
1 procedure strongComponents( $G$ ):
2   // directed graph  $G = (V, E)$  with  $V = [0..n)$ 
3    $G^T = (V, \{vw : vw \in E\})$ 
4    $P[0..n) := \text{dfsPostorder}(G^T)$  // postorder numbers
5   for  $v \in V$  do  $S[P[v]] := v$  end for // postorder sequence
6   // Rest like connectedComponents (with permuted vertices)
7    $C[0..n) := \text{unseen}$ 
8    $\text{SCC}[0..n) := \text{NONE}$ 
9    $id := 0$ 
10  for  $j := n - 1, \dots, 0$  // reverse postorder seq
11     $v := S[j]$ 
12    if  $C[v] == \text{unseen}$ 
13       $\text{dfs}(G, v)$ 
14       $id := id + 1$ 
15  return  $\text{SCC}$ 
16
17 procedure preorderVisit( $v$ ):
18    $\text{SCC}[v] := id$ 
```

---

- ▶ correctness follows from our discussion
- ▶ ordering of SCCs follows reverse topological sort of  $G^{\text{SCC}}$ 
  - ▶ some implementations reverse  $G$  for 2nd DFS, not 1st
  - $\rightsquigarrow$  output in (forward) topological order
  - ▶ but derivation more natural this way?
- ▶ as all our traversals:
  - $\Theta(n + m)$  time,
  - $\Theta(n)$  extra space

## 9.6 Network flows

# Networks and Flows – Informal



Informally, imagine a network of water pipes.

- ▶ Water can flow through the pipes up to a flow capacity limit (up to  $c(e)$  liters per second, say).
- ▶ There's infinite water pressing into the source  $s$  and infinite drain capacity at the sink / target  $t$
- ▶ At all other junctions, inflow = outflow (no leakage)

~> How much water can flow through the network?

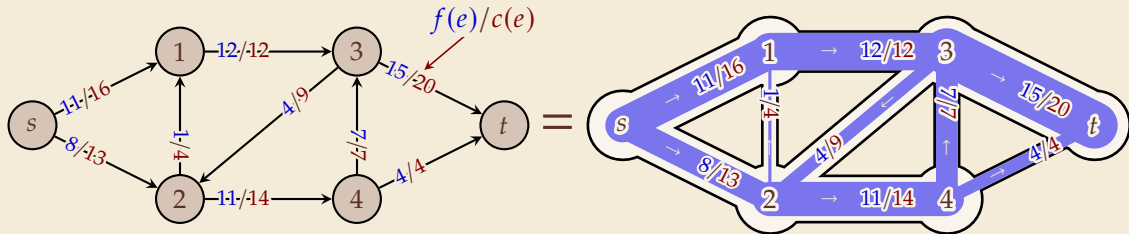
In this example:

- ▶ not more than  $5 + 2 + 3 = 10$  units of flow out of  $\{0, 2\}$  possible
- ~> not more than 10 units out of  $s$  possible
- ~> shown flow is maximal

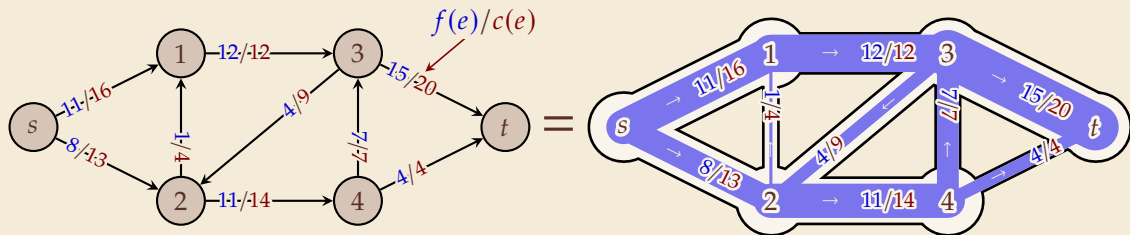
Remainder of this unit: general version of above (+ efficient algorithms)

# Networks and Flows – Definitions

- ▶ *s-t-(flow) network*: for notational convenience only
  - ▶ **simple, directed, connected** graph  $G = (V, E)$ , no antiparallel edges ( $vw \in E \rightsquigarrow wv \notin E$ )
  - ▶ *edge capacities*  $c : E \rightarrow \mathbb{R}_{\geq 0}$
  - ▶ distinguished vertices: *source*  $s \in V$ , target/*sink*  $t \in V$
- ▶ *(network) flow (in  $G$ )*:  $f : E \rightarrow \mathbb{R}_{\geq 0}$
- ▶ flow  $f$  is *feasible* if it satisfies notational convenience: set  $f(vw) = c(vw) = 0$  for  $vw \notin E$ 
  - ▶ *capacity constraints*:  $\forall v, w \in V : 0 \leq f(vw) \leq c(vw)$
  - ▶ *flow conservation*:  $\forall v \in V \setminus \{s, t\} : \sum_{w \in V} f(w, v) = \sum_{w \in V} f(v, w)$
- ▶ *value*  $|f|$  of flow  $f$ :  $|f| = \sum_{v \in V} f(s, v) - \sum_{v \in V} f(v, s)$



# Max-Flow Problem



## ► Maximum-Flow Problem:

► **Given:**  $s$ - $t$ -flow network

► **Goal:** Find feasible flow  $f^*$  with maximum  $|f^*|$  among all feasible flows

## ► $\mathbb{N}$ vs $\mathbb{R}$

► We focus on integral capacities here as we will see can restrict ourselves to integral flows

► but: ideally want algorithms that work with arbitrary real numbers, too



# Multiple Sources & Sinks, Antiparallel Edges

- ▶ Some of the restrictions can be generalized easily.
