11 Greedy Algorithms

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CS566 (Wintersemester 2024/25)
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Learning Outcomes

Unit 11: ***Greedy Algorithms***

1. Describe informally what greedy algorithms are.
2. Know exemplary problems and their greedy solutions: Change-Making Problem, MSTs, SSSPP, Assignment Problem.
3. Be able to design and proof correctness of greedy algorithms for (simple) algorithmic problems.
4. Be able to explain the matroid properties and its relation to greedy algorithms.

Outline

11 Greedy Algorithms

11.1 Introduction

Myopic Optimization

- In a ***"greedy" algorithm***,
  we assemble a solution to an **optimization** problem **step by step**
  always picking the next step to maximize **current** gain,
  and we **never take back** earlier steps.

"Take what you can, give nothing back!"

Myopic Optimization

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  and we **never take back** earlier steps.

"Take what you can, give nothing back!"

- reminiscent of *gradient-descent* algorithms
  but discrete and even more unwilling to undo mistakes

⇝ greedy algorithms only yield optimal solutions for certain problems

  - but where they do, their speed is usually unbeatable

  ⇝ it is understanding where they succeed

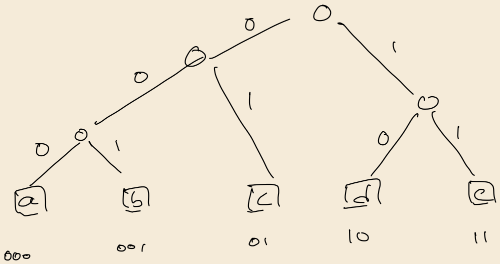
- even where they are not optimal, greedy approaches can be efficient heuristics (unknown quality) or approximation algorithms (c-approximation = at most factor c worse than optimum)

Plan for the Unit

- We will first see a few examples (known and new) of greedy algorithms to make the vague generic description concrete
  - in particular minimum spanning trees and shortest paths in graphs
- Unlike other algorithm design techniques, greedy algorithms have a formal basis: *matroids* (and *greedoids*)
  - The second part will introduce these and how they can unify correctness proofs

A First Example: Reunion With An Old Friend

- We have seen an example of a Greedy Algorithm in Unit 7: *Huffman Codes!*
- Recall the problem:
  - **Given:** Set of symbols $\Sigma = [0..\sigma)$, weights $w : \Sigma \to \mathbb{R}_{\geq 0}$
  - **Goal:** prefix code E (= code trie) that minimizes $\sum_{c \in \Sigma} w(c) \cdot |E(c)|$



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$\rightsquigarrow$ Since only *code tries* are valid, all solutions consist in repeatedly merging tries (starting from single-leaf tries, until single trie left)

- each merge contributes the subtree's total weight to overall cost (since all leaves in merged tries move one level down / all codewords get one extra bit)
- **Huffman's Algorithm:** Always choose current cheapest merge.

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- each merge contributes the subtree's total weight to overall cost (since all leaves in merged tries move one level down / all codewords get one extra bit)
- **Huffman's Algorithm:** Always choose current cheapest merge. w.r.t
- In the correctness proof, we had to show: There is always an optimal code trie where the two lowest-weight symbols are siblings.

This is typical: To show that Greedy is optimal, we need a structural insight into optimal solutions.

11.2 How Can Greed Succeed?

Greed For Change

The Change-Making Problem (a. k. a. Coin-Exchange Problem)

- **Given:** a set of integer denominations of coins $w_1 < w_2 < \cdots < w_k$ with $w_1 = 1$, target value $n \in \mathbb{N}_{\geq 1}$

  (we have sufficient supply of all coins . . .)

- **Goal:** "fewest coins to give change n", i. e., multiplicities $c_1, \ldots, c_k \in \mathbb{N}_{\geq 0}$ with $\sum_{i=1}^{k} c_i \cdot w_i = n$ minimizing $\sum_{i=1}^{k} c_i$

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For Euro coins, denominations are (1¢), (2¢), (5¢), (10¢), (20¢), (50¢), (1€), and (2€).

formally: $\underset{w_1}{1}$, $\underset{w_2}{2}$, $\underset{w_3}{5}$, $\underset{w_4}{10}$, $\underset{w_5}{20}$, $\underset{w_6}{50}$, $\underset{w_7}{100}$, and $\underset{w_8}{200}$.

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formally:	1 ,	2 ,	5 ,	10 ,	20 ,	50 ,	100 , and	200 .
	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8

⇝ Simple greedy algorithm: largest coins first

- optimal time ($O(k)$ if coins sorted)
- is $\sum c_i$ minimal?

```
procedure greedyChange(w[1..k], n):
    // Assumes 1 = w[1] < w[2] < ··· < w[k]
    for i := k, k − 1, . . . , 1:
        c[i] := ⌊n / w[i]⌋
        n := n − c[i] · w[i]
    // Now n == 0
    return c[1..k]
```

Clicker Question

Does greedyChange give the optimal answer for the Euro coins change-making problem?

- (A) Always
- (B) Sometimes
- (C) Never

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Optimality of Greedy Euro-Change

- **Theorem:** greedyChange computes an optimal $c[1..8]$ for $w[1..8] = [1, 2, 5, 10, 20, 50, 100, 200]$ for every $n \in N_{\geq 1}$.

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  - The greedy algorithm can be interpreted as picking one coin at a time, each time choosing the largest possible denomination $\hat{w}(n) = \max\{w[i] : w[i] \leq n\}$.
  - We prove by induction over n: Any optimal solution for n must contain $\hat{w}(n)$.
    - $n = 1$: can only use $\hat{w}(n) = 1$ ✓

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    - $n \in [2..5)$: Assume we had a solution without 2¢ $\rightsquigarrow$ must be $n \times$ 1¢ with $n \geq 2$; $\rightsquigarrow$ we can make this strictly better by replacing 1¢ 1¢ by 2¢ ↯

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  - We prove by induction over n: Any optimal solution for n must contain (ŵ(n)).
    - $n = 1$: can only use $\hat{w}(n) = 1$ ✓
    - $n \in [2..5)$: Assume we had a solution without (2¢) ⇝ must be $n \times$ (1¢) with $n \geq 2$;
      ⇝ we can make this strictly better by replacing (1¢)(1¢) by (2¢) ϟ
    - $n \in [5..10)$: Assume solution without (5¢) summing to $n \geq 5$.
      The solution must fall into one of the following cases:
      (a) $\geq 3 \times$ (2¢) ⇝ replacing (2¢)(2¢)(2¢) by (5¢)(1¢) strictly better ϟ
      (b) $\leq 1 \times$ (2¢) ⇝ value $n - 2 \geq 3$ without (2¢) ϟ by IH
      (c) $2 \times$ (2¢) and $\geq 1 \times$ (1¢) ⇝ (2¢)(2¢)(1¢) $\to$ (5¢) strictly better ϟ
      (d) $2 \times$ (2¢), no (1¢) ⇝ only obtain value $\leq 4 < n$ ϟ

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    - $n \in [5..10)$: Assume solution without (5¢) summing to $n \geq 5$.
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      (b) $\leq 1 \times$ (2¢) $\rightsquigarrow$ value $n - 2 \geq 3$ without (2¢) ↯ by IH
      (c) $2 \times$ (2¢) and $\geq 1 \times$ (1¢) $\rightsquigarrow$ (2¢)(2¢)(1¢) $\rightarrow$ (5¢) strictly better ↯
      (d) $2 \times$ (2¢), no (1¢) $\rightsquigarrow$ only obtain value $\leq 4 < n$ ↯
    - $n \in [10, 20)$: Any solution without (10¢) contains
      (a) (5¢)(5¢) $\rightsquigarrow$ replace by (10¢); or
      (b) at most one (5¢) $\rightsquigarrow$ at least value 5 without (5¢) ↯ by IH

Optimality of Greedy Euro-Change [2]

- … proof continued
  - $n \in [20..50)$ Without (20¢), we must have
    (a) (10¢)(10¢) $\rightarrow$ (20¢) ϟ
    (b) at most one (10¢) $\leadsto$ value $n - 10 \geq 10$ without (10¢) ϟ by IH

Optimality of Greedy Euro-Change [2]

- ... proof continued
  - $n \in [20..50)$ Without (20¢), we must have
    (a) (10¢)(10¢) $\longrightarrow$ (20¢) ↯
    (b) at most one (10¢) $\rightsquigarrow$ value $n - 10 \geq 10$ without (10¢) ↯ by IH
  - $n \in [50..100)$ Without (50¢), we must have
    (a) $\geq 3 \times$ (20¢) $\rightsquigarrow$ (20¢)(20¢)(20¢) $\longrightarrow$ (50¢)(10¢) ↯
    (b) $\leq 1 \times$ (20¢) $\rightsquigarrow$ value $n - 20 \geq 30$ without (20¢) ↯ by IH
    (c) $2 \times$ (20¢) and $\geq 1 \times$ (10¢) $\rightsquigarrow$ (20¢)(20¢)(10¢) $\longrightarrow$ (50¢) ↯
    (d) $2 \times$ (20¢), no (10¢) $\rightsquigarrow$ value $n - 40 \geq 10$ without (10¢) ↯ by IH

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    (d) $2 \times$ (20¢), no (10¢) $\rightsquigarrow$ value $n - 40 \geq 10$ without (10¢) ϟ by IH
  - $n \in [100..200)$: as for $n \in [10, 20)$, *mutatis mutandis*.
  - $n \geq 200$: as for $n \in [20, 50)$. $\square$

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  - $n \in [20..50)$ Without (20¢), we must have
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  - $n \in [50..100)$ Without (50¢), we must have
    (a) $\geq 3 \times$ (20¢) $\rightsquigarrow$ (20¢)(20¢)(20¢) $\to$ (50¢)(10¢) ↯
    (b) $\leq 1 \times$ (20¢) $\rightsquigarrow$ value $n - 20 \geq 30$ without (20¢) ↯ by IH
    (c) $2 \times$ (20¢) and $\geq 1 \times$ (10¢) $\rightsquigarrow$ (20¢)(20¢)(10¢) $\to$ (50¢) ↯
    (d) $2 \times$ (20¢), no (10¢) $\rightsquigarrow$ value $n - 40 \geq 10$ without (10¢) ↯ by IH
  - $n \in [100..200)$: as for $n \in [10, 20)$, *mutatis mutandis*.
  - $n \geq 200$: as for $n \in [20, 50)$.
- The same arguments work for adding coins $1 \cdot 10^m, 2 \cdot 10^m, 5 \cdot 10^m$ for $m = 3, 4, \ldots$

Optimality of Greedy Euro-Change [2]

- … proof continued
  - $n \in [20..50)$ Without (20¢), we must have
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  - $n \in [50..100)$ Without (50¢), we must have
    (a) $\geq 3 \times$ (20¢) $\rightsquigarrow$ (20¢)(20¢)(20¢) $\longrightarrow$ (50¢)(10¢) ϟ
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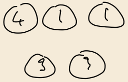
- The same arguments work for adding coins $1 \cdot 10^m, 2 \cdot 10^m, 5 \cdot 10^m$ for $m = 3, 4, \ldots$

That went smoothly!
And we proved a nice structural statement about how optimal solutions look like as a bonus.

Maybe Greedy always works?

Greed Can Mislead

- *Unfortunately, No.* See $w = (1, 3, 4)$ and $n = 6$.



Greed Can Mislead

- *Unfortunately, No.* See $w = (1, 3, 4)$ and $n = 6$.
  or $w = (1, 4, 9)$ and $n = 12$

Where/Why does our proof from above fail?

Greed Can Mislead

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  or $w = (1, 4, 9)$ and $n = 12$
- Indeed, Greedy can be **arbitrarily bad** compared to the optimal solution:
  See $w = (1, 999, 1000)$ and $n = 1998$.

$\rightsquigarrow$ Need to be careful about the details of a correctness argument for greedy algorithms.

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- Indeed, Greedy can be **arbitrarily bad** compared to the optimal solution:
  See $w = (1, 999, 1000)$ and $n = 1998$.

$\rightsquigarrow$ Need to be careful about the details of a correctness argument for greedy algorithms.

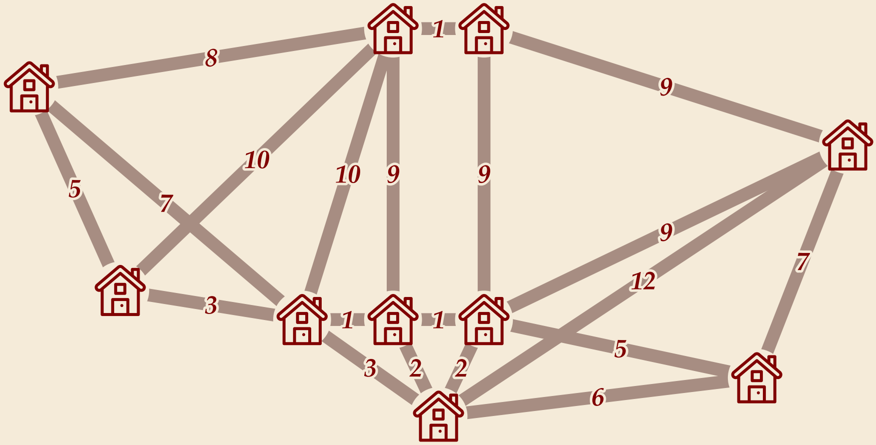
- The Change-Making problem is still only partially understood.
  - Exactly characterizing the denomination sequences that are optimally handled by greedyChange is an **open research problem**.
    - Sufficient criteria for "greed-compatible" denominations found in the literature.
  - The general problem is (weakly) NP-hard
  - Yet, for moderate n, we will see a solution for general denomination sequences later!