- ▶ We forbid **loops** and **antiparallel** edges.
  - ▶ The presented algorithms actually work fine with both!
  - ▶ but proofs are cleaner to write without them
  - ▶ also: can always remove loops and (anti)parallel edges by adding a new vertex in the middle of the edge
- ~> same maximum  $|f|$
- ▶ We only allow a **single source** and a **single sink**
  - ▶ can add a “**supersource**” and “**supersink**” with capacity- $\infty$  edges to all sources resp. sinks
- ~> same maximum  $|f|$

# Reductions

- ▶ Apart from directly modeling (data, traffic, etc.) flow, a key reason to study network flows are reductions of other problems

## 1. Disjoint Paths

- ▶ **Given:** Unweighted (di)graph  $G = (V, E)$ , vertices  $s, t \in V$
- ▶ **Goal:** How many edge-disjoint paths are there from  $s$  to  $t$ ?

## 2. Assignment Problem, Maximum Bipartite Matching

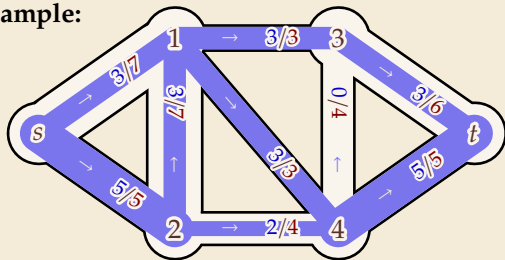
- ▶ **Given:** workers  $W = \{w_1, \dots, w_k\}$  tasks  $T = \{t_1, \dots, t_\ell\}$ , *qualified-for* relation  $Q \subseteq W \times T$
- ▶ **Goal:** Assignment  $a : W \rightarrow T \cup \{\perp\}$  of workers to tasks such that
  - ▶ workers are qualified:  $\forall w \in W : a(w) \neq \perp \implies (w, a(w)) \in Q$
  - ▶  $|a(W)|$ , the number of tasks assigned, is maximized
- ▶ Both problems can be solved by (in both cases, 1. and 3. are very efficient)
  1. constructing a specific flow network from their input data
  2. computing a maximum flow in that network
  3. “reading off” a solution for the original problem from the max flow

## 9.7 The Ford-Fulkerson Method

# Push Push Push!?

- **Simple Idea:** Iteratively find a path from  $s$  to  $t$  that we can push more flow over.

Example:

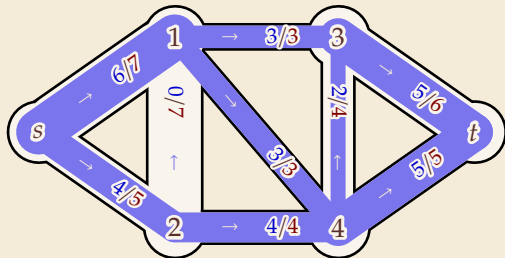


1. Push 3 units of flow over  
 $s \rightarrow 2 \rightarrow 1 \rightarrow 3 \rightarrow t$

2. Push 3 units of flow over  
 $s \rightarrow 1 \rightarrow 4 \rightarrow t$

3. Push 2 units of flow over  
 $s \rightarrow 2 \rightarrow 4 \rightarrow t$

⇒ Every  $s$ - $t$  path now has a saturated edge.