11.3 Greed in Graphs I: MSTs

Metaphor: Planning an electricity grid

Given: Houses to be connected to central power grid
Possible connections with building costs given

Goal: Cheapest way to get every house connected



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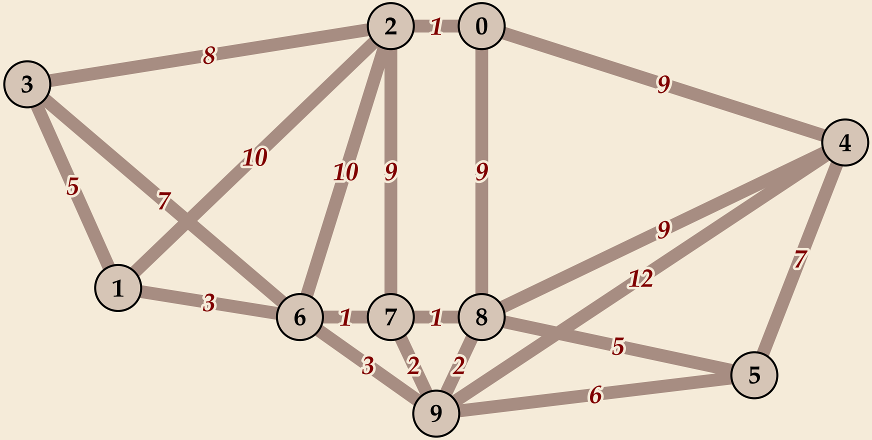

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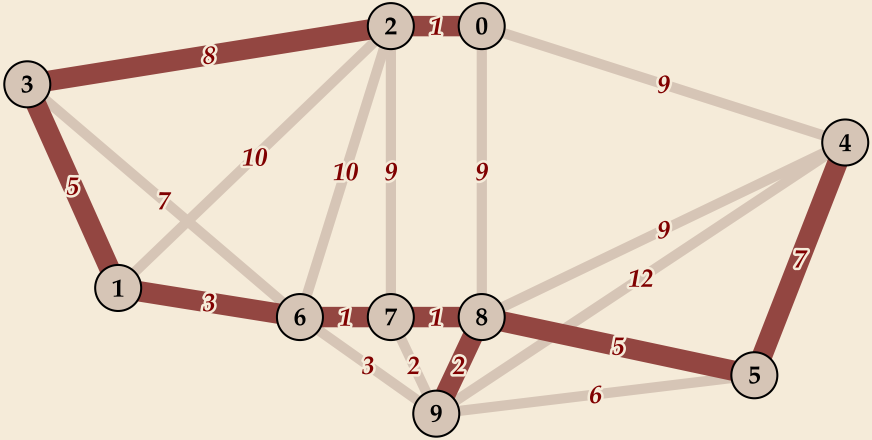

Clicker Question

Which algorithm allows to efficiently test whether a given (undirected) graph is connected?

- A bubble sort
- B depth-first search
- C breadth-first search
- D generic tricolor search
- E Kosaraju-Sharir's algorithm
- F Dijkstra's algorithm
- G Edmonds-Karp algorithm

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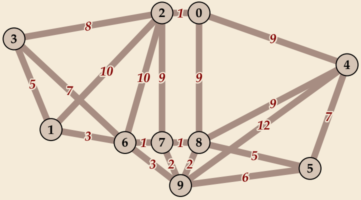
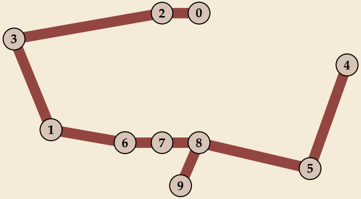
The Minimum Spanning Tree (MST) Problem

Given: **un**directed, edge-**weighted**, simple, **connected** graph $G = (V, E, c)$ ↖ no self loops, no parallel edges



Formally: Recall assumption $V = [0..n)$ ($\rightsquigarrow$ array indices)
edges $E \subseteq \big\{\{u, v\} : u, v \in V \wedge u \neq v\big\}$
edge weights (costs) $c : E \to \mathbb{R}_{\geq 0}$
for all $u, v \in V$ there exists a path $u \rightsquigarrow v$ in (V, E)

Goal: a **spanning tree** (V, T)
with **minimal** total cost $c(T) := \sum_{e \in T} c(e)$



Formally: $T \subseteq E$
(V, T) is connected and acyclic (“spanning tree”)
for every spanning tree (V, T') of G we have $c(T') \geq c(T)$.

Further MST Applications

Direct Applictions

- single-linkage hierarchical clustering
- Bottleneck-shortest paths
- Approximation algorithms, e. g.,
  - Christofides's Metric TSP Approximation
  - Steiner-tree problem

As a cheap subroutine

- Routing protocols
- medical image processing
- ...

Interlude: On Varieties of Trees

We freely use "tree" to mean different things in different contexts . . . mind the confusion.

- here: "tree" = *undirected, nonplane tree* = an undirected, connected and acyclic graph

in spanning *tree*

no order on edges

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  in spanning *tree* (↖ here) no order on edges (↖ nonplane)

The digraph flavor is a rooted tree: (hence undirected trees sometimes called *unrooted*)

- *rooted (nonplane/unordered) tree* = **digraph** (V, E) with *root* $r \in V$ s.t.
  $\forall v \in V \setminus \{r\} : d_{\text{out}}(v) = 1$ and $d_{\text{out}}(r) = 0$

  out-degree = #outgoing edges (↖ d_{out})

We draw trees with the single(!) root on top . . .

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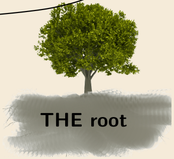

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ordered rooted

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  ↑ in spanning *tree* ↑ no order on edges

  worst-case Θ-class

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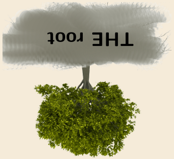

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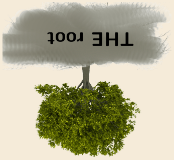
  $\forall v \in V \setminus \{r\} : d_{\text{out}}(v) = 1$ and $d_{\text{out}}(r) = 0$

out-degree = #outgoing edges



We draw trees with the single(!) root on top . . .

Other "trees" don't originate from graphs naturally, but rather from recursion / terms:

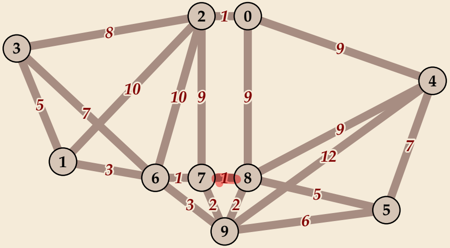
- *binary tree* = a null pointer or a node with left and right children, each a binary tree
  (formally: the set of binary trees is the smallest fixed point of that construction)
- *ordinal trees* = a node with a sequence of 0 or more children, each ordinal trees
  = rooted ordered trees (rooted unordered + total order on children)
- plus many more variants out there . . . $\rightsquigarrow$ if in doubt, double check definitions!

A Naive Approach

How to start finding an MST?

Using the **cheapest** edge shouldn’t hurt . . .

```
procedure greedyMST(V, E, c):
    // Assume (V, E) is simple & connected, c : E → ℝ≥0
    T := ∅
    while (V, T) not connected
        e := cheapest edge that doesn't close a cycle in T
        T := T ∪ {e}
    return T
```



A Naive Approach Works – Kruskal's Algorithm

How to start finding an MST?

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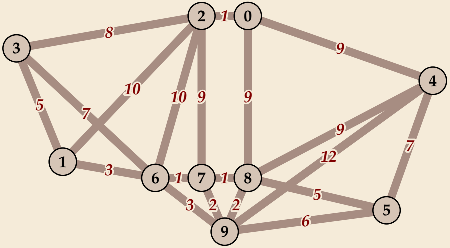
```
1 procedure kruskalMST(V, E, c):
2     // Assume (V, E) is simple & connected, c : E → ℝ≥0
3     T := ∅
4     while (V, T) not connected
5         e := cheapest edge that doesn't close a cycle in T
6         T := T ∪ {e}
7     return T
```



Apart from implementing line 4 and line 5 efficiently, this *is* **Kruskal's Algorithm**!

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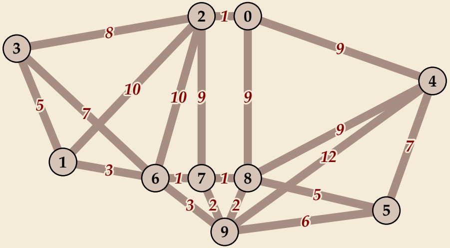


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Apart from implementing line 4 and line 5 efficiently, this *is* **Kruskal's Algorithm**!

As so often with greedy algorithms, the hardest bit is the correctness argument. We have:

Theorem: Kruskal's Algorithm finds a minimum spanning tree.
This immediately follows from proving the following invariant:

Kruskal's Invariant: There is some MST T^* with $T \subseteq T^*$.

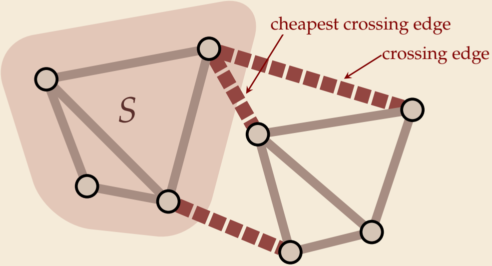
henceforth: identify MST with its edge set

Crossing Edges and the MST-Cut Lemma

To prove the correctness of Kruskal's Algorithm, we need a tool.

Notation:

- **Cut** S:
  non-trivial set of vertices $\emptyset \neq S \subsetneq V$
- **crossing edge** e wrt. cut S:
  $e = \{u, v\}$ with $u \in S, v \in \bar{S} := V \setminus S$

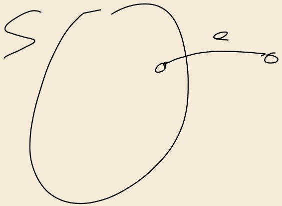


The MST-Cut Lemma:
Let T^* be an MST und $W \subseteq T^*$.
For every cut S, not cutting any edges in W, and
every *cheapest* crossing edge e wrt. S
there is an MST $\hat{T}^*$ that contains $W \cup \{e\}$.

Proof of MST-Cut Lemma

Proof:

- Case 1: $e \in T^*$.
  Then picking $\hat{T}^* = T^*$ proves the claim.



Proof of MST-Cut Lemma

Proof:

- Case 1: $e \in T^*$.
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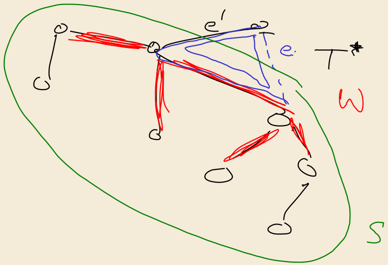
- Case 2: $e \notin T^*$.
  - $\rightsquigarrow$ $T^* \cup \{e\}$ contains unique cycle C using e.
  - Since e crosses cut S, C crosses S
  - $\rightsquigarrow$ There is a second crossing edge $e' \in C$.



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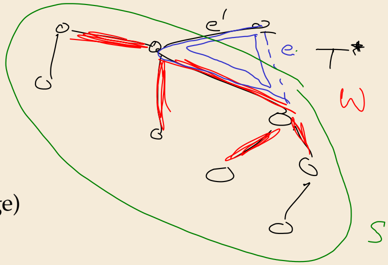
  - Since e crosses cut S, C crosses S
  - $\rightsquigarrow$ There is a second crossing edge $e' \in C$.
  - Since e' is crossing, $e' \notin W$
  - by assumption, $c(e) \le c(e')$ (we pick cheapest crossing edge)
  - $\rightsquigarrow$ $\hat{T}^* = T^* \cup \{e\} \setminus \{e'\}$ is a spanning tree, and $W \cup \{e\} \subseteq \hat{T}^*$
  - $c(\hat{T}^*) = c(T^*) + c(e) - c(e') \le c(T^*)$
  - $\rightsquigarrow$ $\hat{T}^*$ is an **MST**.



Kruskal's Algorithm – Correctness

With these preparations, we can prove

Kruskal's Invariant: There is some MST T^* with $T \subseteq T^*$.

Proof: by induction over the loop iterations

- IB: initially $T = \emptyset$ and $\emptyset \subseteq T^*$ for every MST T^*.
- IH: Assume the invariant is after the ith iteration.

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  - Case 1: $w \in S$.
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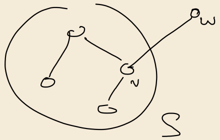

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    Then e is a crossing edge wrt. S; must be a cheapest crossing edge by choice of e.
    $\rightsquigarrow$ by inv. $\exists$ MST $T^* \supseteq T$ and by MST-Cut Lemma, there is an MST $\hat{T}^* \supseteq T \cup \{e\}$
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there is an MST $\hat{T}^*$ that contains $W \cup \{e\}$.

$W = T$

S

Since we only terminate when T is spanning, upon termination $T = T^*$ for an MST T^*.

Kruskal's Algorithm – Data Structures

For an efficient implementation of Kruskal's algorithm, we need to efficiently

1. check whether T is spanning
2. find the next cheapest edge to consider
3. test whether an edge closes a cycle

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Each can be supported as follows:

1. Since we maintain T acyclic, checking $|T| = n - 1$ suffices!
2. It suffices to pre-sort E by weight!
   - We only ever grow T, so if e is closing a cycle now, it will for good.