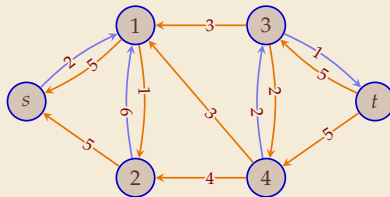
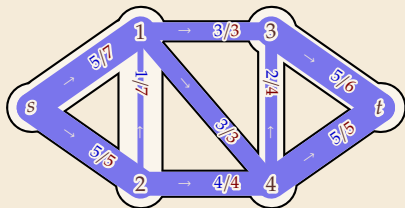
⚠ **But:** resulting flow is **not** optimal!

**Problem:** Cannot undo mistakes.  
Here: shouldn't have put so much flow on  $(1, 2) \dots$

# Residual Networks

- Goal: Allow undoing flow (without backtracking)
- *Residual network*  $G_f$ : given network  $G = (V, E)$  and feasible flow  $f$

►  $G_f = (V, E_f)$  with capacities  $c_f(vw) = \begin{cases} c(vw) - f(vw) & vw \in E \quad // \text{add flow} \\ f(wv) & wv \in E \quad // \text{revert flow} \\ 0 & \text{else} \end{cases}$   
 $E_f = \{vw : c_f(vw) > 0\}$



- *residual flow*  $f'$ : feasible flow in  $G_f$   
 $\rightsquigarrow$  for any  $f$  and residual flow  $f'$  in  $G_f$ , flow  $f + f'$  is a feasible flow in  $G$
- *augmenting path*  $p$ :  $s$ - $t$ -path  $G_f$  particularly simple  $f'$ !

$$(f + f')(vw) = f(vw) + f'(vw) - f'(wv)$$

# Cuts

- **Goal:** Certificate for maximum flows
- **$s$ - $t$ -cut**  $(S, T)$ : partition  $S \dot{\cup} T = V$ ,  $s \in S$ ,  $t \in T$

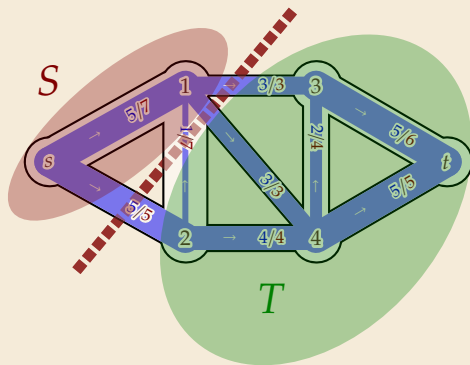
- **net flow** across cut:

$$f(S, T) = \sum_{v \in S} \sum_{w \in T} (f(vw) - f(wv))$$

- **capacity** of cut:

$$c(S, T) = \sum_{v \in S} \sum_{w \in T} f(vw)$$

- **Lemma:** For any cut  $(S, T)$ , we have  $f(S, T) = |f|$ .  
(flow conservation!)
- **Corollary:**  $|f| \leq c(S, T)$  for any  $s$ - $t$ -cut  $(S, T)$



- $f(S, T) = 5 + 3 + 3 - 1 = 10$
- $c(S, T) = 5 + 3 + 3 = 11$

# The Max-Flow Min-Cut Theorem

## ► Max-Flow Min-Cut Theorem:

Let  $f$  be a feasible flow in  $s$ - $t$ -network  $G = (V, E)$ . Then the following conditions are equivalent:

1.  $|f| = c(S, T)$  for some cut  $(S, T)$  of  $G$ .
2.  $f$  is a maximum flow in  $G$
3. The residual network  $G_f$  has no augmenting path.

# Generic Ford-Fulkerson Method

---

```
1 procedure genericFordFulkerson( $G = (V, E), s, t, c$ ):  
2   //  $G$  is a flow network with source  $s \in V$ , sink  $t \in V$  and capacities  $c : E \rightarrow \mathbb{R}_{\geq 0}$   
3   for  $vw \in E$  do  $f(vw) := 0$  end for  
4   while  $\exists$  path  $p$  from  $s$  to  $t$  in  $G_f$  // Freedom: Which augmenting path?  