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3. Use a **Union-Find data structure** (see Algorithmen & Datenstrukturen!)
   - dynamically maintain connected components
   - initially, every vertex has its own id
   - adding vw to T $\rightsquigarrow$ call `union(`v, w`)`
   - vw closes a cycle *iff* `find(`v`) == find(`w`)`

$\notin$ exam

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$\rightsquigarrow$ $O(m \log m) = O(m \log n)$ time and $O(m)$ extra space.

Clicker Question

What is the running time of Prim's algorithm?

A $\Theta(\log(n+m))$

B $\Theta(n\sqrt{m})$

C $\Theta(n+m)$

D $\Theta(n^2+m)$

E $\Theta(m+n\log n)$

F $\Theta(n+m\log n)$

G $\Theta(m\log n)$

H $\Theta(m\log m)$

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→ sli.do/cs566

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- F ~~$\Theta(n + m\log n)$~~
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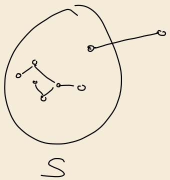
- I ~~$\Theta(n\log(n+m))$~~
- J ~~$\Theta(m^2)$~~

→ sli.do/cs566

11.4 Greed in Graphs II: Prim's MST Algorithm

Prim's Algorithm

- An alternative greedy approach that tries to consider only crossing edges.
  - start with $S = \{s\}$ for some vertex s
  - only consider edges vw for some $v \in S$, $w \notin S$ (crossing edges)
  - add cheapest crossing edge vw to T and add w to S
  - repeat until $|T| = n - 1$
  - $\rightsquigarrow$ T invariably a single tree



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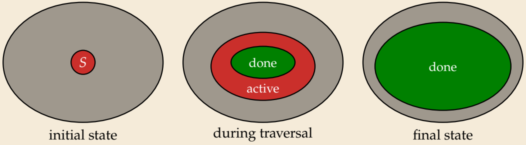
⇝ a graph traversal with tree edges T!



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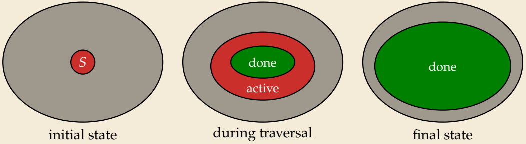

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⇝ Correctness as for Kruskal's algorithm: **Invariant:** $\exists$ MST T^* with $T \subseteq T^*$.

IB: initially true with $T = \emptyset$

IS: whenever we add an edge, it is the cheapest crossing edge w.r.t. cut $(S, \bar{S})$.

Prim's Algorithm – Lazy Implementation

How to efficiently find the cheapest crossing edge?

- **Option 1:** Maintain priority queue Q of **edges**, ordered by weight.

```
procedure lazyPrimMST(G):
    // Assume G = (V, E, c) simple & connected, c : E → ℝ≥0
    T := ∅;  inS[0..n) := false
    Q := new MinPQ()
    visit(0)
    while |T| < n − 1:
        vw := Q.delMin()
        if ¬inS[w] then visit(w); T.insert(vw) end if
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procedure visit(v):
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- Lazy Prim: check if vw is crossing *lazily*
  i. e., only after `delMin`

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- An instance of tricolor graph traversal
  - $v \in$ ***done*** iff $inS[v]$
  - all edges *to* ***active*** vertices are in Q
  - ⇝ visit every edge at most once
- size of Q always $\leq m$ ⇝ **space** $O(m)$

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- An instance of tricolor graph traversal
  ↳ $O(n+m)$ contribution
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- **Running time:**
  $(n \le m+1)$
  - need m calls to `insert` and $n-1$ `delMin`s
  - ⇝ with binary heaps, total time $O(m \log m) = O(m \log n)$
  $m \le n^2$

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Easy modification: store parent in tree rooted at vertex 0

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  - with Fibonacci heaps even $O(m + n \log n)$ (insert amortized $O(1)$ time)

Prim's Algorithm – Eager Implementation

We can reduce the extra space to $O(n)$ if we avoid storing multiple edges to the same $w \in \bar{S}$.

- **Option 2:** Maintain priority queue Q of **vertices** in $\bar{S}$, ordered by **weight of cheapest edge** connecting them to S.
  - call that weight the *distance*, $dist[w]$, of $w \in \bar{S}$ from S.
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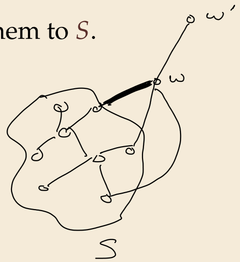

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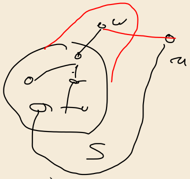

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    - implementation hassle: efficient implementations require a "pointer" into data structure
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⇝ **IndexMinPQ**: (use ST otherwise) (use amortized doubling otherwise)

- **Assumption:** stored objects are from $[0..n)$ **and** n known/fixed at construction time
- IndexMinPQ implementations maintain array positions
  e. g., for binary heaps, maintain $heapIndex[0..n)$, update whenever heap modified

⇝ easy to support `decreaseKey`(i, p') and `contains`(i)
(for a full implementation see *Sedgewick & Wayne* or *Nebel & Wild*)

Prim's Algorithm – Eager Implementation Code

```
procedure primMST(G):
    // Assume G = (V, E, c) is simple & connected, c : E → ℝ≥0
    father[0..n) := NONE;  inS[0..n) := false;  dist[0..n) := ∞
    Q := new IndexMinPQ(n)
    Q.insert(0,0)
    while ¬Q.isEmpty()
        visit(Q.delMin())
    return {{father[v], v} : v ∈ [1..n)}