5      $\Delta := \min\{c_f(e) : e \in p\}$  // bottleneck capacity  
6     for  $e \in p$   
7       if  $e \in E$  // forward edge  
8          $f(e) := f(e) + \Delta$   
9       else // backward edge  
10         $f(e) := f(e) - \Delta$   
11   return  $f$ 
```

---

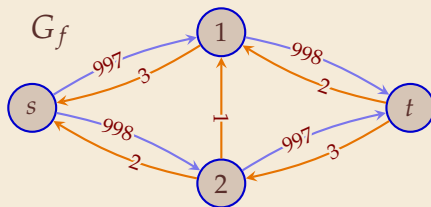
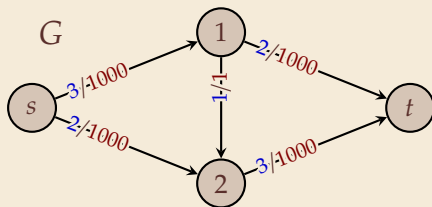
- ▶ Returned flow is a maximum flow  $f^*$  (Max-Flow Min-Cut Theorem)
- ▶ If  $c : E \rightarrow \mathbb{N}_0$ , also  $f : E \rightarrow \mathbb{N}_0$ : For all  $v, w \in V$  holds:
  - ▶ initially  $f(vw) = 0 \in \mathbb{N}_0$
  - ▶  $c_f(vw)$  is difference of  $c(vw) \in \mathbb{N}_0$  and  $f(vw) \in \mathbb{N}_0$
  - ▶  $\Delta$  equal to some  $c_f(v'w') \in \mathbb{N}_{\geq 1}$  ( $E_f$  contains only non-zero capacity edges!)
- $\rightsquigarrow$  new flow  $f(vw) \pm \Delta \in \mathbb{N}_0$

$\rightsquigarrow$  For integral capacities, always terminate after  $\leq |f^*|$  iterations



## Bad Example

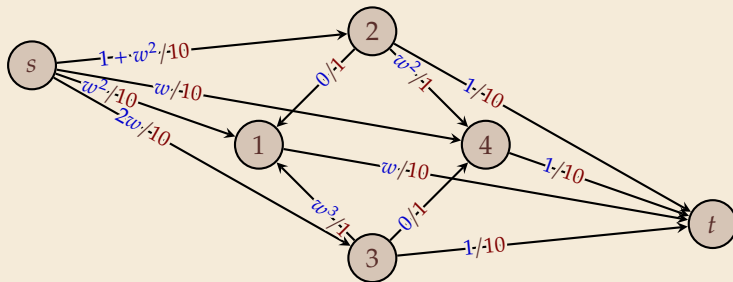
- Unfortunately, we might also take  $|f^*|$  iterations!



- (2 iterations with smarter augmenting paths would have sufficed here)

## A Very Bad Example

- ▶ for irrational flows, might not even terminate
- ▶ example network with irrational initial flow
- ▶  $w = \varphi - 1 = (\sqrt{5} - 1)/2 \approx 0.618 \rightsquigarrow 1 - w = w^2 \approx 0.382$



- ▶ after 2 paths, situation in 1-2-3-4 restored (rotated), but flows multiplied by  $w$

$\rightsquigarrow$  augmenting paths have capacities  $w, w, w^2, w^2, w^3, w^3 \dots$

$\rightsquigarrow$  never terminate, never exceed  $|f| \geq 5$

## 9.8 The Edmonds-Karp Algorithm

# Edmonds-Karp

- ▶ It turns out, many ways to choose augmenting paths systematically work fine
- ▶ Edmonds & Karp: take a shortest path (in #edges)

---

```
1 procedure EdmondsKarp( $G = (V, E), s, t, c$ ):
2   //  $G$  is a flow network with source  $s \in V$ , sink  $t \in V$  and capacities  $c : E \rightarrow \mathbb{R}_{\geq 0}$ 
3   for  $vw \in E$  do  $f(vw) := 0$  end for
4   while true
5     bfs( $G_f, \{s\}$ )
6     if  $\text{distFrom}[t] == \infty$  return  $f$ 
7     else  $p := \text{pathTo}(t)$ 
8      $\Delta := \min\{c_f(e) : e \in p\}$  // bottleneck capacity
9     for  $e \in p$ 
10      if  $e \in E$  // forward edge
11         $f(e) := f(e) + \Delta$ 
12      else // backward edge
13         $f(e) := f(e) - \Delta$ 
14    end while
```

---

# Edmonds-Karp – Analysis

- ▶ **Theorem:** The Edmonds-Karp algorithm terminates after  $O(nm)$  iterations with a maximum flow. The total running time is in  $O(nm^2)$ .