procedure visit(v):
    for (w, c) ∈ G.adj[v] // edge vw with cost c
        if ¬inS[w]
            if c < dist[w] // vw currently cheapest edge to w
                father[w] := v;  dist[w] := c
                if Q.contains(w) // w already active
                    Q.decreaseKey(w, c)
                else // w now becoming active
                    Q.insert(w, c)
                end if
            end if
    end for
    inS[v] := true;  dist[v] := 0 // v now done
```

- Eager Prim: filter edges eagerly!
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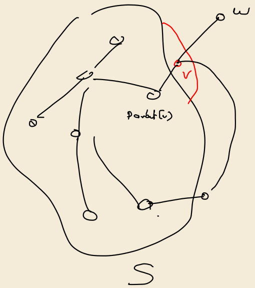

Prim's Algorithm – Eager Implementation Code

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procedure primMST(G):
    // Assume G = (V, E, c) is simple & connected, c : E → ℝ≥0
    father[0..n) := NONE;  inS[0..n) := false;  dist[0..n) := ∞
    Q := new IndexMinPQ(n)
    Q.insert(0,0)
    while ¬Q.isEmpty()
        visit(Q.delMin())
    return {{father[v], v} : v ∈ [1..n)}

procedure visit(v):
    for (w, c) ∈ G.adj[v] // edge vw with cost c
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- Prototypical tricolor traversal variant
  - $v \in$ ***done*** iff $inS[v]$ == true
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  with Fibonacci heaps $O(m + n \log n)$

Minimum Spanning Trees – Discussion

- 👍 MSTs are a versatile modeling tool
- 👍 very efficient to compute even for arbitrary weights
- 👍 Prim's Algorithm (eager version) give best time and space and is efficient in practice

lower bound $\Omega(n+m)$ to "see" entire graph

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  - stronger results known, as well
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- Yes, if **randomization** is allowed (Karger, Klein, Tarjan 1995)
  - uses that linear time suffices to *verify* a given ST as minimal(!)
- General (deterministic, comparison-based, on sparse graphs)? **Open research problem!**
  - Best known general time $O(m\alpha(m,n))$ where α is an "inverse Ackermann function"

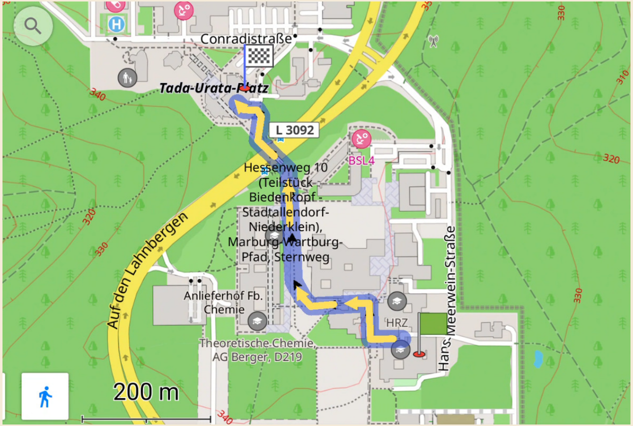
$\alpha(m,n) = \min\{z \ge 1 : A(z, 4\lceil m/n \rceil) > \lg n\}$
$A(0,x) = 2x,\ A(i,0) = 0,\ A(i,1) = 2,\ (i \ge 1),$
$A(i,x) = A\big(i-1, A(i,x-1)\big);\ (i \ge 1, x \ge 2)$

11.5 Greed in Graphs III: Shortest Paths

Metaphor: Route Planning

Given: Road network (map), current location, target location
crossings = vertices, roads = edges, road length = edge weight

Goal: Find shortest path from current location to target



SSSPP

It turns out that a cleaner algorithmic problem is to find shortest paths to *all* vertices.

Single Source Shortest Path Problem (SSSPP)

- **Given:** directed, edge-weighted, simple graph $G = (V, E, c)$
  with edge costs $c : E \to \mathbb{R}$, a start vertex $s \in V$
- **Goal:** a data structure that reports for every $v \in V$:
  $\delta_G(s, v)$: the shortest-path distance from s to v
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- $\mathtt{spath}(v)$ returns a walk w with $c(w) = \delta_G(s, v)$ if such a walk exists

The Trouble with Negative Cycles

- The complications in the definition all stem from **negative-weight edges**

$$\delta_G(s,v) \;=\; \boxed{\inf\big(\{+\infty\} \cup \{c(w) : w \text{ an } s\text{-}v\text{-walk in } G\}\big)}$$

- In general, $\delta_G(s,v)$ can be
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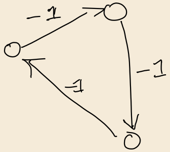

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⇝ In the absense of negative cycles, all shortest walks are **shortest paths** (of at most $n - 1$ edges).

Variants of Shortest Path Problems

Important special cases

1. **Positive SSSPP**
   - $c : E \to \mathbb{R}_{>0}$
   - most relevant case for many applications $\rightsquigarrow$ focus of this section
2. **Unweighted SSSPP**
   - $c(e) = 1$ for $e \in E$ $\rightsquigarrow$ $c(w) =$ #edges for every walk w

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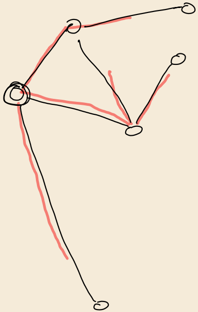
- For the rest of this section, we will assume $c(e) > 0$.
- But: The general case of cyclic graphs with negative edge weights is also relevant
  - We will come back to this case in Unit 12!

Dijkstra's Algorithm

- **Intuition:** Imagine sending out many little pioneers, walking at unit speed from s across all edges in G. The first pioneer to reach a vertex v "claims" v and proclaims the current time (= distance).

  Dijkstra's Algorithm is a event-driven simulation of this process!



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  - Maintain priority queue of events, sorted by time.
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  ⇝ overall process is a graph traversal! claimed = ***done***

Dijkstra's Algorithm – Code & Correctness

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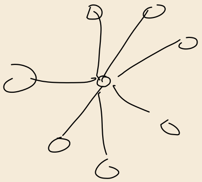
  3. $dist[u] = \delta_G(s, u)$ for all $u \in$ ***done***

Shortest Paths Discussion

- Simple and efficient solution if edge weights are positive
- Dijkstra's Algorithm (with Fibonacci heaps) is worst-case optimal $\Theta(m + n \log n)$
  - (for sorting vertices by distance from s in a comparison-addition model)
  - another fine example of a greedy algorithm!

can reduce sorting to SSSPP



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- 👍 Simple and efficient solution if edge weights are positive
- 👍 Dijkstra's Algorithm (with Fibonacciy heaps) is worst-case optimal
  - (for sorting vertices by distance from s in a comparison-addition model)
  - another fine example of a greedy algorithm!

- improvements often possible for s-t shortest paths (although worst case remains same)
  - in SSSPP Dijkstra, can stop once t is ***done***
  - bidirectional Dijkstra (alternatingly work from both ends until we "meet")
  - A^*/goal-directed search (use cheap lower bound for $\delta_G(v,t)$ in vertex selection)
- we will revisit the general SSSPP (with negative weights)

11.6 Greedy Schedules

Scheduling

- A rich class of optimization problems deals with *scheduling*.
  - **Given:** Jobs (a. k. a. tasks, processes) and machines (a. k. a. workers, processors);
    optionally: constraints (e. g., order of certain jobs)
  - **Common Goal:** Find an optimal schedule, i. e.,
    decide which machine does which jobs, and when,
    such that a given objective is optimized (e. g., shortest makespan)

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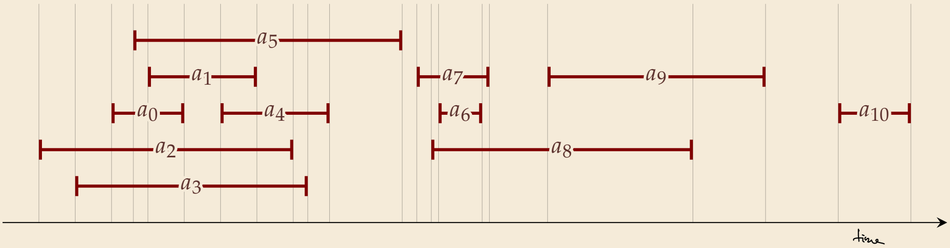
  - **Common Goal:** Find an optimal schedule, i. e.,
    decide which machine does which jobs, and when,
    such that a given objective is optimized (e. g., shortest makespan)
- exact properties change computational complexity of scheduling dramatically
  - can jobs be preempted (paused)?
  - are all machines equally fast on all jobs?
  - can we choose to drop certain jobs (at a cost) or must we schedule all?
  - do jobs have a hard deadline after which they are useless?
  - ...

⇝ *Could fill a module of its own ... Here: one exemplary special case*

The Activity selection problem

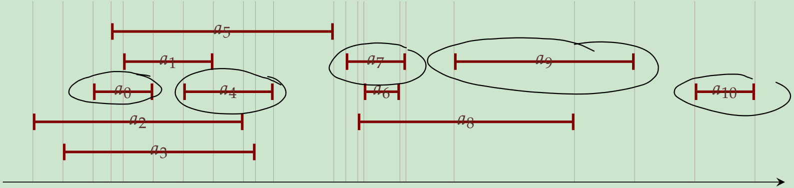
- **Activity Selection:** scheduling for *single* machine, jobs with *fixed* start and end times
  pick a *subset* of jobs without *conflicts*
  Formally:
  - **Given:** Activities $A = \{a_0, \ldots, a_{n-1}\}$, each with a start time s_i and finish time f_i $(0 \le s_i < f_i < \infty)$
  - **Goal:** Subset $I \subseteq [0..n)$ of tasks such that $i, j \in I \wedge i \neq j \implies [s_i, f_i) \cap [s_j, f_j) = \emptyset$
    and $|I|$ is maximal among all such subsets
  - We further assume that jobs are sorted by finish time, i. e., $f_0 \le f_1 \le \cdots \le f_{n-1}$
    (if not, easy to sort them in $O(n \log n)$ time)

$I = \{0, 4, 7, 9, 10\}$



Clicker Question

What is the maximal number of independent (non-overlapping) tasks you can find?



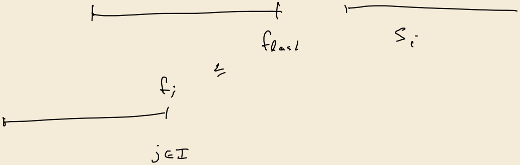
→ sli.do/cs566

Greedy Activity Selection