- ▶ *Proof Plan:*
  - ▶ every augmenting path has a *critical* edge  $vw$  contributing the bottleneck capacity
  - ▶ we will show:
    - (1) distances of vertices from  $s$  in  $G_f$  weakly increase over time
    - (2) before  $vw$  can be a *critical* edge *again*,  $v$ 's distance increases by at least 2
  - ↪ each edge  $vw$  is critical for at most  $n/2$  augmenting paths ( $v$ 's distance  $\in [1..n-2]$ )
  - ↪  $O(nm)$  augmenting paths
  - ▶ each iteration runs one BFS, which costs  $O(n+m) = O(m)$  times since  $G$  is connected.
- ▶ **Notation:**
  - ▶ Write  $f_0, f_1, \dots$  for values of  $f$  during iterations of while loop
  - ↪  $G_{f_i}$  residual network after  $i$ th augmentation
  - ▶ Write  $\delta_i(v)$  for shortest-path distance from  $s$  to  $v$  in  $G_{f_i}$

# Edmonds-Karp – Analysis [2]

- ▶ **EK Monotonicity Lemma:** For all  $i$  and  $v \in V$ , we have  $\delta_{i+1}(v) \geq \delta_i(v)$ .

*Proof:*

▶  $f_i$ : flow after  $i$ th augmentation

▶  $\delta_i(v)$  distance from  $s$  to  $v$  in  $G_{f_i}$

- ▶ by induction over  $k$ , the value of  $\delta_i(v)$
- ▶ IB:  $k = 0$ : only  $v = s$  possible;  $\delta_{i+1}(s) = 0 \geq 0 = \delta_i(s) \checkmark$
- ▶ IH: Assume the claim is true for all shortest paths up to length  $k$
- ▶ IS: Suppose  $\delta_{i+1}(v) = k + 1$ .

$\rightsquigarrow \exists$  shortest path  $p[0..k+1]$  in  $G_{f_{i+1}}$  with  $p[0] = s$  and  $p[k+1] = v$ .

$\rightsquigarrow$  For  $w = p[k]$ ,  $p[0..k]$  is a shortest path from  $s$  to  $w$   $\rightsquigarrow k = \delta_{i+1}(w) \underset{\text{IH}}{\geq} \delta_i(w)$

▶ Case 1:  $vw \in E_{f_i}$   $\rightsquigarrow \delta_i(v) \leq \delta_i(w) + 1$

▶ Case 2:  $vw \notin E_{f_i}$   $\rightsquigarrow$  reverse edge  $vw$  in  $i$ th augmenting path, a shortest  $s$ - $t$ -path  
 $\rightsquigarrow \delta_i(v) = \delta_i(w) - 1 \leq \delta_i(w) + 1$

▶ in both cases:  $\delta_{i+1}(v) = \delta_{i+1}(w) + 1 \geq \delta_i(w) + 1 \geq \delta_i(v)$

## Edmonds-Karp – Analysis [3]

- ▶ **Critical Distance Lemma:** When critical edge  $vw$  becomes a critical again,  $\delta(v)$  has increase by at least 2.

*Proof:*

- ▶ Suppose  $vw$  is critical in  $i$ th iteration  $\rightsquigarrow$  lies on shortest path

$$\rightsquigarrow \delta_i(w) = \delta_i(v) + 1$$

- ▶ before  $vw$  reappears in  $G_f$ , need to have had  $wv$  in augmenting path;  
say this first happens in iteration  $j > i \rightsquigarrow \delta_j(v) = \delta_j(w) + 1$

- ▶ by EK Monotonicity Lemma:

$$\delta_j(v) = \delta_j(w) + 1 \geq \delta_i(w) + 1 = \delta_i(v) + 2$$

This concludes the proof of the theorem.

# Maximum Flow – Discussion



Edmonds-Karp is a robust choice



easy to implement (see Sedgewick Wayne for an elegant Java version!)



worst-case time  $O(n^5)$  for dense graphs quickly prohibitive

- ▶ but: worst-case results typically overly pessimistic
- ▶ other choices of augmenting flows possible
- ▶ in practice: push-relabel methods often faster

- ▶ 2022 theory breakthrough: almost linear(!)  $O(m^{1+o(1)})$  time max flow algorithm  
Chen, Kyng, Liu, Peng, Gutenberg & Sachdeva, FOCS 2022
- ▶ max-flow min-cut theorem is a special case of LP duality
- ▶ can also solve generalization of min-cost flows
  - ▶ each edge  $vw$  has a cost  $a(vw)$
  - ▶ cost of a flow  $f$ :  $\sum a(vw) \cdot f(vw)$
  - ▶ demand  $d$  at sink becomes part of constraints:  $|f| \geq d$