```
procedure greedyActivitySelection(s[0..n), f[0..n))
    I := {0}
    last := 0
    for i := 1, ..., n − 1
        if s[i] ≥ f[last] // no conflict, add it!
            I := I ∪ {i}
            last := i
    return I
```

- running time $O(n)$ trivial (assumes that tasks already sorted!)

always chooses feasible set I by sorted order



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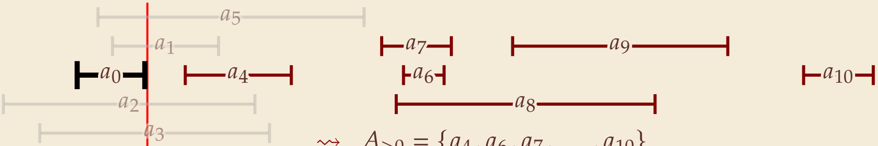
- running time $O(n)$ trivial (assumes that tasks already sorted!)
- **Correctness:** greedyActivitySelection (gAS) is effectively recursive:

$$\text{gAS}(A) = \{0\} \cup \text{gAS}(A_{\geq 0})$$
for $A_{\geq 0} = \{a_i : s_i \geq f_0\}$



$\rightsquigarrow \quad A_{\geq 0} = \{a_4, a_6, a_7, \ldots, a_{10}\}$

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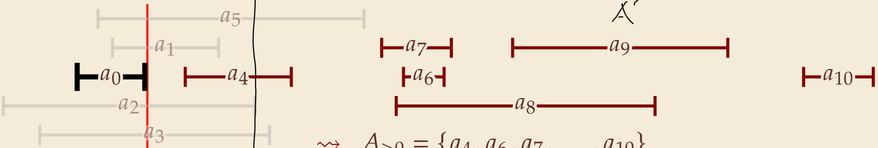

$\rightsquigarrow$ $A_{\geq 0} = \{a_4, a_6, a_7, \ldots, a_{10}\}$

We prove:

1. $\exists$ optimal solution I^* with $0 \in I^*$
2. I^* with $0 \in I^*$ is an optimal solution iff $I^* \setminus \{0\}$ is an optimal solution for $A_{\geq 0}$.

$\rightsquigarrow$ Correctness of gAS follows by induction on n.

Greedy Activity Selection – Correctness Proof

Proofs:

1. $\exists$ optimal solution I^* with $0 \in I^*$

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1. $\exists$ optimal solution I^* with $0 \in I^*$
   - Let I^* be some optimal solution and let $i = \min I^*$.
   - If $i = 0$, we are done.

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   - Otherwise, since I^* is conflict-free and a_0 finishes earlier than a_i, $I^* \setminus \{i\} \cup \{0\}$ is also conflict-free.

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2. I^* with $0 \in I^*$ is an optimal solution iff $I^* \setminus \{0\}$ is an optimal solution for $A_{\geq 0}$.

"$\Rightarrow$" by contraposition. $A \Rightarrow B$ $\neg B \Rightarrow \neg A$

Let $I_{\geq 0} = I \setminus \{0\}$ be a non-optimal solution for $A_{\geq 0}$, i. e.,
$\exists$ solution $I^*_{\geq 0}$ for $A_{\geq 0}$ with $|I^*_{\geq 0}| > |I_{\geq 0}|$.

Greedy Activity Selection – Correctness Proof

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Then also $|I| = \underbrace{|I_{\geq 0} \cup \{0\}|} < |I^*_{\geq 0} \cup \{0\}|$.

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"$\Leftarrow$" by contraposition. Let I be non-optimal for A, i. e., $|I^*| > |I|$ exists.
By Claim 1, we can assume that $0 \in I^*$.
Then $|I \setminus \{0\}| < |I^* \setminus \{0\}|$.

11.7 The Essence of Greed: Matroids

Set Systems

We will now see a formalism to unify the study a whole class of Greedy algorithms.

- **Hereditary Set System**:
  $(S, \mathcal{I})$ for a finite set S and a set of *"independent" sets* $\mathcal{I} \subseteq 2^S$ is a *hereditary* set system if $B \in \mathcal{I} \wedge A \subseteq B \implies A \in \mathcal{I}$
- **Weighted hereditary set system:**
  $(S, \mathcal{I}, w)$ with a hereditary set system $(S, \mathcal{I})$ and weight $w : S \to \mathbb{R}_{\geq 0}$
- We extend w from S to 2^S via $w(A) := \sum_{x \in A} w(x)$

$S = [1..10]$

$w(x) = x$

$w(\{1,2,3\}) = 6$

⇝ Natural *optimization problem* for weighted set system: $\boxed{\max_{A \in \mathcal{I}} w(A)}$

- usually also: find this set A, i. e., $\arg\max_{A \in \mathcal{I}} w(A)$

Canonical Greedy Algorithm

- Given a weighted set system, we can try to greedily optimize $w(A)$:

```
procedure canonicalGreedy(S, 𝓘, w)
    // Assume S = {s_1, ..., s_n} sorted by weight: w(s_1) ≥ w(s_2) ≥ ··· ≥ w(s_n)
    A := ∅  ∈ 𝓘
    for i := 1, ..., n
        if A ∪ {s_i} ∈ 𝓘
            A := A ∪ {s_i}
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```

⇝ *When does this greedy algorithm succeed, i. e., find* $\arg\max_{A \in \mathcal{I}} w(A)$?

Canonical Greedy Algorithm

- Given a weighted set system, we can try to greedily optimize $w(A)$:

---

procedure canonicalGreedy($S, \mathcal{I}, w$)
  // Assume $S = \{s_1, \ldots, s_n\}$ *sorted by weight:* $w(s_1) \ge w(s_2) \ge \cdots \ge w(s_n)$
  $A := \emptyset$
  for $i := 1, \ldots, n$
    if $A \cup \{s_i\} \in \mathcal{I}$
      $A := A \cup \{s_i\}$
  return A

⇝ *When does this greedy algorithm succeed, i.e., find* $\arg\max_{A \in \mathcal{I}} w(A)$*?*

- Certainly not always:
  $S = \{x, y, z\}, \quad \mathcal{I} = \big\{\emptyset, \{x\}, \{y\}, \{z\}, \{y, z\}\big\}$
  $w(x) = 3, w(y) = w(z) = 2$

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  $w(x) = 3, w(y) = w(z) = 2$

- Indeed: Greedy succeeds **if *and only if*** $(S, \mathcal{I})$ is a ***matroid***.

Matroids

- **Matroid:**
  Hereditary set system $(S, \mathcal{I})$ is a matroid if it satisfies the *exchange property*:
  $A, B \in \mathcal{I} \;\wedge\; |A| < |B| \implies \exists x \in B \setminus A : A \cup \{x\} \in \mathcal{I}$
- Prototypical example (also origin of names):
  - S = rows of a given matrix
  - $\mathcal{I}$ = set of **linearly independent** rows

  $\rightsquigarrow$ $(S, \mathcal{I})$ is a matroid by *Steinitz exchange lemma* („*Austauschlemma der linearen Algebra*“)

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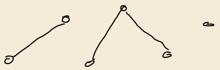
- Further example: *Graphic Matroid*: Given an undirected graph $G = (V, E)$
  - $S = E$
  - $A \in \mathcal{I}$ iff (V, A) is **acyclic**

  $\rightsquigarrow$ check exchange property:
  adding k acyclic edges reduces #connected components by exactly k
  if $|B| > |A|$, some edge in $B \setminus A$ does not close a cycle in A

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  $\rightsquigarrow$ check exchange property:
  adding k acyclic edges reduces #connected components by exactly k
  if $|B| > |A|$, some edge in $B \setminus A$ does not close a cycle in A
  - set $w(e) = W - c(e)$ for c the edge cost and $W > \max c(e)$

  $\rightsquigarrow$ a maximum-weight independent set in $(S, \mathcal{I})$ iff MST of G!

Greedy iff Matroid

Theorem:
Let $(S, \mathcal{I})$ be a hereditary set system. The following statements are equivalent

1. canonicalGreedy$(S, \mathcal{I}, w) = \arg\max_{A \in \mathcal{I}} w(A)$ for *all* weights $w : S \to \mathbb{R}_{\geq 0}$.
2. $(S, \mathcal{I})$ is a matroid.

Proof: A is $\subseteq$-maximal $\Leftrightarrow$ $\nexists B : A \subsetneq B \land B \in \mathcal{I}$

Note: All $\subseteq$-maximal independent sets must have equal cardinality (exchange property!)

(2) ⇒ (1): greedy always chooses $\subseteq$-maximal set A (by construction) ⇝ $|A|$ maximal $= k$

assume $B \in \mathcal{I}$ with $w(B) > w(A)$, where $w(B) = \sum_{x \in B} w(x)$ and $w(A) = \sum_{x \in A} w(x)$

$A = \{s_{i_1}, s_{i_2}, \ldots, s_{i_k}\}$ $\quad i_1 < i_2 < \cdots < i_k$

$B = \{s_{j_1}, s_{j_2}, \ldots, s_{j_k}\}$ $\quad j_1 < j_2 < \cdots < j_k$

⇒ $\exists \mu \quad w(s_{j_\mu}) > w(s_{i_\mu})$

μ smallest such index

greedy: $j_\mu < i_\mu$

$A' = \{s_{i_1}, \ldots, s_{i_{\mu-1}}\}$ $\quad B' = \{s_{j_1}, \ldots, s_{j_\mu}\}$ $\quad |B'| > |A'|$

$\underset{\text{ex. prop.}}{\Rightarrow} \quad \exists\, s_{j_\sigma} \in B' \setminus A' \; : \; A' \cup \{s_{j_\sigma}\} \in \mathcal{J}$

$w(s_{j_\sigma}) \geq w(s_{j_\mu}) > w(s_{i_\mu})$

$\Rightarrow$ at this point, greedy would have chosen s_{j_σ} instead of s_{i_μ} ↯

$\neg(2) \Rightarrow \neg(1)$ $\quad (S, \mathcal{J})$ not matroid

$\Rightarrow \exists\, A, B \in \mathcal{J} : |A| < |B| \quad \forall x \in B \setminus A : A \cup \{x\} \notin \mathcal{J}$

$k := |B|$

$$w(x) = \begin{cases} k+1 & x \in A \\ k & x \in B \setminus A \\ 0 & \text{else} \end{cases}$$

$w(A) = |A|(k+1) \leq (k-1)(k+1) = k^2 - 1$

$w(B) = k \cdot k$

$\Rightarrow$ greedy outputs $w(A)$

$\square$

Discussion

Matroid Theory

- 👍 If we can identify a problem as matroid, Greedy automatically works!
- 👎 unfortunately often necessarily easier than a direct proof

Greedy Algorithms

- 👍 If applicable, Greedy algorithms usually offer linear running time
- 👍 If successful, correctness proof often insightful for problem solved
- 👎 Restricted to "tame" problems